Anomaly on Tabletop

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Research Directions in Quantum Field Theory and String Theory
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Ho-Ung Yee & P.Y. arXiv:1909.12409

Weyl Semi-Metals

Chiral Anomaly with Nonlinear Dispersion

Chern Monopole and Nielsen-Nynomia Theorem

Eta Invariant with Nonlinear Dispersion

Weyl Semi-Metals

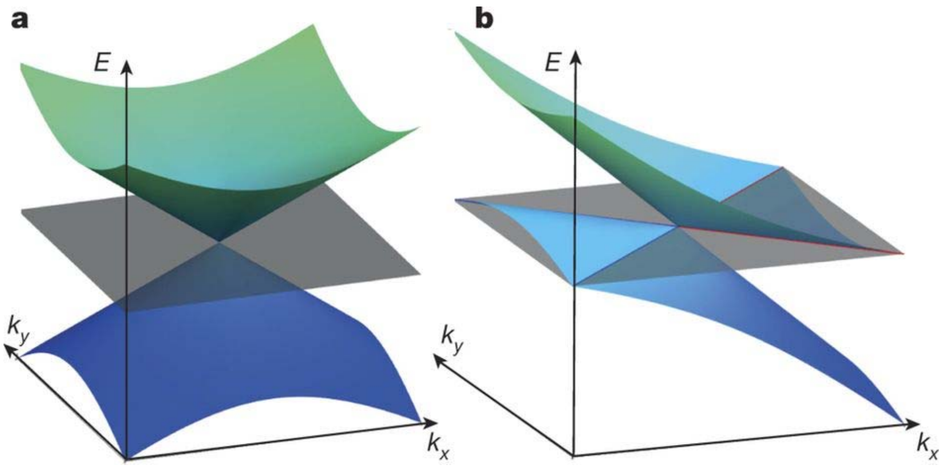
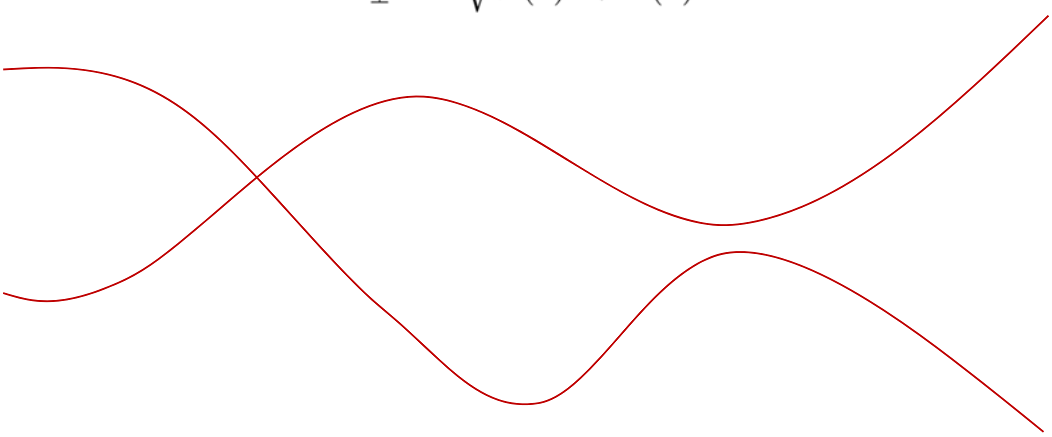


image from Type-II Weyl semimetals, Alexey A. Soluyanov et. al. Nature vol. 527

Weyl semi-metal

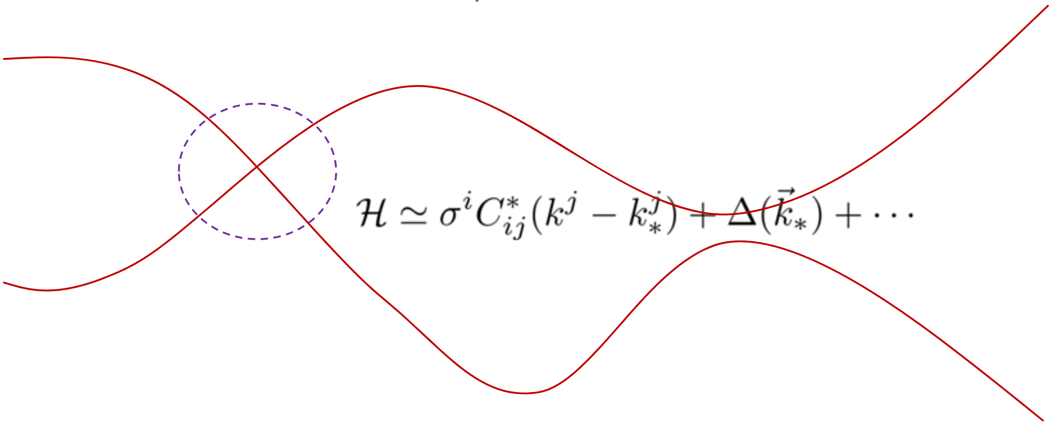
$$E_{\pm} = \pm\sqrt{\vec{\mathcal{P}}(\vec{k})^2} + \Delta(\vec{k})$$



$$\mathcal{H} = \sigma^i \mathcal{P}_i(\vec{k}) + \Delta(\vec{k})$$

Weyl semi-metal

$$E_{\pm} = \pm\sqrt{\vec{\mathcal{P}}(\vec{k})^2 + \Delta(\vec{k})}$$



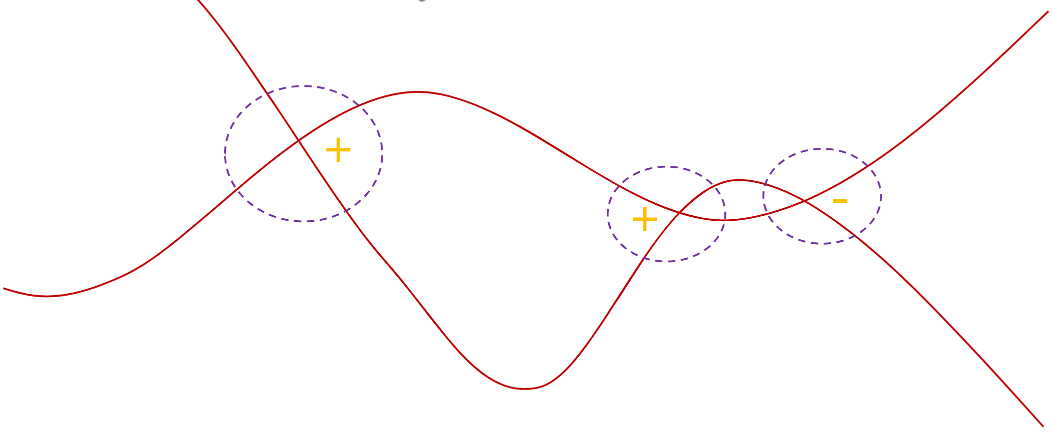
$$\mathcal{H} \simeq \sigma^i C^*_{ij}(k^j - k^j_*) + \Delta(\vec{k}_*) + \cdots$$

$\mathrm{det}C^* > 0$ chiral isospinor

$\mathrm{det}C^* < 0$ anti-chiral isospinor

Weyl semi-metal

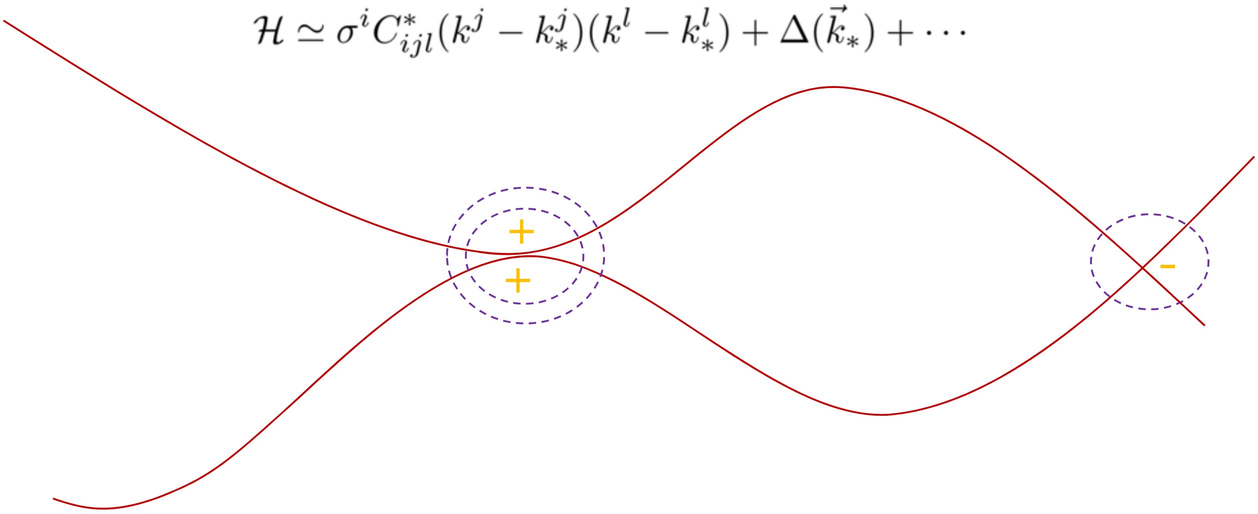
$$\mathcal{H} \simeq \sigma^i C^*_{ij}(k^j - k^j_*) + \Delta(\vec{k}_*) + \cdots$$



