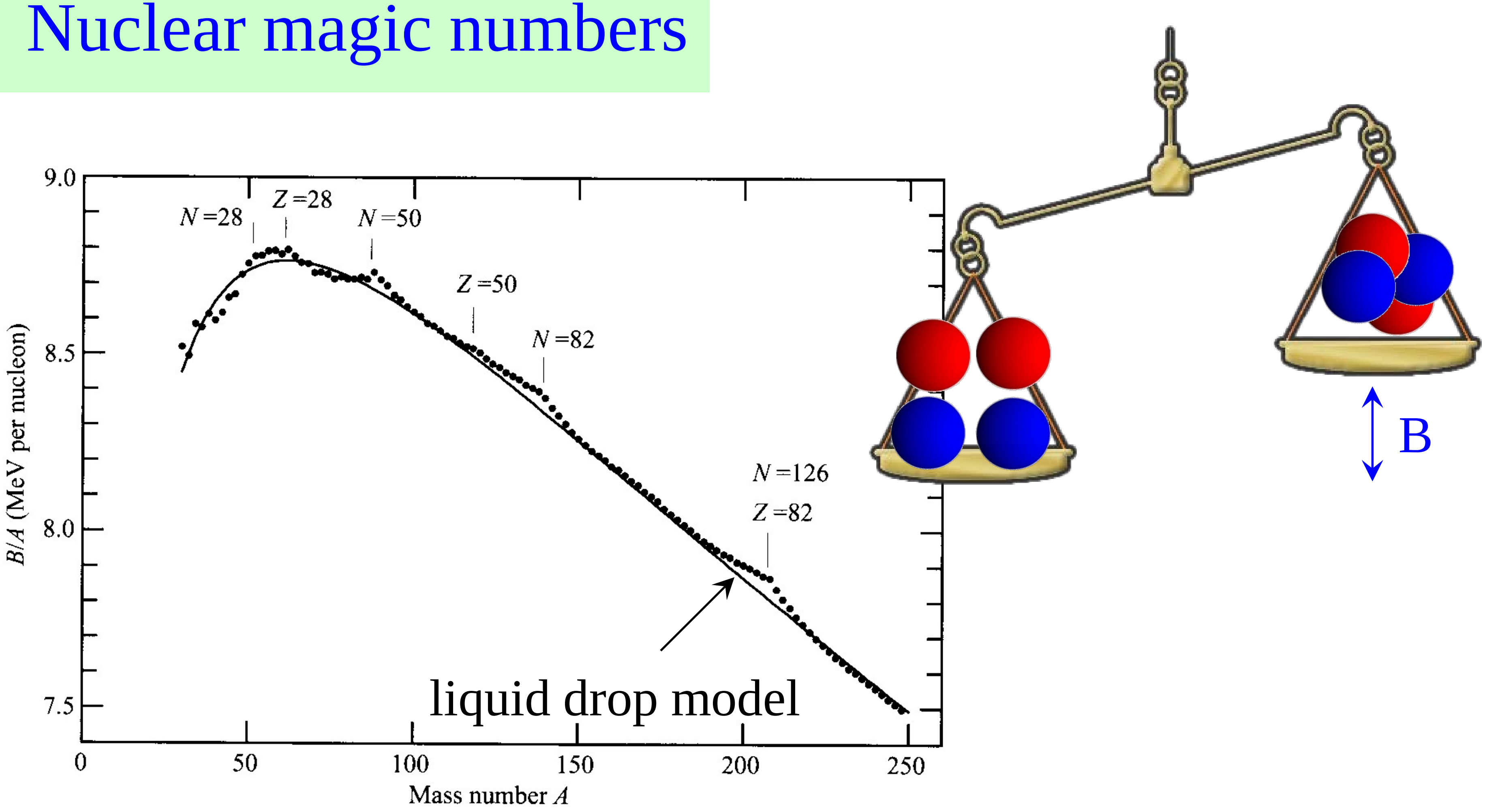
Nuclear magic numbers

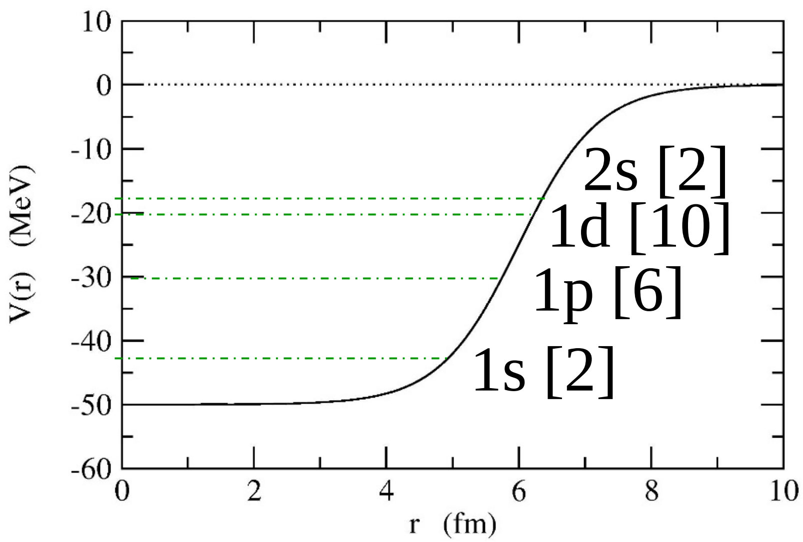


liquid drop model

Extra binding for N or Z = 2, 8, 20, 28, 50, 82, 126 (magic numbers)

Extra binding for N or Z = 2, 8, 20, 28, 50, 82, 126 (magic numbers)

An interpretation: independent particle motion in a potential well



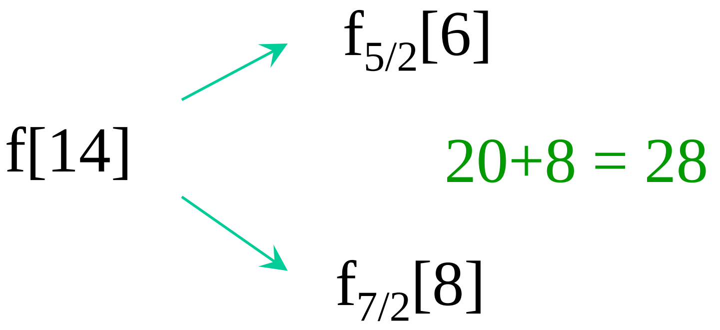
