"Research Directions in
Quantum Field Theory
and String Theory 2020",
Feb. 26, 2020, Sogang U.

The Hamiltonian Dynamics of Horava Gravity

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Collaboration with **Deniz O. Devecioglu**,
[arXiv:2001.02556].

11 years ago…

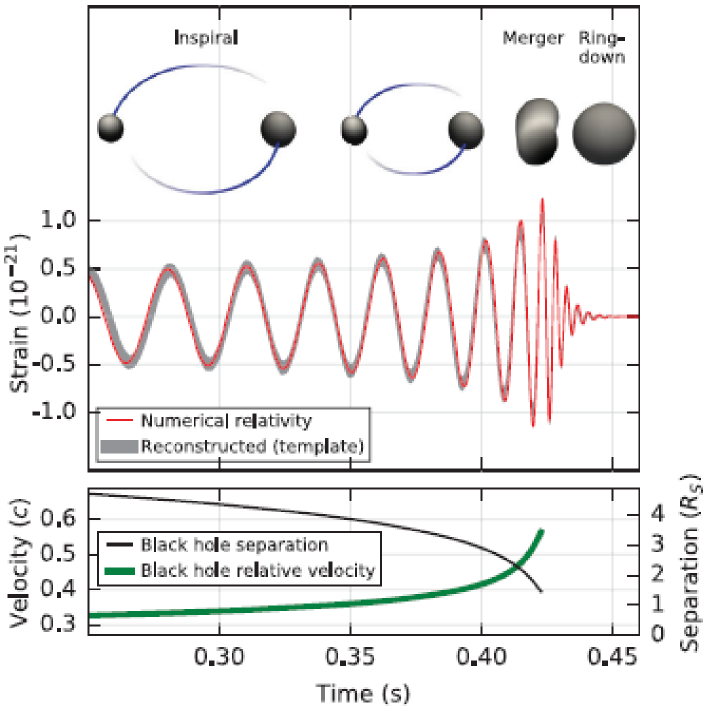
- **arXiv:0901.3775**
- **Quantum Gravity at a Lifshitz Point**
- Petr Horava (UC, Berkeley).
- *(Submitted on **26 Jan 2009** (v1))*

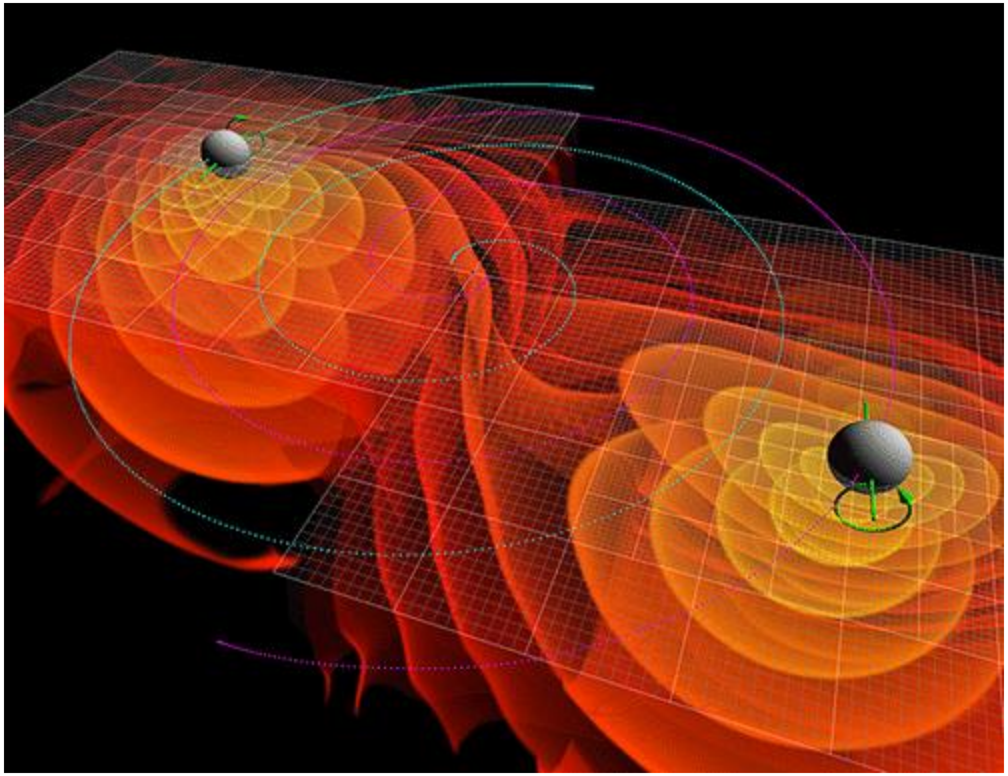
- We present a candidate quantum field theory of gravity with dynamical critical exponent equal to z=3 in the UV. (As in condensed matter systems, z measures the degree of anisotropy between space and time.) This theory, which at short distances describes interacting **nonrelativistic gravitons**, is power-counting renormalizable in 3+1 dimensions. When restricted to satisfy the condition of detailed balance, this theory is intimately related to topologically massive gravity in three dimensions, and the geometry of the Cotton tensor. At long distances, this theory flows naturally to the relativistic value z=1, and could therefore serve as a possible candidate for a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from relevant deformations of the deeply nonrelativistic z=3 theory at short distances.

But now…

Quantum Gravity at a Lifshitz Point
Petr Horava (UC, Berkeley & LBL, Berkeley). Jan 2009. 29 pp.
Published in **Phys.Rev. D79 (2009) 084008**
DOI: 10.1103/PhysRevD.79.084008
e-Print: **arXiv:0901.3775 [hep-th] | PDF**
•Detailed record - Cited by 1821 records 1000+

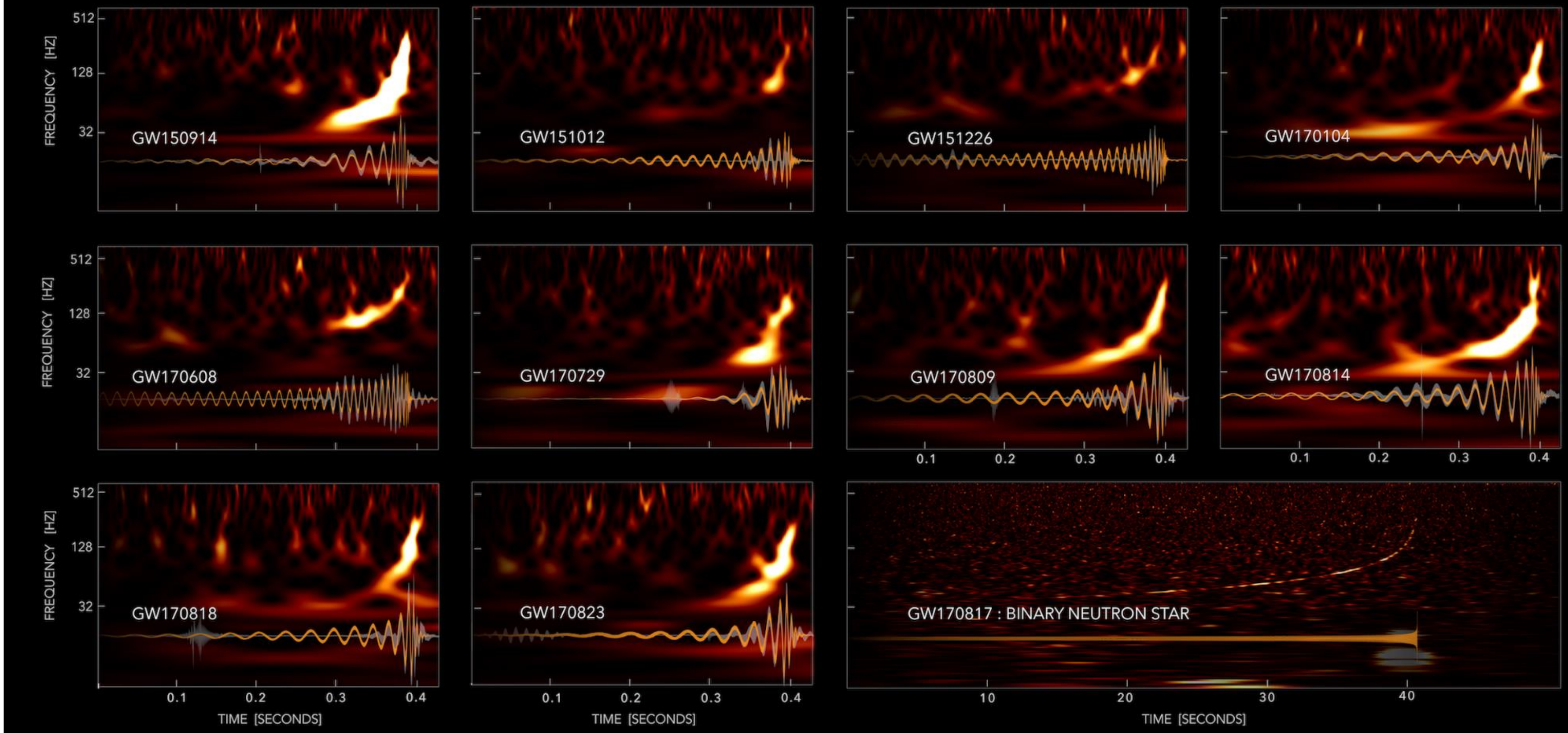
01. Recent LIGO/VIRGO data seem to imply that we need to consider quantum gravity more seriously.