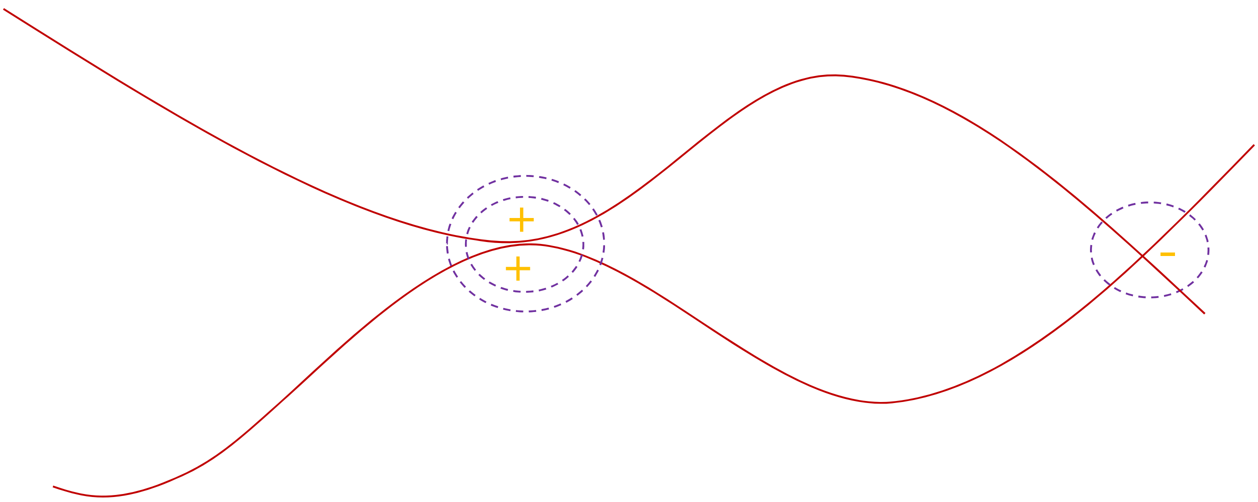
$\mathrm{det}C^* > 0$ chiral isospinor

$\mathrm{det}C^* < 0$ anti-chiral isospinor what happens if two chiral points merge?

$$\mathcal{H} \simeq \sigma^i C^*_{ijl}(k^j - k^j_*)(k^l - k^l_*) + \Delta(\vec{k}_*) + \cdots$$



topological properties
of such isospinors with nonlinear dispersion?



$$\mathcal{H} = \sigma^i \mathcal{P}_i(\vec{k}) + \Delta(\vec{k})$$

Weyl semi-metal

$$\int dx^4 \, \mathcal{L} = \int dx^4 \, \bar{\psi} \, \sigma^\mu \mathcal{P}_\mu(-iD) \, \psi$$

$$\sigma^{\mu=i} = \sigma^i$$

$$\sigma^{\mu=0} = 1_{2\times 2}$$

$$D_\mu = \partial_\mu + A_\mu$$

Weyl semi-metal

$$\int dx^4 \, \mathcal{L} = \int dx^4 \, \bar{\psi} \, \sigma^\mu \mathcal{P}_\mu(-iD) \, \psi$$

$$\mathcal{P}_i = \mathcal{P}_i(-i\vec{D})$$

$$\mathcal{P}_0 = -i\partial_0 + \Delta(-i\vec{D})$$

$$\mathcal{H} = \sigma^i \mathcal{P}_i(\vec{k}) + \Delta(\vec{k})$$

Chiral Anomaly with Nonlinear Dispersion

chiral anomaly

$$\int dx^4\ \mathcal{L} = \int dx^4\ \bar{\psi}\,\sigma^\mu \mathcal{P}_\mu(-i(\partial + A))\,\psi$$

$$\sigma^{\mu=i} = \sigma^i$$

$$\sigma^{\mu=4} = -i_{2\times 2}$$

$$Z(A) \equiv \int [d\psi d\bar{\psi}]\ e^{-\int dx^4\ \bar{\psi}\,\sigma^\mu \mathcal{P}_\mu(-i(\partial + A))\,\psi}$$

chiral basis for 4d fermions

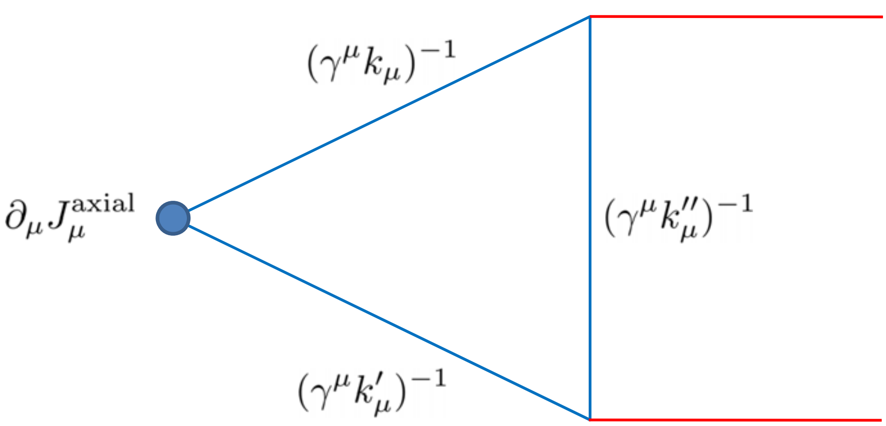
$$\gamma^\mu \mathcal{P}_\mu = \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{D}^\dagger & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i\mathcal{P}_4 + \sigma^k \mathcal{P}_k \\ i\mathcal{P}_4 + \sigma^k \mathcal{P}_k & 0 \end{pmatrix}$$

$$\gamma^{k=1,2,3} = \begin{pmatrix} 0 & \sigma^k_{2\times 2} \\ \sigma^k_{2\times 2} & 0 \end{pmatrix}$$

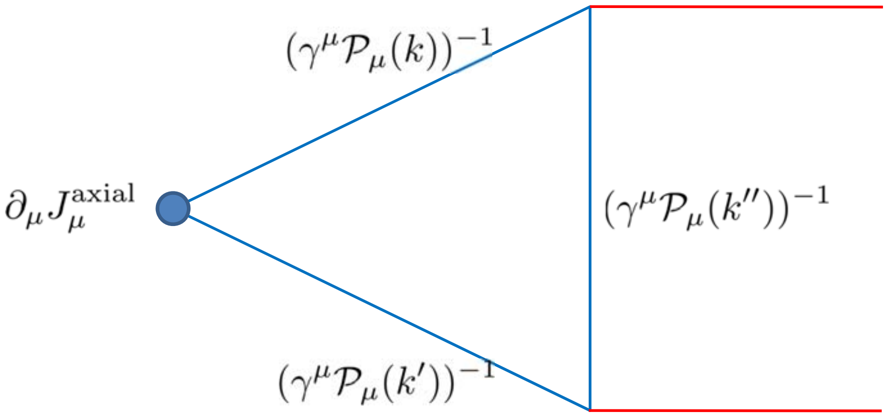
$$\gamma^4 = i \begin{pmatrix} 0 & -1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix}$$

$$\Gamma = -\gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & -1_{2\times 2} \end{pmatrix}$$

recall how one compute axial anomaly via Feynman diagram for a relativistic Dirac fermion



linear divergence → finite anomaly after Pauli-Villar's regularization linear divergence ?