$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r) - \epsilon\right]\psi(\boldsymbol{r}) = 0$$

$$\psi(\boldsymbol{r}) = \frac{u_l(r)}{r} Y_{lm}(\hat{\boldsymbol{r}}) \cdot \chi_{m_s}$$

degeneracy: $2*(2l+1)$

spin-orbit interaction

f[14]	34
s[2],d[10]	20
p[6]	8
s[2]	2



f[14]

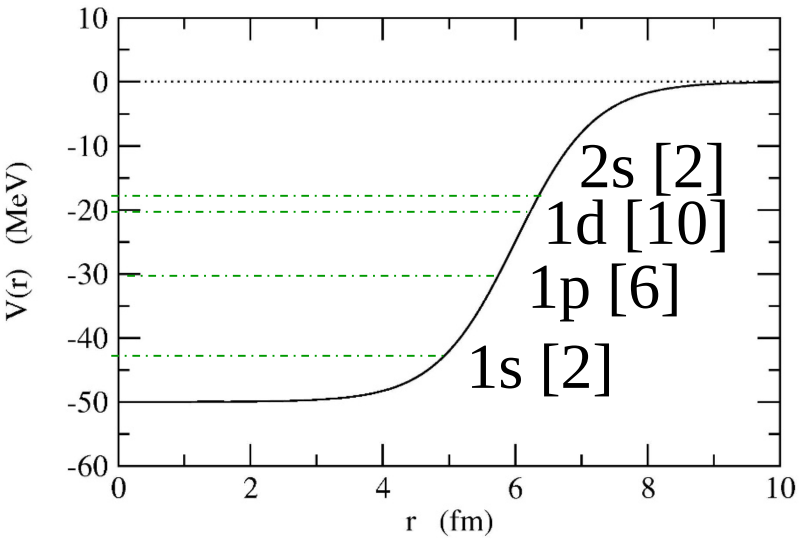
$f_{5/2}[6]$

20+8 = 28

$f_{7/2}[8]$

Extra binding for N or Z = 2, 8, 20, 28, 50, 82, 126 (magic numbers)