GRAVITATIONAL-WAVE TRANSIENT CATALOG-1

LIGO VIRGO NSF Georgia Tech



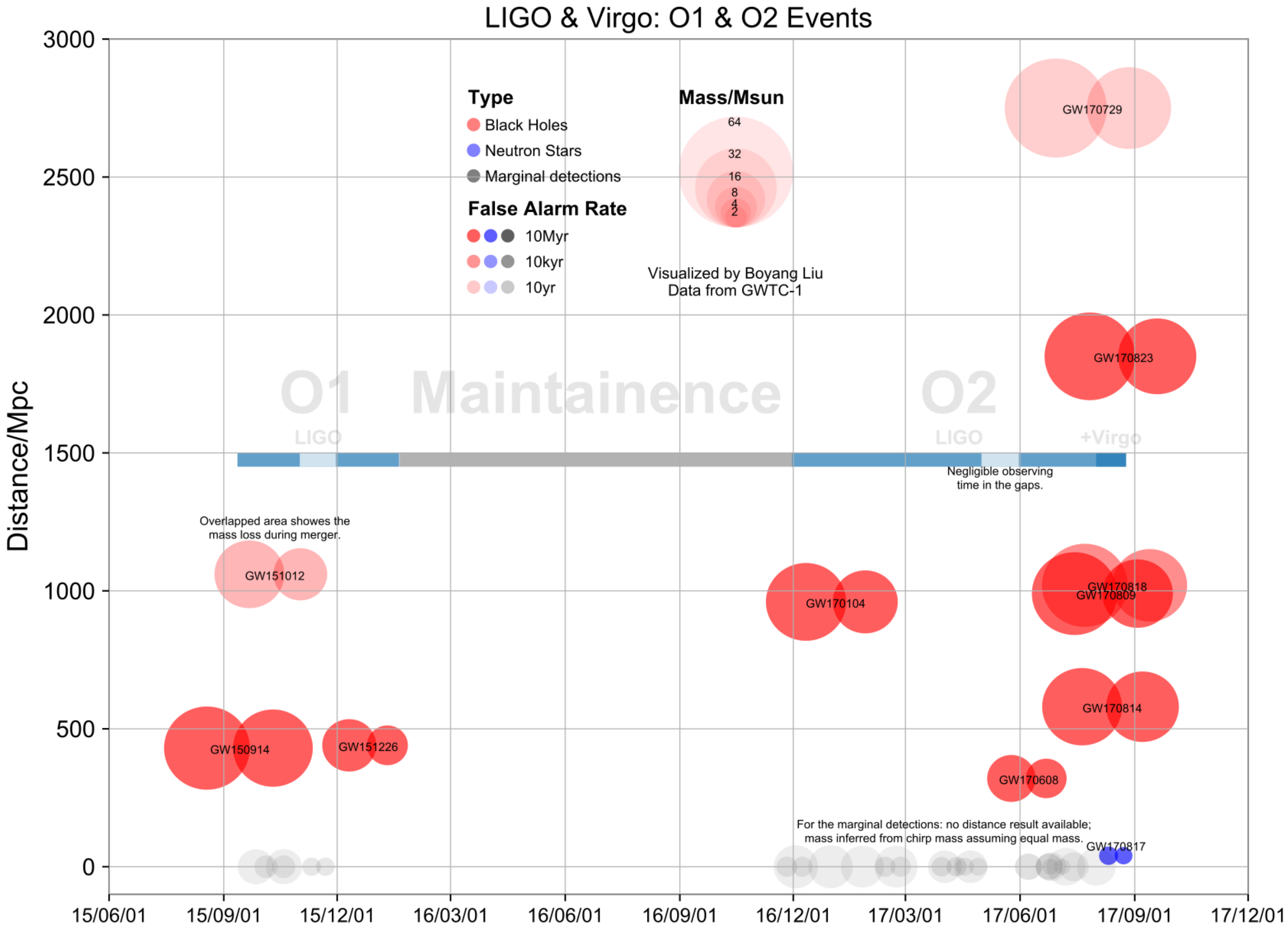
LIGO-VIRGO DATA: HTTPS://DOI.ORG/10.7935/82H3-HH23

▬ WAVELET (UNMODELED)

▬ EINSTEIN'S THEORY

S. GHONGE, K. JANI | GEORGIA TECH

O1 & O2/2015-2017events



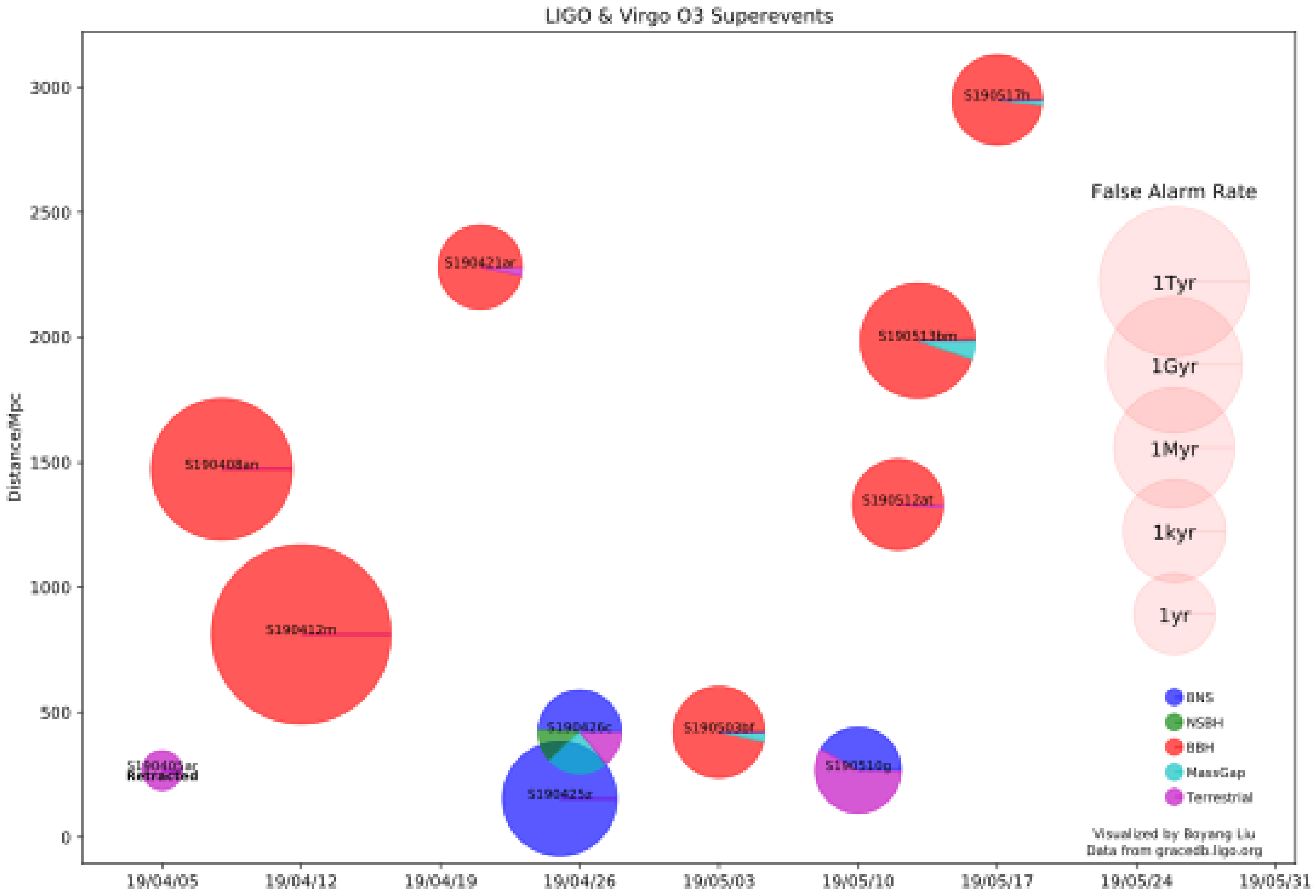
Visualized by Boyang Liu
Data from GWTC-1

Negligible observing time in the gaps.