however, we will first follow Fujikawa:
chiral anomaly ← the measure's failure to be invariant

$$\int [d\bar{\psi} d\psi] \; e^{-\int dx^4 \; \bar{\psi} \mathcal{D} \psi}$$

$$\psi \;\rightarrow\; e^{-i\alpha} \psi$$
$$\bar{\psi} \;\rightarrow\; \bar{\psi}\, e^{i\alpha}$$

$$e^{i\alpha \mathrm{Tr}\Gamma} \int [d\bar{\psi} d\psi] \; e^{-\int dx^4 \; \bar{\psi} \mathcal{D} \psi}$$

path integral of 4d Weyl fermions

$$\psi = \sum a_n \Psi_L^{(n)} \qquad \mathcal{D}\Psi_L^{(n)} = \lambda_n \Psi_R^{(n)}$$

$$\bar{\psi}^\dagger = \sum \bar{a}_m^\dagger (\bar{\Psi}_L^\dagger)^{(m)} \qquad \mathcal{D}^\dagger (\bar{\Psi}_L^\dagger)^{(n)} = \lambda_n (\bar{\Psi}_R^\dagger)^{(n)}$$

$$[d\psi d\bar{\psi}] = \prod_n da_n \prod_m d\bar{a}_m \qquad e^{-i\alpha\Gamma}[d\psi d\bar{\psi}] = \prod_n e^{i\alpha} da_n \prod_m e^{-i\alpha} d\bar{a}_m$$

path integral of 4d Weyl fermions

$$\psi = \sum a_n \Psi_L^{(n)}$$

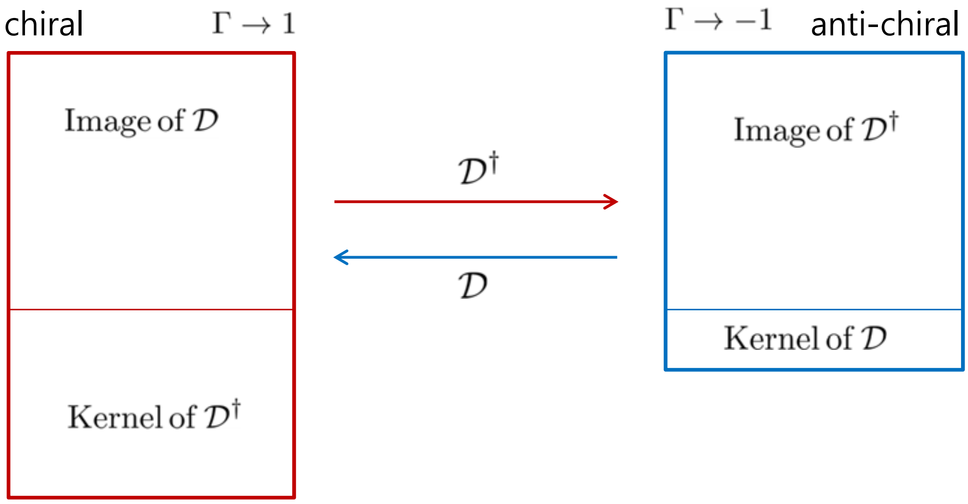
$$\bar{\psi}^\dagger = \sum \bar{a}_m^\dagger (\bar{\Psi}_L^\dagger)^{(m)}$$

$$[d\bar{\psi} d\psi] \simeq \prod_n da_n \prod_m d\bar{a}_m \qquad e^{-i\alpha\Gamma}\left([d\psi d\bar{\psi}]\right) \simeq e^{i\alpha \mathrm{Tr}(\Gamma)} \prod_n da_n \prod_m d\bar{a}_m$$

↑ trace over Dirac spinor even though we consider a Weyl !!!

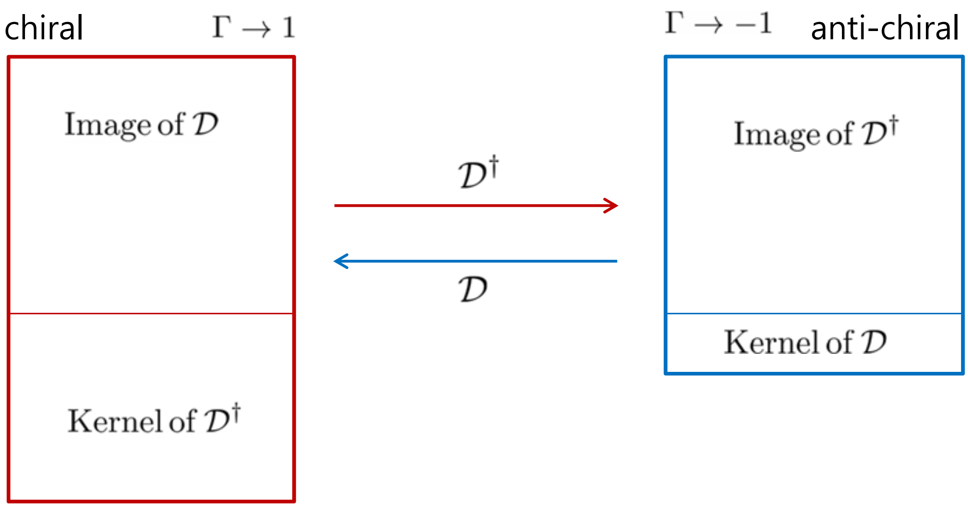
the Dirac operator pairs the opposite chirality,
possibly except at zero eigenvalue sector

$$\gamma^\mu \mathcal{P}_\mu = \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{D}^\dagger & 0 \end{pmatrix}$$



which is finite & topological

$$\gamma^\mu \mathcal{P}_\mu = \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{D}^\dagger & 0 \end{pmatrix}$$



how was this computed for relativistic fermions?

$$\mathrm{Tr}(\Gamma) = \lim_{s\to 0} \mathrm{Tr}\left(\Gamma e^{-sQ}\right) = \lim_{s\to 0} \mathrm{Tr}\left(\Gamma e^{-s(i\gamma^\mu D_\mu)^2}\right)$$

$$\begin{aligned} Q \equiv (i\gamma^\mu D_\mu)^2 &= -\gamma^\mu\gamma^\nu D_\mu D_\nu \\ &= -\delta^{\mu\nu} D_\mu D_\nu + \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} \\ &= \boxed{-\delta^{\mu\nu}\partial_\mu\partial_\nu} + \boxed{\cdots + \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}} \\ &= Q^{(0)} + Q^{(1)} \end{aligned}$$

heat kernel expansion

$$\mathrm{Tr}\left(\Gamma e^{-sQ}\right) = \int dx^4 \,\mathrm{tr}\left(\Gamma\langle x|e^{-sQ}|x\rangle\right) = \int dx^4 \,\mathrm{tr}\left(\Gamma G_s(x;x)\right)$$

$$G_s(x;y) = \langle x|\exp\left[-sQ\right]|y\rangle = G_s^{(0)} + G_s^{(1)} + G_s^{(2)} + \cdots$$

heat kernel expansion

$$\mathrm{Tr}\left(\Gamma e^{-sQ}\right) = \int dx^4 \ \mathrm{tr}\left(\Gamma \langle x | e^{-sQ} | x \rangle\right) = \int dx^4 \ \mathrm{tr}\left(\Gamma G_s(x;x)\right)$$

$$G_s(x;y) = \langle x | \exp\left[-sQ\right] | y \rangle \ = G_s^{(0)} + G_s^{(1)} + G_s^{(2)} + \cdots$$

$$G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z \ G_{\beta-s}^{(0)}(x;z) Q^{(1)}(z) G_s^{(n)}(z;y)$$

$$G_s^{(0)}(x;y) = \langle x | e^{-sQ^{(0)}} | y \rangle \qquad \lim_{s\to 0} G_s^{(0)}(x;y) = \delta(x;y)$$

heat kernel expansion

$$G_s^{(0)}(x;y) = \langle x | \, e^{-sQ^{(0)}} | y \rangle$$

$$= \int \frac{d^d k}{(2\pi)^d} \langle x | k \rangle \langle k | \, e^{-sQ^{(0)}} | k \rangle \langle k | y \rangle$$

$$= \int \frac{d^d k}{(2\pi)^d} e^{-sQ^{(0)}(k)} e^{ik(y-x)}$$

$$G_s^{(0)}(x;y) = \langle x | e^{-sQ^{(0)}} | y \rangle \qquad \lim_{s\to 0} G_s^{(0)}(x;y) = \delta(x;y)$$

heat kernel expansion

$$G_s^{(0)}(x;y) = \langle x|\, e^{-sQ^{(0)}}\, |y\rangle$$

$$= \frac{1}{(4\pi s)^{d/2}} e^{-(x-y)^2/4s} \qquad\qquad Q^{(0)} = -\partial^2$$

$$G_s^{(0)}(x;y) = \langle x| e^{-sQ^{(0)}} |y\rangle \qquad \lim_{s\to 0} G_s^{(0)}(x;y) = \delta(x;y)$$

small s limit allows enormous simplifications !

$$\mathrm{Tr}(\Gamma) = \lim_{s\to 0} \mathrm{Tr}\left(\Gamma e^{-sH}\right) = \lim_{s\to 0} \mathrm{Tr}\left(\Gamma e^{-s(i\gamma^\mu D_\mu)^2}\right)$$

$$= \lim_{s\to 0} \int dx^4 \; \frac{e^{-\vec{0}^2/4s}}{(4\pi s)^2} \mathrm{tr}\left(\Gamma \frac{[s\gamma^\mu \gamma^\nu F_{\mu\nu}/2]^2}{2}\right)$$

$$= \int dx^4 \; \frac{1}{2^7 \pi^2} \mathrm{tr}\left(\Gamma \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta\right) F_{\mu\nu} F_{\alpha\beta}$$

$$= -\int dx^4 \; \frac{1}{2^5 \pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$$

4d chiral anomaly of a single relativistic Weyl fermion
is measured by the Dirac index density, a la Fujikawa