An interpretation: independent particle motion in a potential well



+ spin-orbit interaction

Today: how to construct the potential well?

cf. magic numbers: robust?

neutron-rich nuclei: disappearance of N=8 and 20,
appearance of N=16 (new magic number)

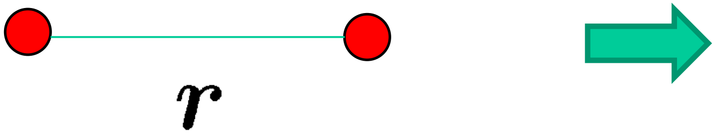
→ needs to know how to construct *V*(r)

$$H = T + V = \sum_{i=1}^{A} t(\vec{r}_i) + \sum_{\substack{ij \\ i<j}}^{A} v(\vec{r}_i , \vec{r}_j)$$

$$= \underbrace{T + \sum_{i} v(r_i)}_{\overset{\shortparallel}{H_{MF}}} + \underbrace{\sum_{i<j} v(\vec{r}_i - \vec{r}_j) - \sum_{i} v(r_i)}_{V_{Res}}$$

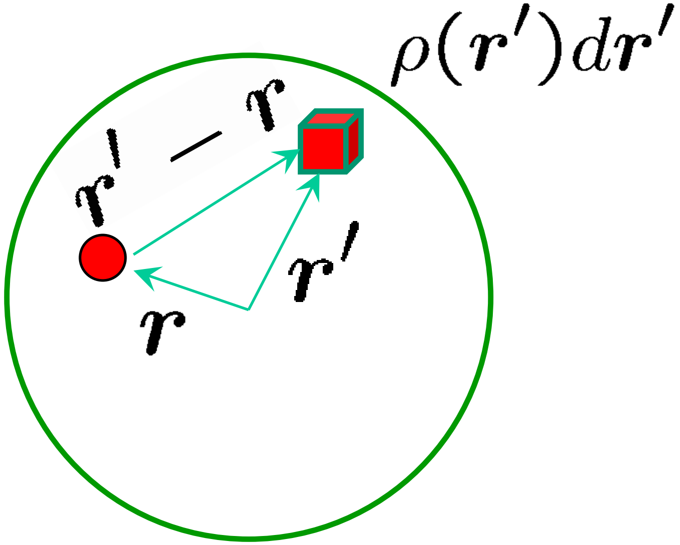
Mean-field (Hartree-Fock) Theory

nucleon-nucleon interaction



$v(\boldsymbol{r})$

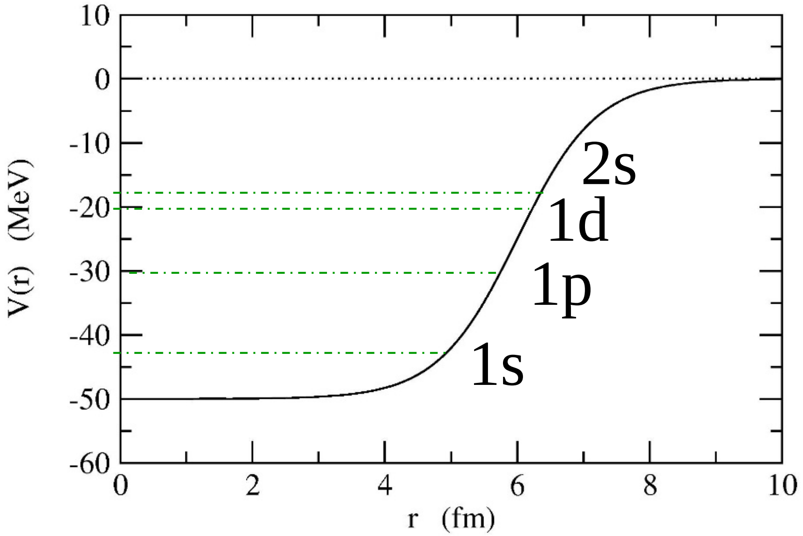
interaction for a nucleon inside a nucleus:

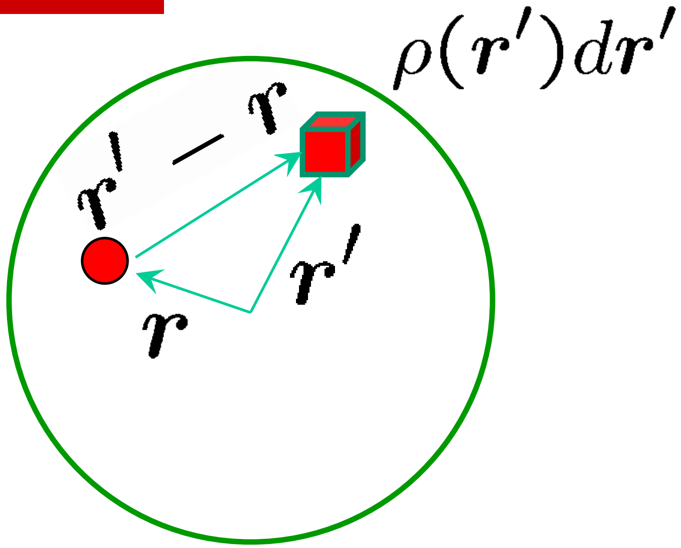


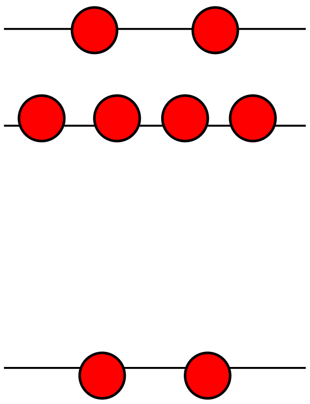
naively speaking,

$$\boxed{V(\boldsymbol{r}) \sim \int v(\boldsymbol{r}-\boldsymbol{r}')\rho(\boldsymbol{r}')d\boldsymbol{r}'}$$

Mean-field (Hartree-Fock) Theory







$1p_{1/2}$

$1p_{3/2}$

$1s_{1/2}$

shell model

naively speaking,

$$\boxed{V(\boldsymbol{r}) \sim \int v(\boldsymbol{r}-\boldsymbol{r}')\rho(\boldsymbol{r}')d\boldsymbol{r}'}$$