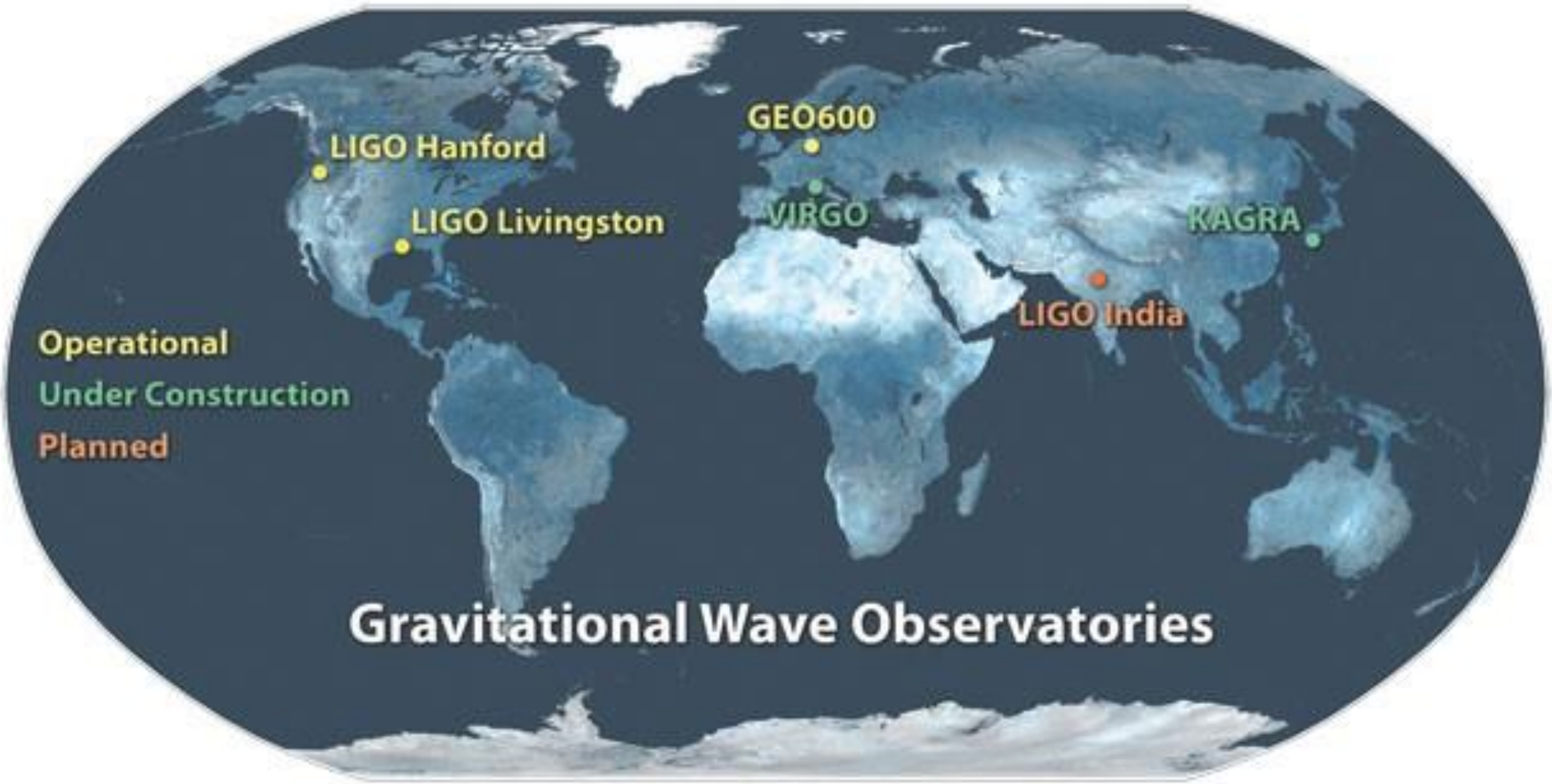
Overlapped area showes the mass loss during merger.

For the marginal detections: no distance result available; mass inferred from chirp mass assuming equal mass.

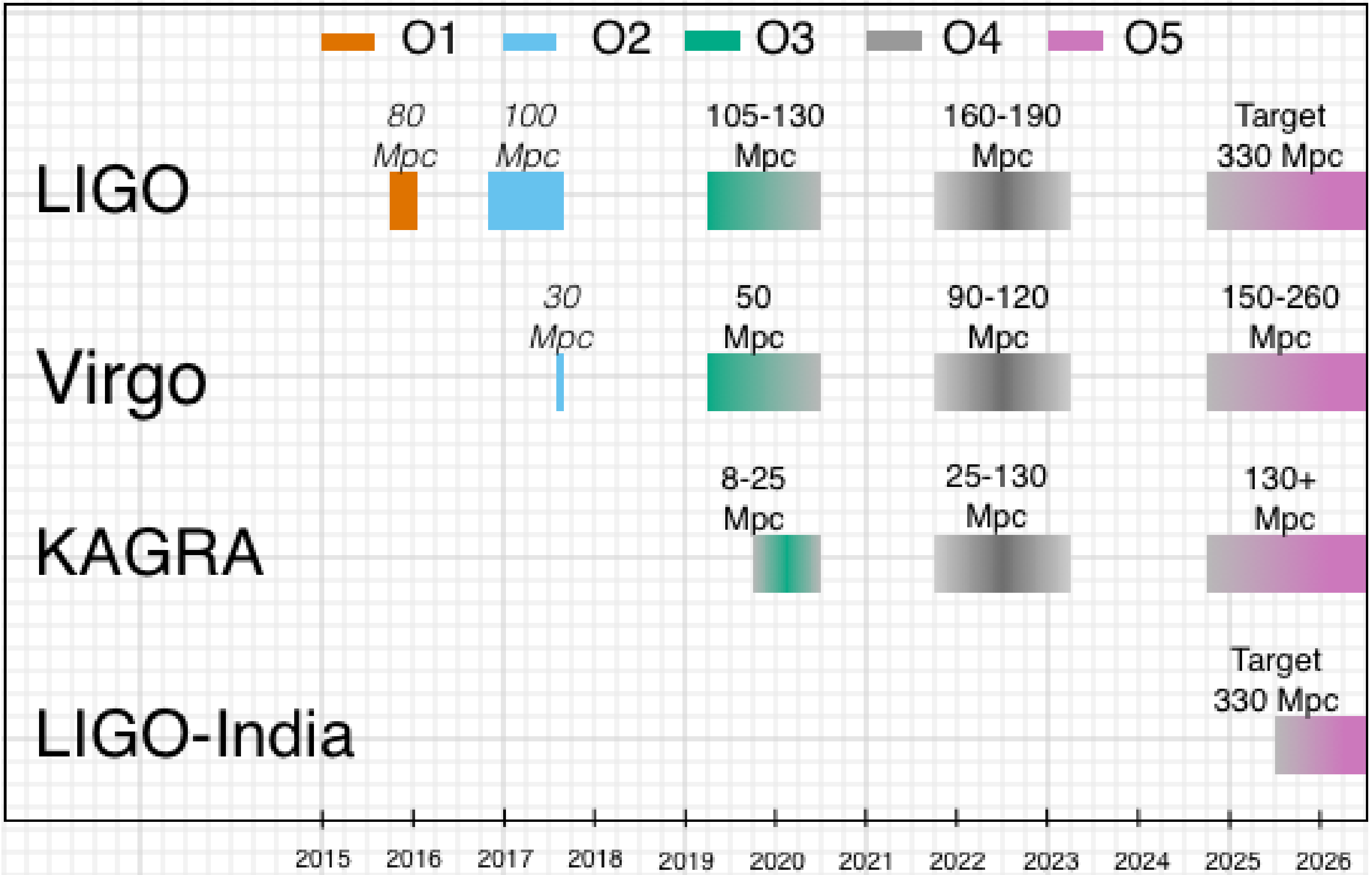
O3/2019 events



Gravitational Wave Observatories



KAGRA will start tomorrow! (plan to start 7-day engineering run from December 17.)



- **We open the strong gravity test era of GR, beyond the weak gravity tests in solar system !!**

- **Cf. Multi-Messenger Astronomy ! (after GW 170817); some strong constraints on alternatives to GR !!**

Cf. Event Horizon Telescope: black hole shadows.