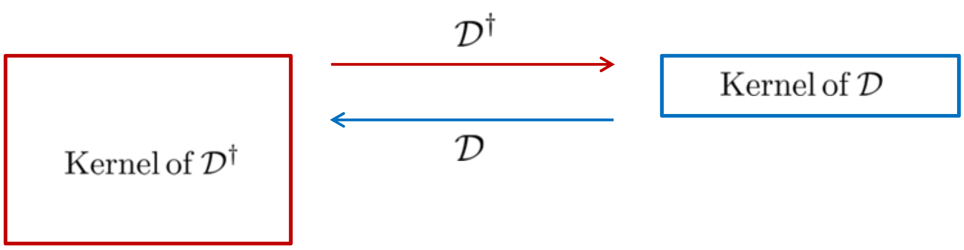
$$\frac{\partial}{\partial \alpha} \log Z(A) = i\,\mathrm{Tr}(\Gamma)$$

$$= -i \int dx^4 \, \frac{1}{2^5 \pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$$

$$= -i \int \frac{1}{8\pi^2} F \wedge F$$

$\mathrm{Tr}(\Gamma)$ would compute the index, an integer,
if the Hilbert space is fully discrete

$$\mathrm{Tr}(\Gamma) \;=\; \dim\left(\text{Kernel of } \mathcal{D}^\dagger\right) - \dim\left(\text{Kernel of } \mathcal{D}\right)$$



what if the Dirac operator is nonlinear in the momenta?

$$\mathrm{Tr}(\Gamma) = \lim_{s\to 0} \mathrm{Tr}\left(\Gamma e^{-s(\gamma^\mu \mathcal{P}_\mu(-iD))^2}\right)$$

$$(\gamma^\mu \mathcal{P}_\mu)^2 = \gamma^\mu\gamma^\nu \mathcal{P}_\mu \mathcal{P}_\nu = \delta^{\mu\nu}\mathcal{P}_\mu\mathcal{P}_\nu + \frac{1}{2}\gamma^\mu\gamma^\nu[\mathcal{P}_\mu, \mathcal{P}_\nu]$$

$$= \underbrace{\boxed{\delta^{\mu\nu}\mathcal{P}_\mu(k)\mathcal{P}_\nu(k)}}_{Q^{(0)}} + \underbrace{\boxed{\cdots + \frac{1}{2}\gamma^\mu\gamma^\nu F_{\alpha\beta}\tilde{\partial}^\alpha \mathcal{P}_\mu(k)\tilde{\partial}^\beta \mathcal{P}_\nu(k) + \cdots}}_{Q^{(1)}}$$

modified heat kernel expansion

$$G_s^{(0)}(x;y) = \langle x|\, e^{-sQ^{(0)}}\,|y\rangle$$

$$= \int \frac{d^d k}{(2\pi)^d}\langle x|k\rangle\langle k|\, e^{-sQ^{(0)}}\,|k\rangle\langle k|y\rangle$$

$$G_s^{(0)}(x;y) = \langle x|e^{-sQ^{(0)}}|y\rangle \qquad \lim_{s\to 0} G_s^{(0)}(x;y) = \delta(x;y)$$

modified heat kernel expansion

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$$= \int \frac{d^d k}{(2\pi)^d} \langle x|k\rangle\langle k|\, e^{-sQ^{(0)}}|k\rangle\langle k|y\rangle$$

$$= \int \frac{d^d k}{(2\pi)^d} e^{-s\mathcal{P}_\mu(k)\mathcal{P}_\mu(k)} e^{ik(y-x)}$$

$$G_s^{(0)}(x;y) = \langle x|e^{-sQ^{(0)}}|y\rangle \qquad \lim_{s\to 0} G_s^{(0)}(x;y) = \delta(x;y)$$

modified heat kernel expansion

$$\mathrm{Tr}(\Gamma) = \lim_{s\to 0} \mathrm{Tr}\left(\Gamma e^{-s(\gamma^\mu \mathcal{P}_\mu(-iD))^2}\right)$$

$$= \lim_{s\to 0} \int dx^4 \int \frac{d^4 k}{(2\pi)^2} e^{-s\mathcal{P}(k)^2} \times \mathrm{tr}\left(\Gamma \frac{\left[(s/2)\gamma^\mu\gamma^\nu F_{\alpha\beta}\tilde{\partial}^\alpha \mathcal{P}_\mu \tilde{\partial}^\beta \mathcal{P}_\nu\right]^2}{2}\right)$$

modified heat kernel expansion

$$\mathrm{Tr}(\Gamma) = \lim_{s\to 0} \mathrm{Tr}\left(\Gamma e^{-s(\gamma^\mu \mathcal{P}_\mu(-iD))^2}\right)$$

$$= \lim_{s\to 0} \int dx^4 \int \frac{d^4k}{(2\pi)^2} e^{-s\mathcal{P}(k)^2} \times \mathrm{tr}\left(\Gamma \frac{\left[(s/2)\gamma^\mu\gamma^\nu F_{\alpha\beta}\tilde{\partial}^\alpha \mathcal{P}_\mu \tilde{\partial}^\beta \mathcal{P}_\nu\right]^2}{2}\right)$$

$$= \lim_{s\to 0} \frac{s^2}{2} \int d^4x \, \mathrm{tr}\left(\frac{1}{4}\Gamma\gamma^\mu\gamma^\nu\gamma^{\mu'}\gamma^{\nu'}\right) F_{\alpha\beta}(x) F_{\alpha'\beta'}(x)$$

$$\times \int \frac{d^4k}{(2\pi)^4} \frac{\partial \mathcal{P}_\mu(k)}{\partial k_\alpha} \frac{\partial \mathcal{P}_\nu(k)}{\partial k_\beta} \frac{\partial \mathcal{P}_{\mu'}(k)}{\partial k_{\alpha'}} \frac{\partial \mathcal{P}_{\nu'}(k)}{\partial k_{\beta'}} e^{-s\mathcal{P}(k)^2}$$

modified heat kernel expansion

$$\mathrm{Tr}(\Gamma) = \lim_{s\to 0} \mathrm{Tr}\left(\Gamma e^{-s(\gamma^\mu \mathcal{P}_\mu(-iD))^2}\right)$$

$$= \lim_{s\to 0} \int dx^4 \int \frac{d^4k}{(2\pi)^2} e^{-s\mathcal{P}(k)^2} \times \mathrm{tr}\left(\Gamma \frac{\left[(s/2)\gamma^\mu\gamma^\nu F_{\alpha\beta}\tilde{\partial}^\alpha \mathcal{P}_\mu \tilde{\partial}^\beta \mathcal{P}_\nu\right]^2}{2}\right)$$

$$= \lim_{s\to 0} \frac{s^2}{2} \int d^4x \, \mathrm{tr}\left(\frac{1}{4}\Gamma\gamma^\mu\gamma^\nu\gamma^{\mu'}\gamma^{\nu'}\right) F_{\alpha\beta}(x) F_{\alpha'\beta'}(x)$$

$$\times \int \frac{d^4k}{(2\pi)^4} \frac{\partial \mathcal{P}_\mu(k)}{\partial k_\alpha} \frac{\partial \mathcal{P}_\nu(k)}{\partial k_\beta} \frac{\partial \mathcal{P}_{\mu'}(k)}{\partial k_{\alpha'}} \frac{\partial \mathcal{P}_{\nu'}(k)}{\partial k_{\beta'}} e^{-s\mathcal{P}(k)^2}$$

$$= \boxed{\lim_{s\to 0} \frac{s^2}{\sqrt{\pi}^4} \int d^4k \, \det\left(\frac{\partial \mathcal{P}_\mu}{\partial k_\alpha}\right) e^{-s\mathcal{P}(k)^2}} \times \left(-\frac{1}{8\pi^2}\int F \wedge F\right)$$

chiral anomaly for Weyl with nonlinear dispersion

$$\mathrm{Tr}(\Gamma) = \lim_{s\to 0} \mathrm{Tr}\left(\Gamma e^{-s(\gamma^\mu \mathcal{P}_\mu(-iD))^2}\right)$$

$$= \lim_{s\to 0} \frac{s^2}{2} \int d^4x \, \mathrm{tr}\left(\frac{1}{4}\Gamma\gamma^\mu\gamma^\nu\gamma^{\mu'}\gamma^{\nu'}\right) F_{\alpha\beta}(x) F_{\alpha'\beta'}(x)$$

$$\times \int \frac{d^4k}{(2\pi)^4} \frac{\partial \mathcal{P}_\mu(k)}{\partial k_\alpha} \frac{\partial \mathcal{P}_\nu(k)}{\partial k_\beta} \frac{\partial \mathcal{P}_{\mu'}(k)}{\partial k_{\alpha'}} \frac{\partial \mathcal{P}_{\nu'}(k)}{\partial k_{\beta'}} e^{-s\mathcal{P}(k)^2}$$

$$= \boxed{\lim_{s\to 0} \frac{s^2}{\sqrt{\pi}^4} \int d^4k \, \det\left(\frac{\partial \mathcal{P}_\mu}{\partial k_\alpha}\right) e^{-s\mathcal{P}(k)^2}} \times \left(-\frac{1}{8\pi^2}\int F \wedge F\right)$$

if we further assume that the map $k \to \mathcal{P}$ is unbounded

$$\mathrm{Tr}(\Gamma) = \lim_{s\to 0} \mathrm{Tr}\left(\Gamma e^{-s(\gamma^\mu \mathcal{P}_\mu(-iD))^2}\right)$$

$$= \boxed{\lim_{s\to 0} \frac{s^2}{\sqrt{\pi}^4} \int d^4k \, \det\left(\frac{\partial \mathcal{P}_\mu}{\partial k_\alpha}\right) e^{-s\mathcal{P}(k)^2}} \times \left(-\frac{1}{8\pi^2}\int F \wedge F\right)$$

$$= \lim_{s\to 0} \frac{1}{\sqrt{\pi}^4} \int_{N_{\mathcal{P}}\cdot\tilde{\mathbf{R}}^4} d^4(s^{1/2}\mathcal{P}) \, e^{-(s^{1/2}\mathcal{P})^2} \quad \to \quad N_{\mathcal{P}}$$

$$= \boxed{N_{\mathcal{P}}} \times \left(-\frac{1}{8\pi^2}\int F \wedge F\right)$$