* strongly interacting many-fermion system

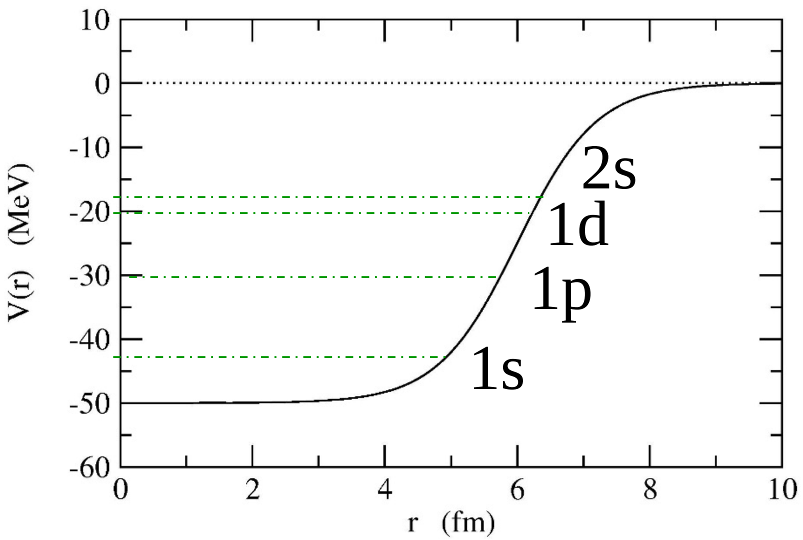
↓

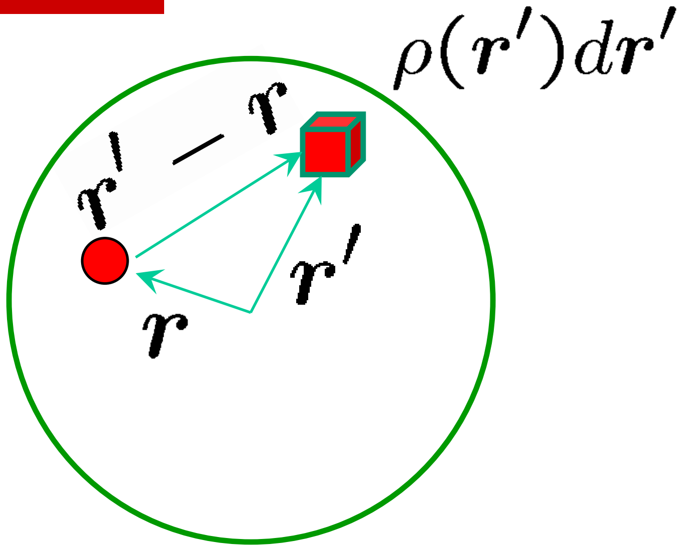
= quasiparticle

non-interacting fermion in MF

+ residual int (perturbation theory)

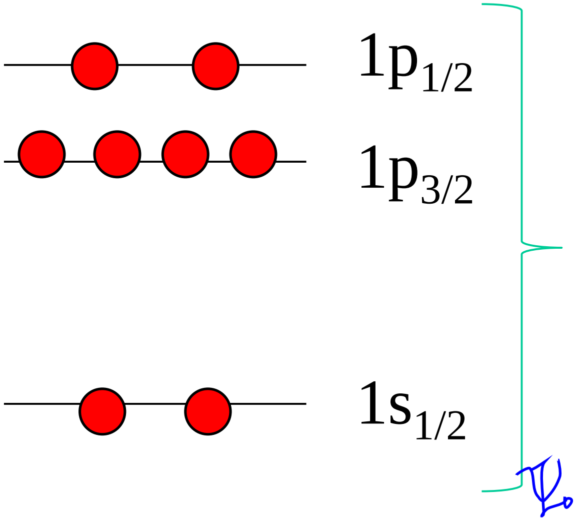
Mean-field (Hartree-Fock) Theory





naively speaking,

$$\boxed{V(\boldsymbol{r}) \sim \int v(\boldsymbol{r}-\boldsymbol{r}')\rho(\boldsymbol{r}')d\boldsymbol{r}'}$$



shell model

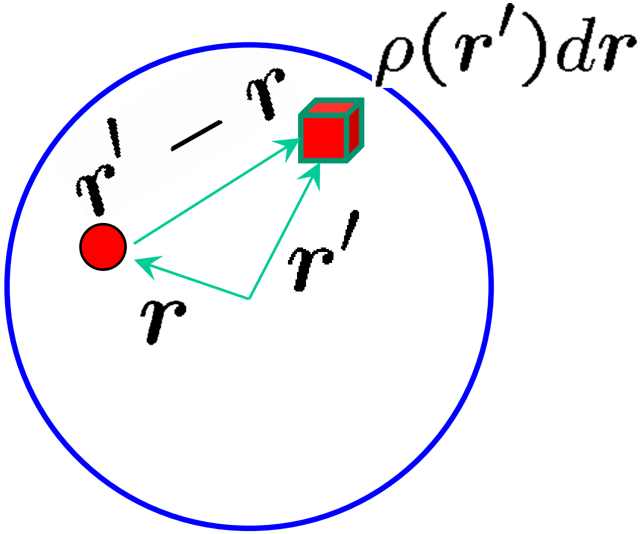
independent motion

$\Psi_0(r_1 r_2 \cdots) = \varphi_{\alpha_1}(r_1)\varphi_{\alpha_2}(r_2)\cdots$

$$\rho(\boldsymbol{r}) = \sum_i |\psi_i(\boldsymbol{r})|^2$$

$H_{MF}\Psi_0(r_1, r_2 \cdots r_A) = E\Psi_0(r_1, r_2 \cdots r_A)$

Mean-field (Hartree-Fock) Theory



naively speaking,

$$\boxed{V(\boldsymbol{r}) \sim \int v(\boldsymbol{r}-\boldsymbol{r}')\rho(\boldsymbol{r}')d\boldsymbol{r}'}$$

$$\rho(\boldsymbol{r}) = \sum_i |\psi_i(\boldsymbol{r})|^2$$

$$\begin{aligned} 0 &= \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V(\boldsymbol{r}) - \epsilon_i\right]\psi_i(\boldsymbol{r}) \\ &= \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + \int v(\boldsymbol{r}-\boldsymbol{r}')\left(\sum_j |\psi_j(\boldsymbol{r}')|^2\right)d\boldsymbol{r}' - \epsilon_i\right]\psi_i(\boldsymbol{r}) \end{aligned}$$