Cf’. Schwarzschild black hole

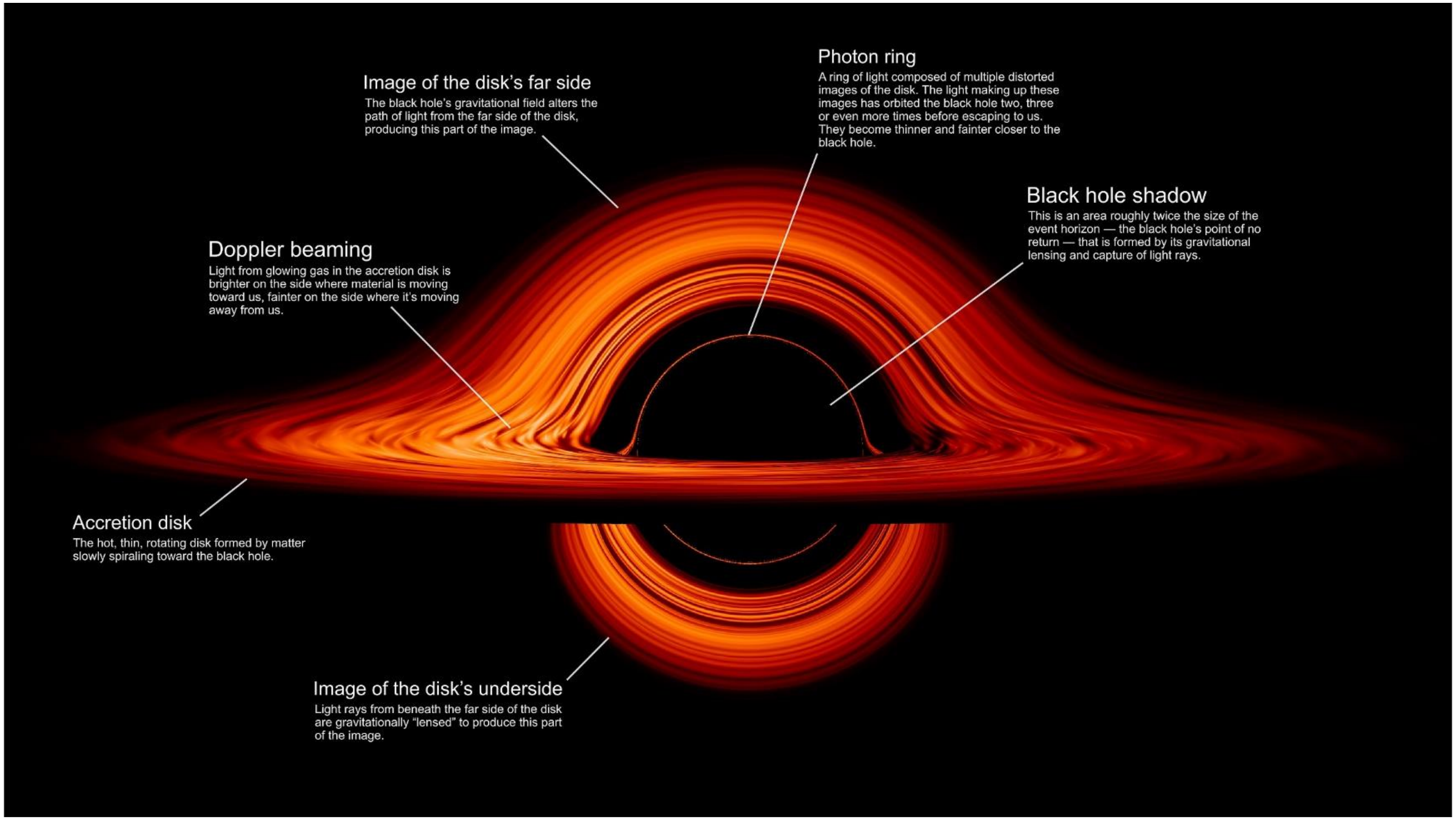


Image of the disk’s far side

The black hole’s gravitational field alters the path of light from the far side of the disk, producing this part of the image.

Photon ring

A ring of light composed of multiple distorted images of the disk. The light making up these images has orbited the black hole two, three or even more times before escaping to us. They become thinner and fainter closer to the black hole.

Black hole shadow

This is an area roughly twice the size of the event horizon — the black hole’s point of no return — that is formed by its gravitational lensing and capture of light rays.

Doppler beaming

Light from glowing gas in the accretion disk is brighter on the side where material is moving toward us, fainter on the side where it’s moving away from us.

Accretion disk

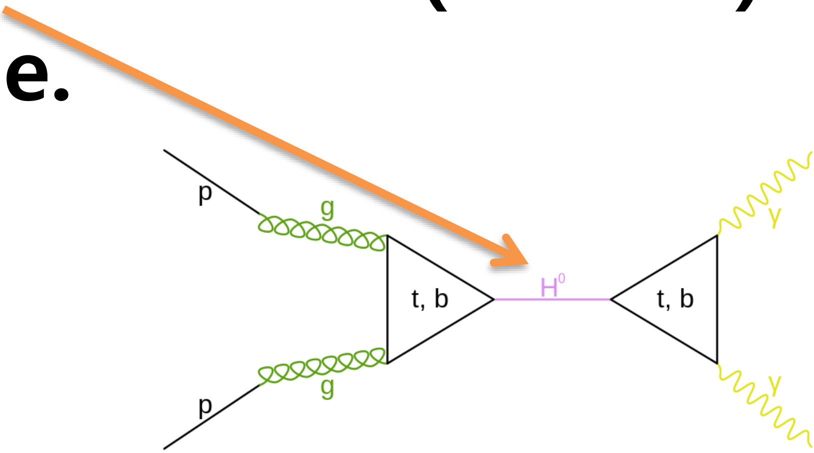
The hot, thin, rotating disk formed by matter slowly spiraling toward the black hole.

Image of the disk’s underside

Light rays from beneath the far side of the disk are gravitationally “lensed” to produce this part of the image.

02. Renormalization has been a powerful constraint on Quantum Theory of Particle Interactions.

- Higgs particle is its a (natural) consequence.



Q: What if we require renormalizability in quantum gravity ?

The renormalizable Q.G. can not be realized in Einstein's gravity or its (relativistic) higher-derivative generalizations (1977,Stelle).

- There are ghosts, in addition to massless gravitons: In R+R^2 gravity, the full (quantum) propagator becomes

$$\frac{1}{k^2} + \frac{1}{k^2} G_N k^4 \frac{1}{k^2} + \frac{1}{k^2} G_N k^4 \frac{1}{k^2} G_N k^4 \frac{1}{k^2} + \ldots = \frac{1}{k^2 - G_N k^4} = \frac{1}{k^2} - \frac{1}{k^2 - 1/G_N}.$$

Massless gravitons

Ghosts (!)

03. Horava (or HL) gravity has been proposed as a renormalizable gravity

- There is no ghost by abandoning equal-footing treatment of space and time (i.e., Lorentz symmetry) in UV (2009, Horava)

- Power counting renormalizable: But no (complete) proof of renormalizability yet !

(Cf. 2015,Barvinsky et al: proof for projectable case)

Cf. Yang-Mills theory (1954,Yang-Mills; Criticized by Pauli;1959,Glashow, Salam-Ward; 1967 Weinberg; 1972,'t Hooft, Veltman)

- Today, I will consider the Hamiltonian dynamics in Horava gravity, which has not been fully understood yet.

- This is related to the long-standing issue of the scalar graviton problem in Horava gravity.

Plan

I. Horava (HL) gravity: Basic Idea

II. Hamiltonian Dynamics of Horava Gravity: Set-Up

III. Dirac's Constraints Analysis

IV. Future Directions

I. Horava (HL) Gravity: Basic Idea

Quick Summary: Renormalizable gravity theory by abandoning Lorentz symmetry in UV : Foliation Preserving Diffeomorphism (*FPDiff*.).

Horava gravity ~ Einstein gravity (with a Lorentz deformation parameter λ)

+ non-covariant deformations with higher spatial derivatives (up to 6 orders)

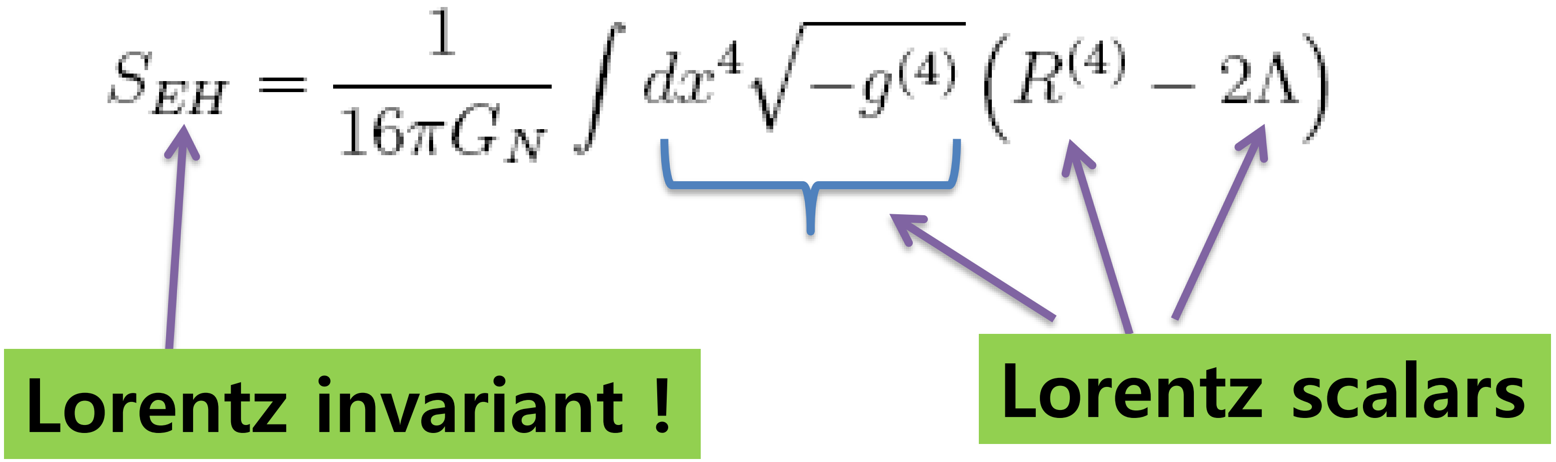
+ "detailed balance" in the coefficients