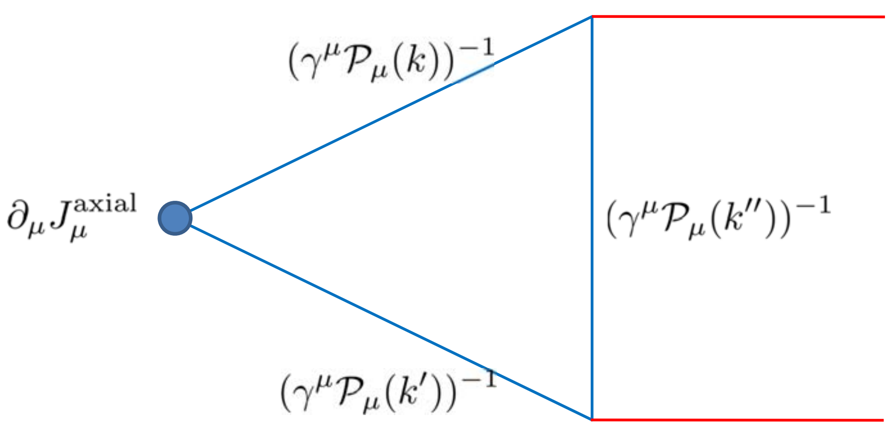
chiral anomaly is enhanced by the "winding number"
of the map $k \to \mathcal{P}$, computed at the large momentum limit

$$\mathrm{Tr}(\Gamma) = \lim_{s\to 0} \mathrm{Tr}\left(\Gamma e^{-s(\gamma^\mu \mathcal{P}_\mu(-iD))^2}\right)$$

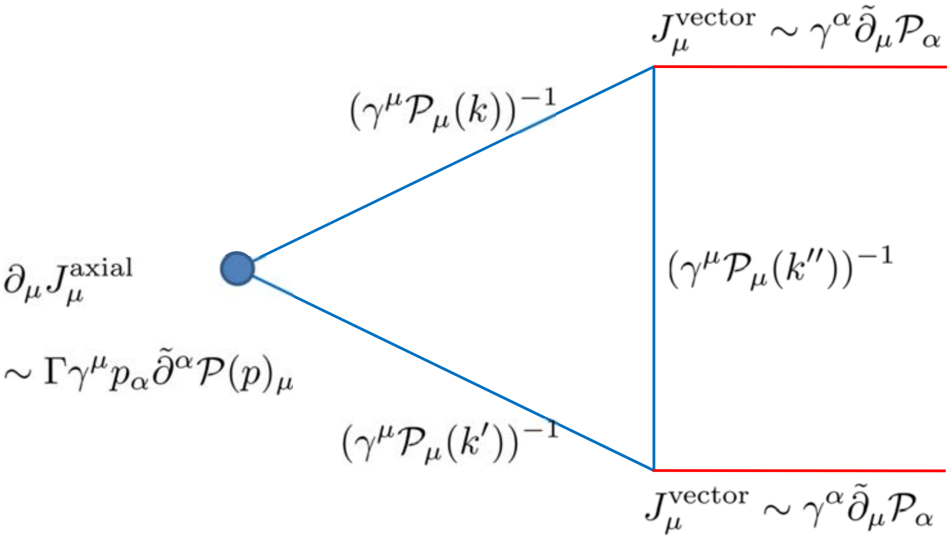
$$= \boxed{N_{\mathcal{P}}} \times \left(-\frac{1}{8\pi^2}\int F\wedge F\right)$$

$$= \lim_{s\to 0}\frac{1}{\sqrt{\pi}^4}\int d^4k \,\det\left(\frac{s^{1/2}\partial\mathcal{P}_\mu}{\partial k_\alpha}\right) e^{-(s^{1/2}\mathcal{P}(k))^2}$$

via triangle diagram? → linear divergence ?

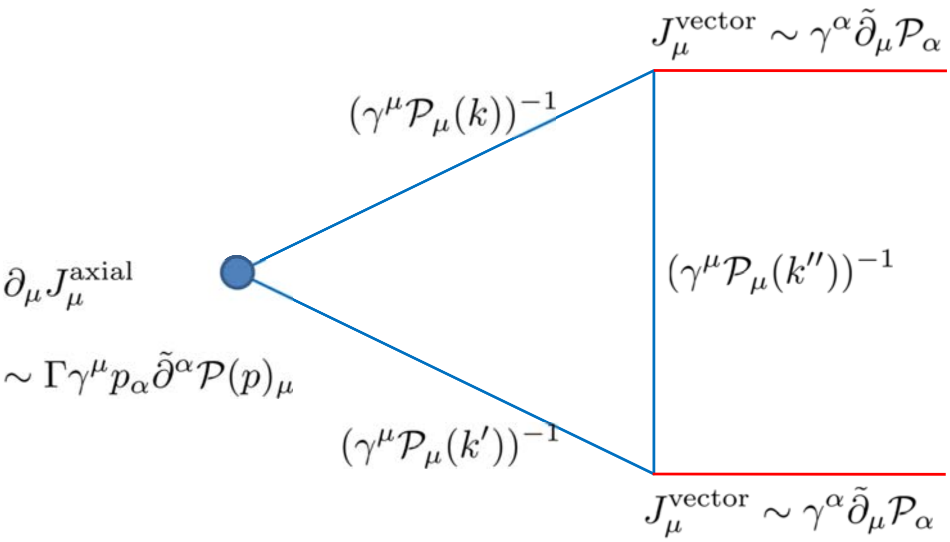


linear divergence, again !!!



$$(\gamma \cdot k)^{-1} \quad \rightarrow \quad (\tilde{\partial}\mathcal{P})(\gamma \cdot \mathcal{P})^{-1}$$

after a careful dance about the Wick rotation



→ 2 x the same axial anomaly, with the same enhancement by the same winding number of $k \rightarrow \mathcal{P}(k)$!

the results can be also interpreted as a new Atiyah-Singer index
if the boundary contribution is absent

$$\mathrm{Tr}(\Gamma) = \dim\left(\text{Kernel of } \mathcal{D}^\dagger\right) - \dim\left(\text{Kernel of } \mathcal{D}\right)$$

$$= \boxed{N_{\mathcal{P}}} \times \left(-\frac{1}{8\pi^2}\int F\wedge F\right)$$

$$= \lim_{s\to 0}\frac{1}{\sqrt{\pi}^4}\int d^4k \det\left(\frac{s^{1/2}\partial\mathcal{P}_\mu}{\partial k_\alpha}\right) e^{-(s^{1/2}\mathcal{P}(k))^2}$$

which generalizes, straightforwardly, to all even dimensions as

$$\mathrm{Tr}(\Gamma) = \dim\left(\text{Kernel of } \mathcal{D}^\dagger\right) - \dim\left(\text{Kernel of } \mathcal{D}\right)$$

$$= \boxed{N_{\mathcal{P}}} \times \left(\frac{1}{(d/2)!(2\pi i)^{(d/2)}}\int F\wedge\cdots\wedge F\right)$$

$$= \lim_{s\to 0}\frac{1}{\sqrt{\pi}^d}\int d^dk \det\left(\frac{s^{1/2}\partial\mathcal{P}_\mu}{\partial k_\alpha}\right) e^{-(s^{1/2}\mathcal{P}(k))^2}$$