(5 constant parameters: $\kappa, \lambda, \nu, \mu, \Lambda_W$)

Cf. Einstein gravity: $\lambda = 1$ $c,\ G,\ \Lambda$

The Action Construction:

- **Einstein-Hilbert action:**



$$S_{EH} = \frac{1}{16\pi G_N} \int \underbrace{dx^4 \sqrt{-g^{(4)}}} \left(R^{(4)} - 2\Lambda \right)$$

$$= \frac{1}{16\pi G_N} \int \underbrace{d^4x \sqrt{g}\, N} \underbrace{\{ \left(K_{ij}K^{ij} - K^2 \right) + R - 2\Lambda \}}_{R^{(4)}}$$

in ADM decomposition

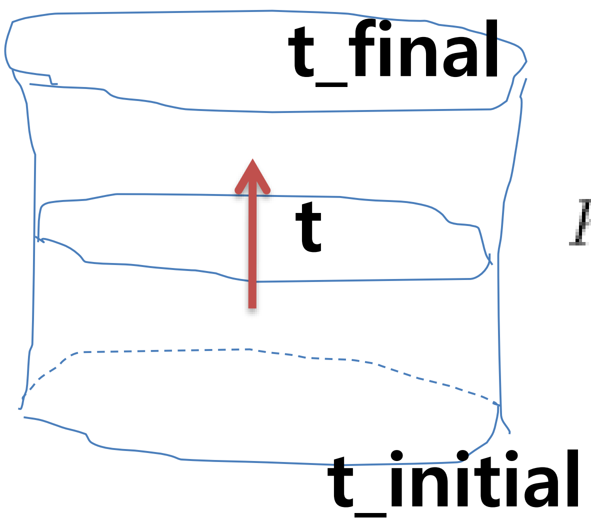
$$ds^2 = -N^2c^2dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$$

- Here, we have used the Gauss-Godacci relation (up to boundary terms)

$$R^{(4)} = \left(K_{ij}K^{ij} - K^2\right) + R$$

Extrinsic curvature of t=constant hypersurface

Intrinsic curvature : 3 curvature



$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right)$$

- In order not to introduce higher-time derivatives to avoid the ``possible" ghost problems, we do not consider "simply" the following terms

$$(K_{ij}K^{ij})^2, K^4 \cdots$$

but only consider

$$R^2, \; R_{ij}R^{ij}, \; \nabla_k R_{ij} \nabla^k R^{ij}, \quad \cdots$$

But in order that this action form is not changed in different coordinates, we need to restrict the coordinate transformations into *FPDiff* ! In more technical terms,..

- In the anisotropic scaling (mom.) dimensions,

$$[\mathbf{x}] = -1, \qquad [t] = -z,$$

we do not need to keep the Lorentz invariant combinations only. (Planck unit)

- For example, we may consider

$$\left(K_{ij}K^{ij} - \lambda K^2\right) + \beta R$$

, in which the Lorentz symmetry is explicitly broken for

$$\lambda \neq 1, \beta \neq 1$$

but there is still Foliation Preserving diffeomorphisms (FPDiff).

- **Then, the action can be written as**

$$S_{\mathrm{Horava}} = \frac{2}{\kappa^2} \int dt\, d^D\mathbf{x}\, \sqrt{g} N \left(K_{ij} K^{ij} - \lambda K^2 \right)$$

Kinetic term

$$+ \int dt\, d^D\mathbf{x}\, \sqrt{g} N\, V[g_{ij}]$$

Potential term

Why 6 order spatial derivatives ?

- The renormalizable quantum gravity can not be realized in Einstein's gravity or its (relativistic) higher-derivative generalizations: There are ghosts, in addition to massless gravitons, and unitarity violation: In R+R^2 gravity, the full propagator becomes

$$\frac{1}{k^2}+\frac{1}{k^2}G_N k^4\frac{1}{k^2}+\frac{1}{k^2}G_N k^4\frac{1}{k^2}G_N k^4\frac{1}{k^2}+\ldots=\frac{1}{k^2-G_N k^4}=\frac{1}{k^2}-\frac{1}{k^2-1/G_N}.$$

Massless gravitons

Ghosts (!)

- But, for anisotropic (scaling) dimensions,

$$[\mathbf{x}] = -1, \qquad [t] = -z,$$

the propagator for V=R+R^z becomes (?)

$$\frac{1}{\omega^2 - \mathbf{k}^2 - G(\mathbf{k}^2)^z}$$

G: Dimensionless coupling

At high energy with (z>1), this expands as,

$$\frac{1}{\omega^2 - c^2\mathbf{k}^2 - G(\mathbf{k}^2)^z} = \frac{1}{\omega^2 - G(\mathbf{k}^2)^z} + \frac{1}{\omega^2 - G(\mathbf{k}^2)^z} c^2\mathbf{k}^2 \frac{1}{\omega^2 - G(\mathbf{k}^2)^z} + \ldots.$$

Improved UV divergences but no ghost, i.e., no unitarity problem.

- Whereas at low energy,

$$\frac{1}{\omega^2 - \mathbf{k}^2 - G(\mathbf{k}^2)^z} = \frac{1}{\omega^2 - \mathbf{k}^2} + \frac{1}{\omega^2 - \mathbf{k}^2} G(\mathbf{k}^2)^z \frac{1}{\omega - \mathbf{k}^2} + \ldots.$$

Flow to z=1

Dimension counting

- For an arbitrary *spatial* dimension D,

$$[g_{ij}] = 0, \qquad [N_i] = z - 1, \qquad [N] = 0.$$

$$[dt\, d^D \mathbf{x}] = -D - z\,,$$

$$[\kappa] = \frac{z - D}{2}.$$

Dimensionless coupling for z=D:
Power counting renormalizable

$$S_V = \int \underbrace{dt\, d^D \mathbf{x}}_{-D-z} \sqrt{g} N \underbrace{V[g_{ij}]}_{D+z}$$

- So, in D=3 (3+1 space-time), we need the potential V with [V]=6: 6'th-order spatial derivatives with "dimension-less" couplings !
- From

$$[\nabla_k] = [\nabla^k] = 1 \qquad [R_{ij}] = [R^{ij}] = 2,$$

we have large numbers of possible terms, which are invariant by themselves, like

$$\nabla_k R_{ij} \nabla^k R^{ij}, \qquad \nabla_k R_{ij} \nabla^i R^{jk}, \qquad R \Delta R, \qquad R^{ij} \Delta R_{ij};$$

$$R^3, \qquad R^i_j R^j_k R^k_i, \qquad R R_{ij} R^{ij},$$

Detailed Balance Condition:

- We need (foliation preserving Diff invariant) potential term having 6th order spatial derivatives at most (power-counting renormalizable with z=3) :

$$S_V = \int dt\, d^D\mathrm{x}\, \sqrt{g} N\, V[g_{ij}]$$

- There are too large numbers of possible terms, which are invariant by themselves.

- Horava **required** the potential to be of

$$S_V = \frac{\kappa^2}{8}\int dt\, d^D\mathrm{x}\, \sqrt{g} N\, E^{ij} \mathcal{G}_{ijk\ell} E^{k\ell},$$

by **demanding**

D-dimensional Euclidean action

$$\sqrt{g}E^{ij} = \frac{\delta W[g_{k\ell}]}{\delta g_{ij}}$$

for some action W, and $\mathcal{G}_{ijk\ell}$, the inverse of **De Witt** metric,

$$G^{ijk\ell} = \frac{1}{2}\left(g^{ik}g^{j\ell} + g^{i\ell}g^{jk}\right) - \lambda g^{ij}g^{k\ell}$$

Cf. **Kinetic** part is also given by

$$S = \frac{1}{2}\int dt\, d^D\mathrm{x}\, \sqrt{g}\left\{\frac{1}{\kappa^2 N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right) G^{ijk\ell}\left(\dot{g}_{k\ell} - \nabla_k N_\ell - \nabla_\ell N_k\right)\right.$$

- For D=3, W is 3-dimensional Euclidean action.
- First, we may consider Einstein-Hilbert action,

$$W = \frac{1}{\kappa_W^2} \int d^D\mathrm{x} \sqrt{g}(R - 2\Lambda_W).$$

then, this gives 4'th-order spatial derivative potential, with a dimensionful coupling,

$$S_V = \frac{\kappa^2}{8\kappa_W^4} \int dt\, d^D\mathrm{x} \sqrt{g} N \left(R^{ij} - \frac{1}{2} R g^{ij} + \Lambda_W g^{ij} \right) \mathcal{G}_{ijk\ell} \left(R^{k\ell} - \frac{1}{2} R g^{k\ell} + \Lambda_W g^{k\ell} \right).$$

- So, this is not enough to get 6'th order !!