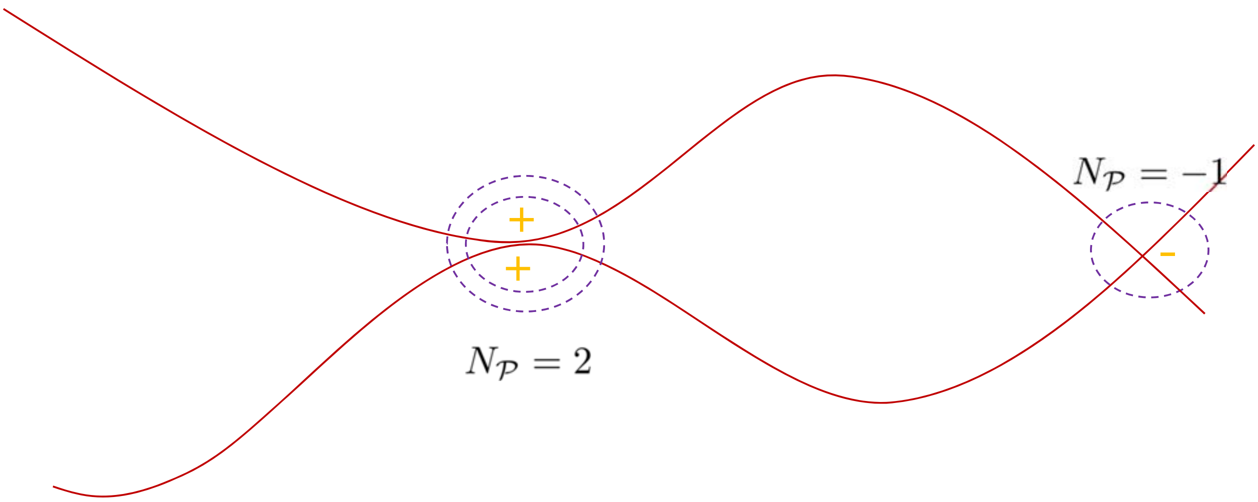
the simplest examples

$$\mathcal{H} \simeq \sigma^{\pm} C_1 \cdot (\delta k_{\pm})^N + \sigma^3 C_3 \cdot (\delta k_3)^L + \Delta(\vec{k}_*) + \cdots$$

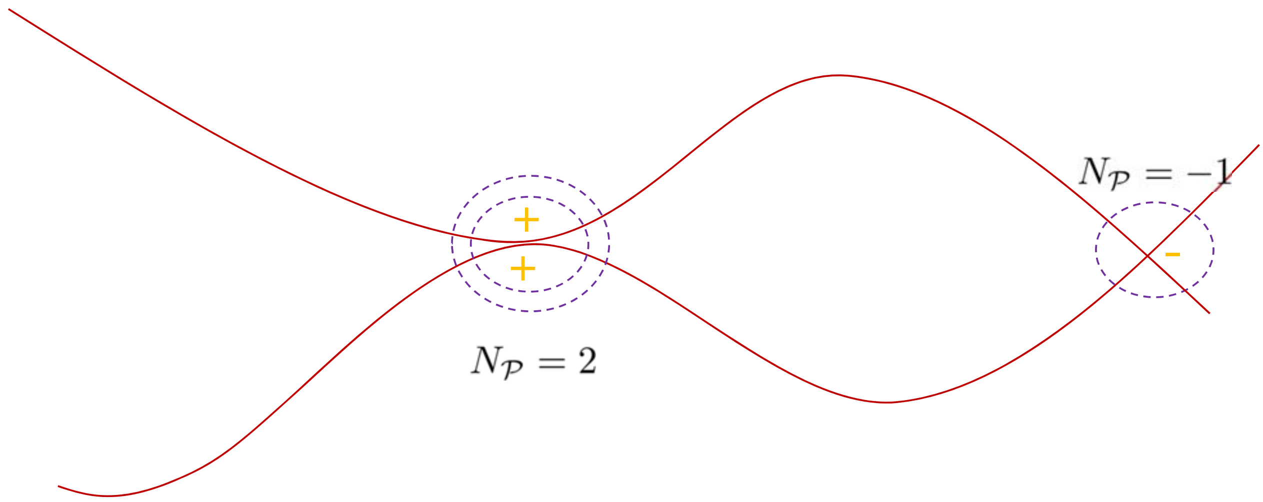
$$N_{\mathcal{P}} = \begin{cases} \mathrm{sgn}(C_3) \cdot N & \text{for odd } L \\ 0 & \text{for even } L \end{cases}$$

cf) N=2,L=0: Z. M. Huang, J. Zhou and S. Q. Shen, Phys. Rev. B 96, no. 8, 085201 (2017)
N=3, L=0: L. Lepori, M. Burrello and E. Guadagnini, JHEP 1806, 110 (2018)

in a Brillouin zone, each (degenerate)
Weyl cone is characterized by a local winding number



what happens if we scan the entire Brillouin zone?

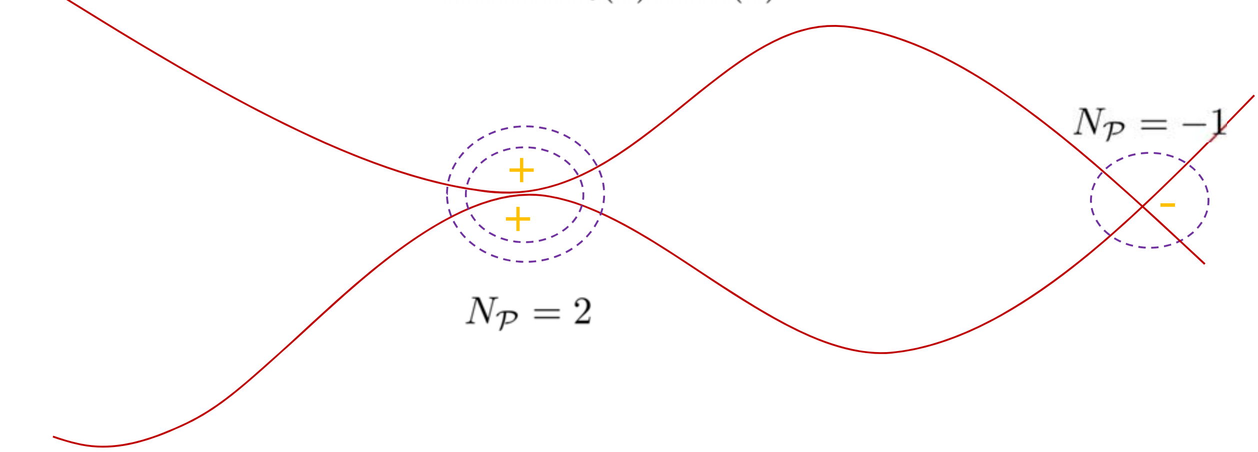


Chern Monopole and Nielsen-Nynomia Theorem

or, UV-IR connection of the anomaly

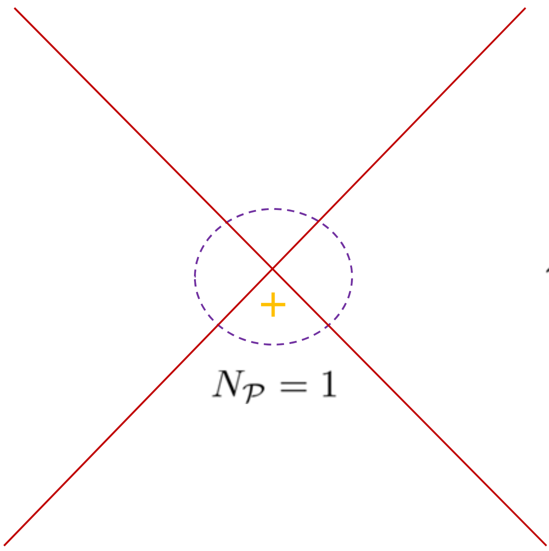
recall Weyl semi-metal for d=3+1

$$\mathcal{H} = \sigma^i \mathcal{P}_i(\vec{k}) + \Delta(\vec{k})$$



consider a simpler problem

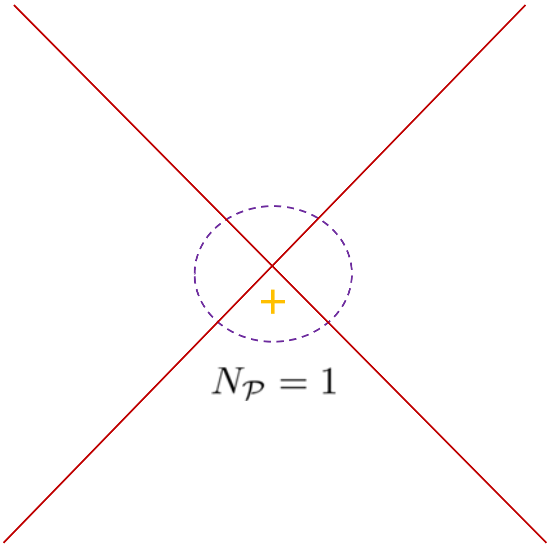
$$\mathcal{H} = \sigma^i k_i + \Delta(\vec{k})$$



$$\mathcal{H}|\pm\rangle_{\vec{k}} = \left(\pm\sqrt{\vec{k}^{\,2}} + \Delta(\vec{k})\right)|\pm\rangle_{\vec{k}}$$

whose two branches are

$$\mathcal{H} = \sigma^i k_i + \Delta(\vec{k})$$

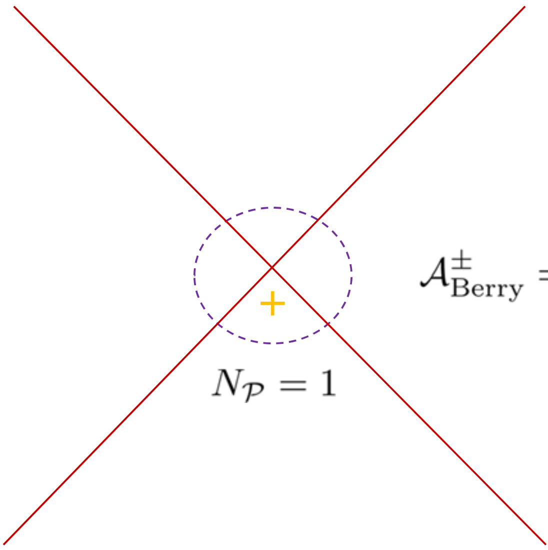


$$|+\rangle_{\vec{k}} = \begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix}$$

$$|-\rangle_{\vec{k}} = \begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ \cos(\theta/2)e^{i\phi/2} \end{pmatrix}$$

with Berry phases constituting $\pm 2\pi$ flux monopole

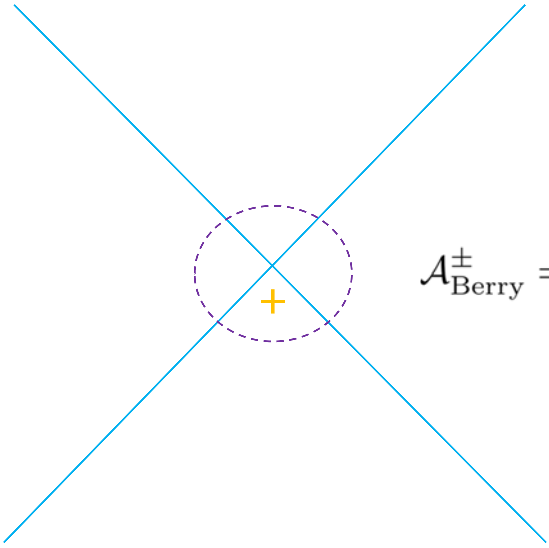
$$\mathcal{H} = \sigma^i k_i + \Delta(\vec{k})$$



$$\mathcal{A}^{\pm}_{\text{Berry}} = -\langle \pm | \frac{\partial}{\partial k_i} | \pm \rangle_{\vec{k}} \, dk_i \quad = \pm \frac{i}{2} \cos\theta \, d\phi$$

going back to the general dispersion is straightforward

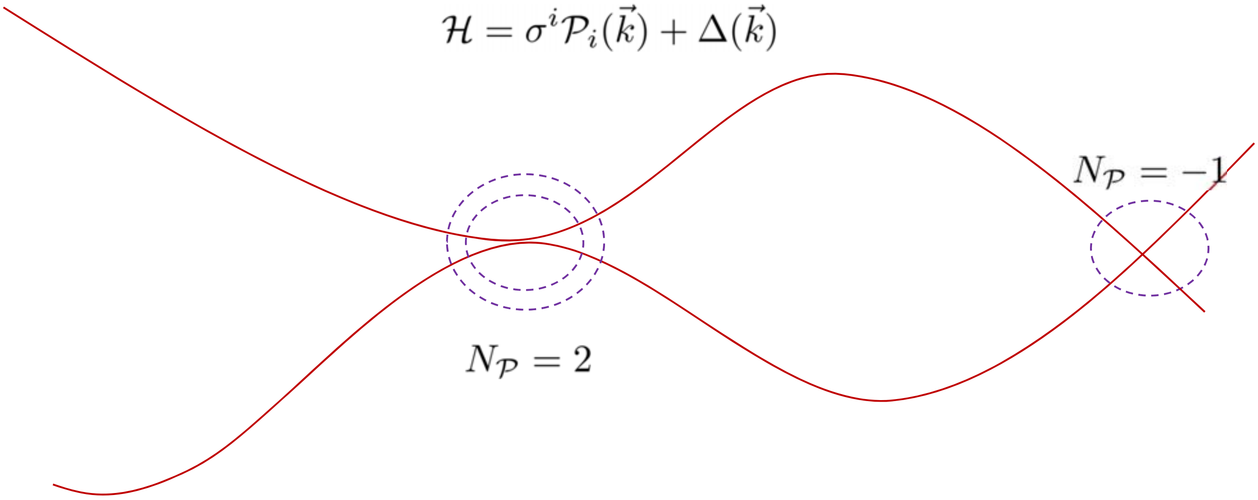
$$\mathcal{H} = \sigma^i \mathcal{P}_i(\vec{k}) + \Delta(\vec{k})$$



$$\mathcal{A}^{\pm}_{\text{Berry}} = -\langle \pm | \frac{\partial}{\partial \mathcal{P}_i} | \pm \rangle_{\vec{\mathcal{P}}} \, d\mathcal{P}_i = \pm \frac{i}{2} \cos\Theta \, d\Phi$$

which translates to $\pm 2\pi \cdot N_{\mathcal{P}}$ flux Berry monopoles
distributed over the momentum space, or the Brillouin zone

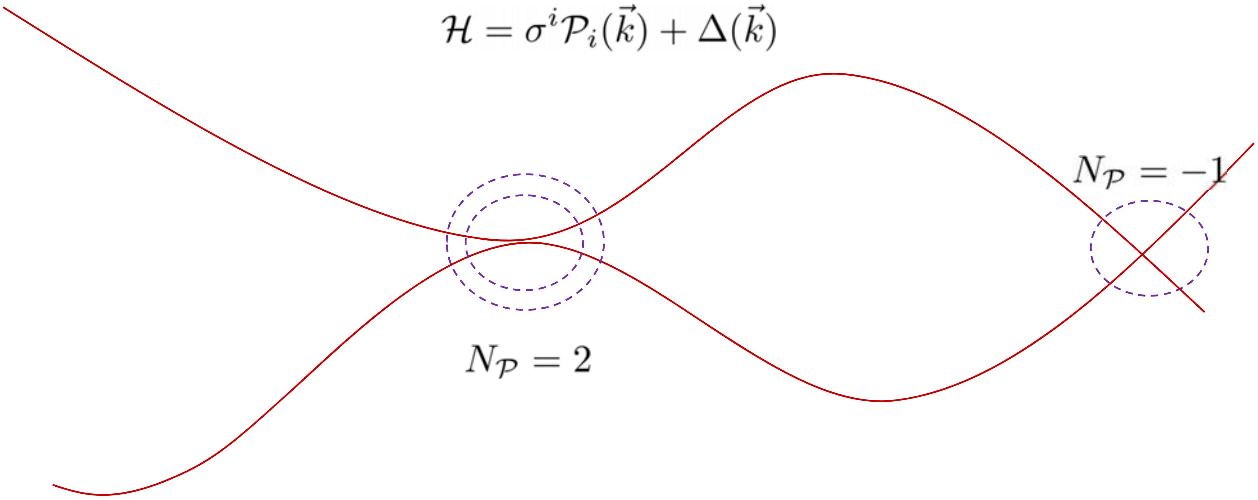
$$\mathcal{H} = \sigma^i \mathcal{P}_i(\vec{k}) + \Delta(\vec{k})$$



$$\mathcal{A}^{\pm}_{\text{Berry}} = -\langle \pm | \frac{\partial}{\partial k_i} | \pm \rangle_{\vec{\mathcal{P}}} \, dk_i = \mathcal{P}^* \left[\pm \frac{i}{2} \cos\Theta \, d\Phi \right]$$

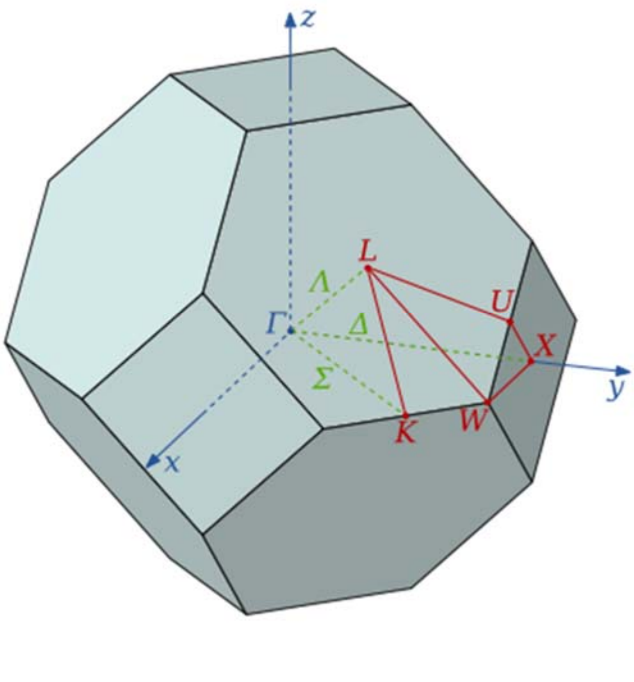
which translates to $\pm 2\pi \cdot N_{\mathcal{P}}$ flux Berry monopoles distributed over the momentum space, or the Brillouin zone

$$\mathcal{H} = \sigma^i \mathcal{P}_i(\vec{k}) + \Delta(\vec{k})$$



$$\mathcal{F}^{\pm}_{\text{Berry}} = d\mathcal{A}^{\pm}_{\text{Berry}} = \pm 2\pi \sum_{k_*} N_{\mathcal{P}}(k_*)\, \delta^{(3)}(k - k_*)$$