- **In 3-dim, we also have a peculiar, 3’rd-derivative-order action, called (gravitational) Chern-Simons action.**

$$W = \frac{1}{w^2} \int_\Sigma \omega_3(\Gamma).$$

$$\omega_3(\Gamma) = \mathrm{Tr}\left(\Gamma \wedge d\Gamma + \frac{2}{3}\Gamma \wedge \Gamma \wedge \Gamma\right) \equiv \varepsilon^{ijk}\left(\Gamma^m_{i\ell}\partial_j\Gamma^\ell_{km} + \frac{2}{3}\Gamma^m_{i\ell}\Gamma^\ell_{jm}\Gamma^m_{kn}\right) d^3\mathrm{x}$$

- **This produces the potential**

$$-\frac{\kappa^2}{2w^4} C_{ij} C^{ij}$$

with the Cotton tensor $C^{ij} = \varepsilon^{ik\ell}\nabla_k\left(R^j_\ell - \frac{1}{4}R\delta^j_\ell\right)$

- Then, in total, we get the 6’th order action

$$S=\int dt\, d^3\mathrm{x}\,\sqrt{g}\,N\left\{\frac{2}{\kappa^2}K_{ij}G^{ijk\ell}K_{k\ell}-\frac{\kappa^2}{2}\left[\frac{1}{w^2}C^{ij}-\frac{\mu}{2}\left(R^{ij}-\frac{1}{2}Rg^{ij}+\Lambda_W\, g^{ij}\right)\right]\right.$$
$$\left.\times\, g_{ijk\ell}\left[\frac{1}{w^2}C^{k\ell}-\frac{\mu}{2}\left(R^{k\ell}-\frac{1}{2}Rg^{k\ell}+\Lambda_W\, g^{k\ell}\right)\right]\right\}.$$

from

$$W=\frac{1}{w^2}\int\omega_3(\Gamma)+\mu\int d^3\mathrm{x}\,\sqrt{g}(R-2\Lambda_W).$$

So, we have 5 constant parameters, which seem to be minimum, from the detailed balancing.

- Some improved UV behaviors, without ghosts, are expected, i.e., renormalizability

→ Predictable Quantum Gravity !!(?)

- But, it seems that the detailed-balance condition is too strong to get a physically viable universe !
- For example, there is no Minkowski , i.e., vanishing c.c. vacuum solution ! (Lu, Mei, Pope '09): There is no Newtonian gravity limit !!
- We need to break the detailed balance but without altering UV behaviors: It is called, soft breaking in IR or "IR modification".

- On the other hand, in UV we need some modification for scale invariant cosmological perturbations:

- With the detailed balance, tensor spectrum is "scale" invariant but scalar spectrum is not ! (Brandenberger et al, Gong et al, 2010).

- We need to break the detailed balance but without altering UV behaviors of scale invariant *tensor* modes: It is called, "UV modification" (S. Shin, MIP, 2017).

II. Hamiltonian Dynamics of Horava Gravity: Set-Up

- We start with the action (up to boundary terms),

$$S = \int_M dt d^D \mathbf{x} \sqrt{g} N \left\{ \frac{2}{\kappa^2} \left(K_{ij} K^{ij} - \lambda K^2 \right) - \mathcal{V}[g_{ij}, \nabla_i] \right\}$$

- with

$$ds^2 = -N^2 dt^2 + g_{ij} \left(dx^i + N^i dt \right) \left(dx^j + N^j dt \right)$$

$$K_{ij} = \frac{1}{2N} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right)$$

- The action is invariant under the foliation-preserving Diff (FPDiff) for an arbitrary lambda:

$$\begin{aligned}
\delta x^i &= -\zeta^i(t, \mathbf{x}), \; \delta t = -f(t), \\
\delta g_{ij} &= \partial_i \zeta^k g_{jk} + \partial_j \zeta^k g_{ik} + \zeta^k \partial_k g_{ij} + f \dot{g}_{ij}, \\
\delta N_i &= \partial_i \zeta^j N_j + \zeta^j \partial_j N_i + \dot{\zeta}^j g_{ij} + f \dot{N}_i + \dot{f} N_i, \\
\delta N &= \zeta^j \partial_j N + f \dot{N} + \dot{f} N.
\end{aligned}$$

- For GR case (lambda=1, V=R), there is an accidental symmetry which do not preserve the foliation so that the full Diff is recovered: Recovery of GR in lambda=1, V=R limit !

- **For simplicity of our analysis, we consider but in arbitrary dimension D,**

$$\mathcal{V}[g_{ij}, \nabla_i] = \mathcal{V}(R)$$

- **with 2xD (spatial) derivatives for the (power-counting) renormalizable theory.**
- **Then, the first-order action is given by**

$$S = \int_M dt d^D \mathrm{x} \left\{ \pi^{ij} \dot{g}_{ij} - N \mathcal{H}^t - N_i \mathcal{H}^i \right.$$

- **with the conjugate momentum**

$$\pi^{ij} \equiv \frac{\delta S}{\delta \dot{g}_{ij}} = \frac{2\sqrt{g}}{\kappa^2} \left(K^{ij} - \lambda K g^{ij} \right)$$

- **and**

$$
\begin{aligned}
\mathcal{H}^t &= \frac{\kappa^2}{2\sqrt{g}}\left[\pi^{ij}\pi_{ij} - \left(\frac{\lambda}{D\lambda - 1}\right)\pi^2\right] + \sqrt{g}\mathcal{V}, \\
\mathcal{H}^i &= -2\nabla_j \pi^{ij},
\end{aligned}
$$

III. Dirac's Constraints Analysis

1. **Primary constraints:**

$$\Phi_t \equiv \pi_N \approx 0, \;\; \Phi_i \equiv \pi_{N^i} \approx 0$$

from the definition of conjugate mom.

$$\pi_N \equiv \delta S/\delta \dot{N} \text{ and } \pi_{N^i} \equiv \delta S/\delta \dot{N}^i.$$

2. Preservation of primary constraints

$$\dot{\Phi}_\mu = \{\Phi_\mu, H_c\} \approx 0,$$

with the **canonical Hamiltonian,**

$$H_c = \int_{\Sigma_t} d^D x \left\{ N\mathcal{H}^t + N_i\mathcal{H}^i \right\}$$

gives the **secondary constraints,**

$$\mathcal{H}^t \approx 0, \quad \mathcal{H}^i \approx 0.$$

- With the primary constraints, one can consider the extended Hamiltonian (with Lagrange multipliers u_μ),

$$H_E = H_c + \int_{\Sigma_t} d^D\mathbf{x}\left[u^\mu \Phi_\mu\right]$$

- Dynamical Eq. (Cf. Wald for GR case):

$$\begin{aligned}
\dot{g}_{ij} &= \{g_{ij}, H_c\} = \frac{\delta H_c}{\delta \pi^{ij}} \\
&= \left(\frac{\kappa^2}{2}\right)\frac{2N}{\sqrt{g}}\left(\pi_{ij} - \tilde{\lambda} g_{ij}\pi\right) + \nabla_i N_j + \nabla_j N_i, \\
\dot{\pi}^{ij} &= \{\pi^{ij}, H_c\} = -\frac{\delta H_c}{\delta g_{ij}} \\
&= \left(\frac{\kappa^2}{2}\right)\frac{N}{\sqrt{g}}\left[\frac{1}{2}g^{ij}\left(\pi_{mn}\pi^{mn} - \tilde{\lambda}\pi^2\right) - 2\left(\pi^{im}\pi^j_m - \tilde{\lambda}\pi\pi^{ij}\right)\right] \\
&\quad -N\sqrt{g}\left[\frac{1}{2}g^{ij}\mathcal{V}(R) - R^{ij}\mathcal{V}'(R)\right] - \sqrt{g}\left[\nabla^i\nabla^j\left(N\mathcal{V}'(R)\right) - g^{ij}\nabla_m\nabla^m\left(N\mathcal{V}'(R)\right)\right] \\
&\quad +\nabla_m\left(N^m\pi^{ij}\right) - \left(\nabla_m N^i\right)\pi^{jm} - \left(\nabla_m N^j\right)\pi^{im},
\end{aligned}$$

where, $\tilde{\lambda} = \lambda/(D\lambda - 1)$.

- **After tedious computations, we obtain**

$$\begin{aligned}\{\langle\eta^t\mathcal{H}^t(x)\rangle,\langle\zeta^t\mathcal{H}^t(y)\rangle\} &= \int d^Dz\left(\eta^t\nabla_i\zeta^t-\zeta^t\nabla_i\eta^t\right)C^i(z),\\ \{\langle\eta^t\mathcal{H}^t(x)\rangle,\langle\zeta^i\mathcal{H}_i(y)\rangle\} &= -\int d^Dz\,\zeta^i\nabla_i\eta^t\mathcal{H}^t(y),\\ \{\langle\eta^i\mathcal{H}_i(x)\rangle,\langle\zeta^j\mathcal{H}_j(y)\rangle\} &= \int d^Dz\left(\eta^i\nabla_i\zeta^j-\zeta^i\nabla_i\eta^j\right)\mathcal{H}_j(y),\end{aligned}$$

with ($\hat{\lambda}\equiv(\lambda-1)/(D\lambda-1)$**)**

$$C^i=\left(-\frac{\kappa^2}{2}\right)\left[\left(\mathcal{H}^i+2\hat{\lambda}\nabla^i\pi\right)\mathcal{V}'(R)+2\left(\pi^{ij}-\hat{\lambda}g^{ij}\pi\right)\nabla_j\mathcal{V}'(R)\right]$$

for the smeared constraint, $\langle\eta\mathcal{H}\rangle\equiv\int d^Dx\eta\mathcal{H}$

- **For** $-\mathcal{V}(R)=\Lambda+\alpha R+\xi R^n$ **,**

$$C^i=\left(\frac{\kappa^2}{2}\right)\left[\left(\mathcal{H}^i+2\hat{\lambda}\nabla^i\pi\right)\left(\alpha+\xi nR^{n-1}\right)+2\left(\pi^{ij}-\hat{\lambda}g^{ij}\pi\right)\xi n\nabla_jR^{n-1}\right]$$

- For GR case ($\lambda = 1,\ i.e.,\ \hat{\lambda} = 0,\ \xi = 0,$),

$$C^i = (\kappa^2/2)\alpha\mathcal{H}^i \approx 0$$

and Hamiltonian constraint $\mathcal{H}^t \approx 0$ becomes the first-class constraints, with a closed constraints algebra

For a more general case with arbitrary lambda but $\xi = 0,$ (no higher-derivative potential), called λR model or lambda-deformed GR, we have the same results with $\pi \approx 0$ (maximal slicing) !

- But, for the most general cases

  (i) with the higher derivatives or

  (ii) non-maximal slicing in λR model, the Hamiltonian constraint becomes second-class!

- Whereas, the momentum constraint $\mathcal{H}_i \approx 0$ is still the first-class constraint, as in GR.

- For the local constraints, we obtain

$$
\begin{aligned}
\{\mathcal{H}^t(x), \mathcal{H}^t(y)\} &= C^i(x)\nabla_i^x \delta^D(\mathrm{x}-\mathrm{y}) - C^i(y)\nabla_i^y \delta^D(\mathrm{x}-\mathrm{y}), \\
\{\mathcal{H}^t(x), \mathcal{H}_i(y)\} &= -\mathcal{H}^t(y)\nabla_i^y \delta^D(\mathrm{x}-\mathrm{y}), \\
\{\mathcal{H}_i(x), \mathcal{H}_j(y)\} &= \mathcal{H}_i(y)\nabla_j^x \delta^D(\mathrm{x}-\mathrm{y}) + \mathcal{H}_j(x)\nabla_i^x \delta^D(\mathrm{x}-\mathrm{y}),
\end{aligned}
$$

- 3. Preservation of the secondary constraints,

$$\begin{aligned}\dot{\mathcal{H}}^t(x) &= \{\mathcal{H}^t(x), H_E\}\\ &= \frac{1}{N}\nabla_i(N^2C^i) + \nabla_i(N^i\mathcal{H}^t) \approx \frac{1}{N}\nabla_i(N^2\tilde{C}^i),\\ \dot{\mathcal{H}}_i(x) &= \{\mathcal{H}_i(x), H_E\}\\ &= \mathcal{H}^t\nabla_i N + \nabla_j(N^j\mathcal{H}_i) + \mathcal{H}_j\nabla_i N^j \approx 0,\end{aligned}$$

produces a tertiary constraint,

$$\tilde{\Omega} \equiv \nabla_i(N^2\tilde{C}^i) \approx 0,$$

where

$$\tilde{C}^i \equiv \left(\frac{\kappa^2}{2}\right)\left[2\hat{\lambda}\nabla^i\pi\left(\alpha + \xi n R^{n-1}\right) + 2\left(\pi^{ij} - \hat{\lambda}g^{ij}\pi\right)\xi n\nabla_j R^{n-1}\right].$$

- **4. Preservation of the tertiary constraint gives,**

$$\begin{aligned}
\dot{\tilde{\Omega}}(x) &= \{\tilde{\Omega}(x), H_E\} \\
&= \{\tilde{\Omega}(x), H_c\} + 2\tilde{\Omega}\left(\frac{u^t}{N}\right) + 2\tilde{C}^i N^2 \nabla_i \left(\frac{u^t}{N}\right), \\
&\approx \{\tilde{\Omega}(x), H_c\} + 2\tilde{C}^i N^2 \nabla_i \left(\frac{u^t}{N}\right) \approx 0.
\end{aligned}$$

- **Then, there are two cases depending on whether $\tilde{C}^i \approx 0$ or not.**

A. Case $\tilde{C}^i \approx 0$

- The Lagrange multiplier u^t is not determined, but we have

$$
\begin{aligned}
\dot{\tilde{\Omega}}(x) &\approx \{\tilde{\Omega}(x), H_c\} \\
&\approx \nabla_i \left(N^2 B^i[N]\right) \equiv \Phi[N] \approx 0
\end{aligned}
$$

with

$$
\begin{aligned}
B^i[N] &= \frac{2\alpha\hat{\lambda}}{\lambda D - 1}\left(\frac{\kappa^2}{2}\right)^2 \frac{1}{\sqrt{g}}\left\{\left[(2\lambda+1)g^{ij}\pi - 2(\lambda D - 1)\pi^{ij}\right] N\nabla_j \pi + \pi^2 \nabla^i N\right\} \\
&+2\alpha\hat{\lambda}(D-1)\left(\frac{\kappa^2}{2}\right)\sqrt{g}\nabla^i\left[\left(\alpha R + \frac{\Lambda D}{D-1}\right)N - \alpha\nabla^2 N\right].
\end{aligned}
$$

- For GR case $(\hat{\lambda} = 0, \xi = 0)$, this is trivially satisfied, as it should be.
- Otherwise, we have a new constraint

$$\Phi[N] \approx 0.$$

- **Preservation of the new constraint,**

$$\begin{aligned}\dot{\Phi}(x) &= \{\Phi(x), H_E\} \\ &= \{\Phi(x), H_c\} + \nabla_i \left(N^2 B^i[u^t]\right)\end{aligned}$$

gives, after a long computation,

$$\begin{aligned}\nabla_i \left(N^2 B^i[u^t]\right) \approx\ & 4\alpha\hat{\lambda}(D-1)\left(\frac{\kappa^2}{2}\right)^2 \nabla_i \left\{N^2 \left[2N(D-1)\sqrt{g}\pi^{ij}\nabla_j \left(\left(\alpha R + \frac{D\Lambda}{D-1}\right) N - \alpha\nabla^2 N\right)\right.\right. \\ & \left.\left. + \alpha\nabla^i \left(\pi^{jk}\left(R_{jk}N^2 - 2N\nabla_j\nabla_k N - \nabla_j N\nabla_k N\right)\right)\right]\right\} \\ & + \left(\pi, \nabla_i\pi - \text{dependent terms}\right). \qquad (35)\end{aligned}$$

- **This would determine u^t and no more constraints !**

- **The constraints** $\chi_A \equiv (\pi_N, \mathcal{H}^t, \tilde{\Omega}, \tilde{\Phi}) \approx 0$ **are the second-class,**

$$
\begin{aligned}
&\{\pi_N(x), \mathcal{H}^t(y)\} = 0, \\
&\{\pi_N(x), \tilde{\Omega}(y)\} = -2\nabla_i^y \left(N\tilde{C}^i(y)\delta^D(x-y)\right) \approx 0, \\
&\{\pi_N(x), \tilde{\Phi}(y)\} = \Delta(x-y), \\
&\{\mathcal{H}^t(x), \mathcal{H}^t(y)\} = C^i(x)\nabla_i^x\delta^D(x-y) - C^i(y)\nabla_i^y\delta^D(x-y) \approx 0, \\
&\{\mathcal{H}^t(x), \tilde{\Omega}(y)\} \approx \{\pi_N(x), \tilde{\Phi}^i(y)\},\ etc.
\end{aligned}
$$