the Brillouin zone is compact for real crystaline material

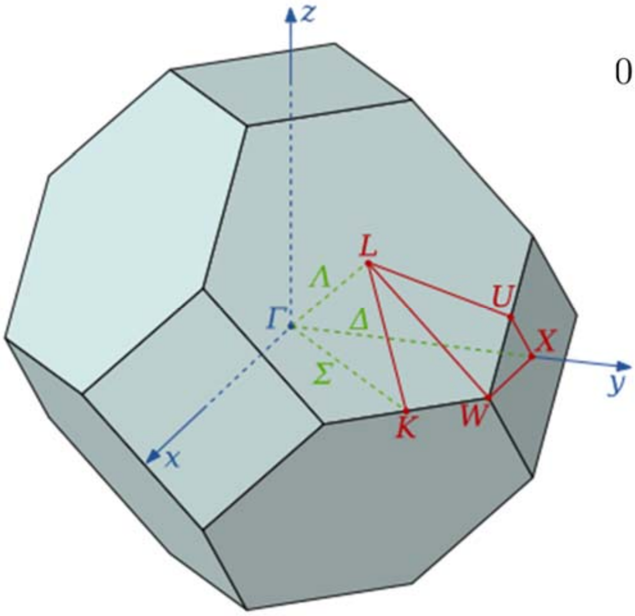


$$\vec{k} \simeq \vec{k} + 2\vec{L}$$

$$\vec{k} \simeq \vec{k} + 2\vec{X}$$

$$\vdots$$

there cannot be a net number of Berry monopoles, given the magnetic flux conservation



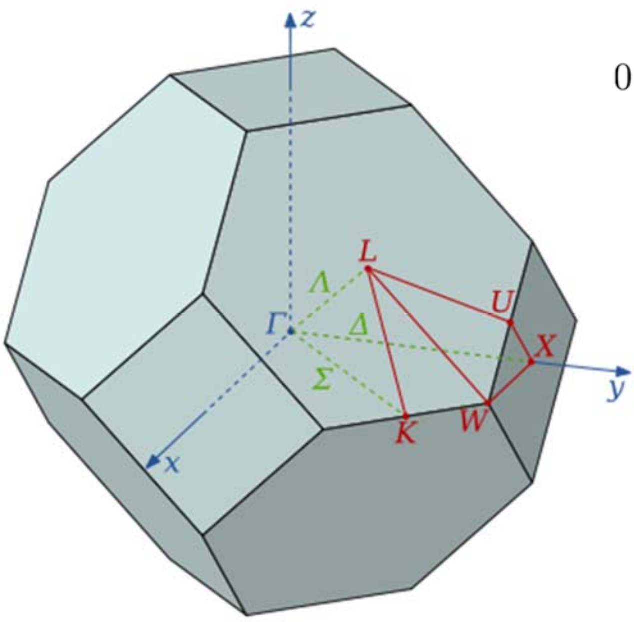
$$0 = \int d\mathcal{F}^{\pm}_{\text{Berry}} = \pm 2\pi \sum N_{\mathcal{P}}$$

$$\vec{k} \simeq \vec{k} + 2\vec{L}$$

$$\vec{k} \simeq \vec{k} + 2\vec{X}$$

$$\vdots$$

→ a Nielsen-Nynomia theorem :
net chiral spinor is impossible in any lattice theory



$$0 = \int d\mathcal{F}^{\pm}_{\text{Berry}} = \pm 2\pi \sum N_{\mathcal{P}}$$

↓

#chiral = #anti-chiral

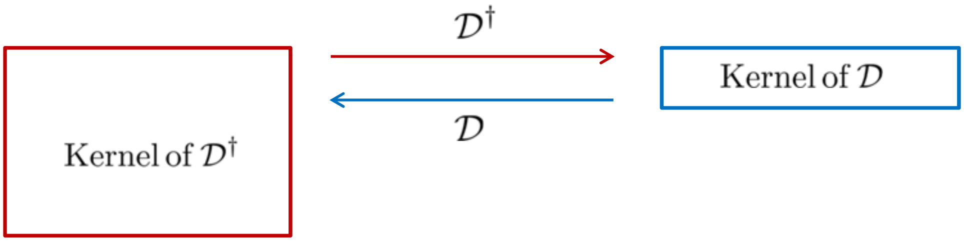
X. L. Qi, T. Hughes and S. C. Zhang,
"Topological Field Theory of Time-Reversal Invariant Insulators,"
Phys. Rev. B 78, 195424 (2008)

Eta Invariant with Nonlinear Dispersion

if the sample has a boundary, as is always the case with real material, the relevant index is that of Atiyah-Patodi-Singer

$$\dim\left(\text{Kernel of } \mathcal{D}^\dagger\right) - \dim\left(\text{Kernel of } \mathcal{D}\right)$$

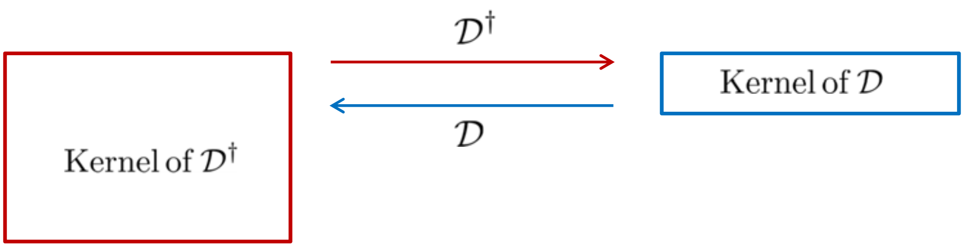
$$= \quad -\frac{1}{8\pi^2}\int_M F\wedge F - \frac{\eta(\partial M)}{2}$$



the bulk part is enhanced by the winding number, so what happens to the boundary contribution?

$$\dim\left(\text{Kernel of } \mathcal{D}^\dagger\right) - \dim\left(\text{Kernel of } \mathcal{D}\right)$$

$$= \quad N_{\mathcal{P}} \times \left(-\frac{1}{8\pi^2}\int_M F\wedge F\right) - \quad ? \times \frac{\eta(\partial M)}{2}$$

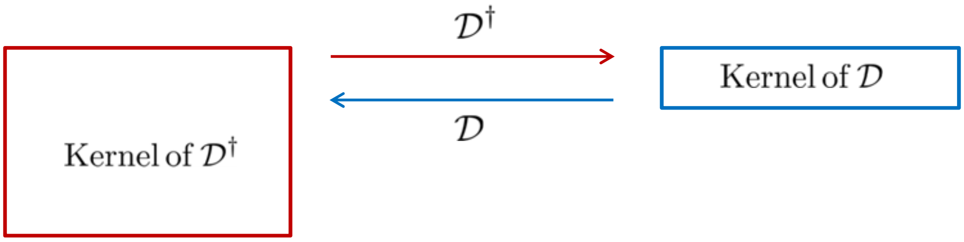


→ the entire index formula is enhanced by the common winding number, with the operator restricted somewhat

$$\mathcal{D} = -D_4 + \vec{\sigma} \cdot \vec{P}_k(-i\vec{D})$$

$$\dim\left(\text{Kernel of } \mathcal{D}^\dagger\right) - \dim\left(\text{Kernel of } \mathcal{D}\right)$$

$$= \; N_{\vec{P}} \times \left(-\frac{1}{8\pi^2}\int_M F\wedge F\right) - \overset{!}{N_{\vec{P}}} \times \frac{\eta(\partial M)}{2}$$

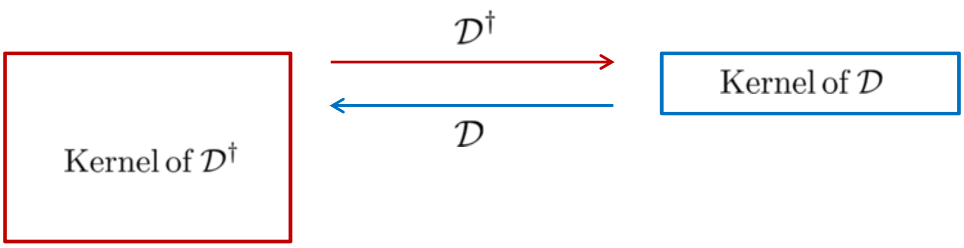


→ the entire index formula is enhanced by the common winding number, with the operator restricted somewhat

$$\mathcal{D} = -D_4 + \vec{\sigma} \cdot \vec{P}_k(-i\vec{D})$$

$$\dim\left(\text{Kernel of } \mathcal{D}^\dagger\right) - \dim\left(\text{Kernel of } \mathcal{D}\right)$$

$$= \; N_{\vec{P}} \times \left(-\frac{1}{8\pi^2}\int_M F\wedge F - \frac{\eta(\partial M)}{2}\right)$$



where are we headed?

a new batch of topological quantities?

another UV/IR nature of anomaly

topology of the Brillouin zone/momentum space:
does it lead us somewhere new?