- **with a non-vanishing determinant generally,**

$$
\begin{aligned}
det(\{\chi_A(x), \chi_B(y)\}) &\approx \{\pi_N(x), \tilde{\Phi}(y)\}\{\mathcal{H}^t(x), \tilde{\Omega}(y)\}\{\tilde{\Omega}(x), \mathcal{H}^t(y)\}\{\tilde{\Phi}(x), \pi_N(y)\} \\
&\approx \left(\Delta(x-y)\Delta(y-x)\right)^2
\end{aligned}
$$

$$
\begin{aligned}
\Delta(x-y) &\equiv -\nabla_i^y\left[2N\tilde{\Theta}^i(y)\ \delta^D(x-y) + N^2(y)\left(\frac{\delta\tilde{\Theta}^i(y)}{\delta N(x)}\right)\right] \\
&= -2\tilde{\Phi} - 2N^2\tilde{\Theta}^i(x)\nabla_i^x\left(\frac{\delta^D(x-y)}{N(x)}\right) - \nabla_i^y\left[N^2(y)\left(\frac{\delta\tilde{\Theta}^i(y)}{\delta N(x)}\right)\right]
\end{aligned}
$$

- The constraints $\Gamma_A \equiv (\pi_{N_i}, \mathcal{H}_i) \approx 0$ are 1st-class, as in GR.
- DOF $= \frac{1}{2}(P - 2N_1 - N_2)$

$$\begin{aligned} &= \frac{1}{2}[(D+1)(D+2) - 2 \times 2D - 4] \\ &= \frac{1}{2}(D+1)(D-2), \end{aligned}$$

$$P = (D+1)(D+2)$$

$$(N, \pi_N, N_i, \pi_{N_i}, g_{ij}, \pi_{ij})$$

- GR: $N_1 = 2(D+1), N_2 = 0$
- Horava gravity: $N_1 = 2D \quad N_2 = 4$

- 2 first-class constraints $\pi_N, \mathcal{H}^t$ in GR become 4 second-class constraints in Horava gravity, maintaining the total 2 degrees of freedom !!

B. Case $\tilde{C}^i \neq 0$

- **The constraints** $\tilde{\chi}_A \equiv (\pi_N, \mathcal{H}^t, \Omega) \approx 0$ **are 2nd-class,**

$$\begin{aligned} det(\{\tilde{\chi}_A(x), \tilde{\chi}_B(y)\}) &= -\{\pi_N(x), \tilde{\Omega}(y)\}\{\mathcal{H}^t(x), \mathcal{H}^t(y)\}\{\tilde{\Omega}(x), \pi_N(y)\} \\ &= 4\nabla^y_j\left(N\tilde{C}^j(y)\delta^D(x-y)\right)\nabla^x_k\left(N\tilde{C}^k(x)\delta^D(x-y)\right)C^i(x)\nabla^x_i\delta^D(x-y) \\ &\quad - (x \leftrightarrow y) \end{aligned}$$

with the same first-class constraints, $\Gamma_A \equiv (\pi_{N_i}, \mathcal{H}_i)$

$$\begin{aligned} s &= \frac{1}{2}[(D+1)(D+2) - 2\times 2D - \boxed{3}] \\ &= \frac{1}{2}(D+1)(D-2) \boxed{+\frac{1}{2}}, \end{aligned}$$

1 DOF in phase space

GR's DOF

VI. Future Directions

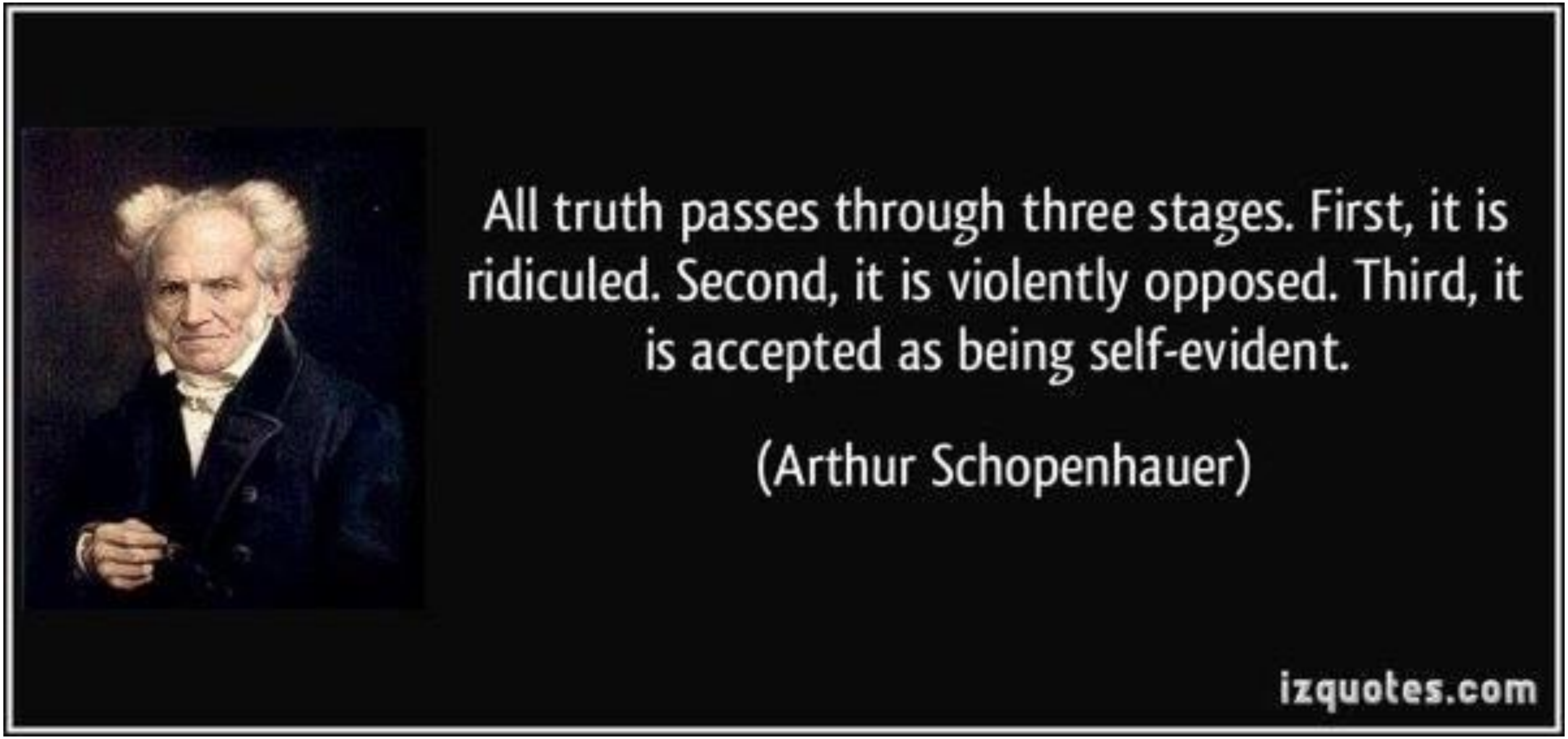
- Other Basic Problems:
- 1. Birkhoff's Theorem ?
- (arXiv: 1804.05698 [PRD])

- 2. Bianchi Identity ? (to be appeared)

- 3. Boundary actions ? (in progress)
- (cf. Gibbons Hawking term in GR)

- 4. We have identified a new (extended) constraints algebra for Horava gravity. It seems that this new constraints structure could be valid more generally, i.e., with Ricci, etc. The general proof for FPDiff gravity theory would be a challenging problem !

$$
\begin{aligned}
\{\mathcal{H}^t(x),\mathcal{H}^t(y)\} &= C^i(x)\nabla^x_i\delta^D(\mathrm{x}-\mathrm{y}) - C^i(y)\nabla^y_i\delta^D(\mathrm{x}-\mathrm{y}),\\
\{\mathcal{H}^t(x),\mathcal{H}_i(y)\} &= -\mathcal{H}^t(y)\nabla^y_i\delta^D(\mathrm{x}-\mathrm{y}),\\
\{\mathcal{H}_i(x),\mathcal{H}_j(y)\} &= \mathcal{H}_i(y)\nabla^x_j\delta^D(\mathrm{x}-\mathrm{y}) + \mathcal{H}_j(x)\nabla^x_i\delta^D(\mathrm{x}-\mathrm{y}),
\end{aligned}
$$



All truth passes through three stages. First, it is ridiculed. Second, it is violently opposed. Third, it is accepted as being self-evident.

(Arthur Schopenhauer)

Thank you !!

- " 장론 이렇게 잘 하는 사람 처음 봤다! "

(서울대 이론물리전공 L 교수님의 말씀...)

대학에서는 은퇴하시지만 학문에서는 계속 후배들을 지켜봐 주시고, 지도해 주시길 바랍니다...

임채호 교수님, 그동안 감사했읍니다.

퇴임 축하드립니다!