the potential depends on the solutions

Mean-field (Hartree-Fock) Theory

$$\boxed{\begin{aligned} 0 &= \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V(\boldsymbol{r}) - \epsilon_i\right]\psi_i(\boldsymbol{r}) \\ &= \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + \int v(\boldsymbol{r}-\boldsymbol{r}')\left(\sum_j |\psi_j(\boldsymbol{r}')|^2\right)d\boldsymbol{r}' - \epsilon_i\right]\psi_i(\boldsymbol{r}) \end{aligned}}$$

the potential depends on the solutions

→ self-consistent solutions

ψ_i ; determined by minimizing the $E_{g.s}$

$E_{g.s} = \langle \Psi_0 | \underset{\underset{T + V_{MF} + V_{RES}}{\uparrow}}{H} | \Psi_0 \rangle$

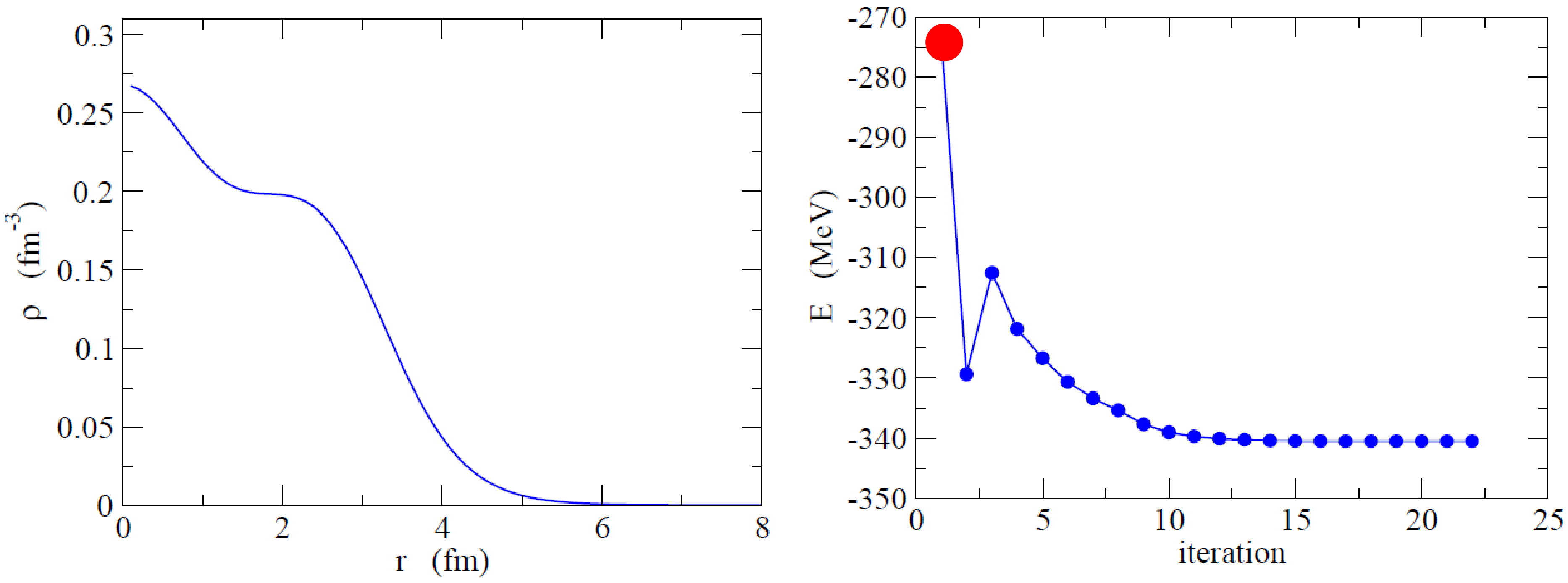
Iteration: $\{\psi_i\} \to \rho \to V \to \{\psi_i\} \to \cdots$

$\psi_i^{(n-1)} - \psi_i^{(n)} \ll$ preset limit

repeat until the first and the last wave functions are the same.

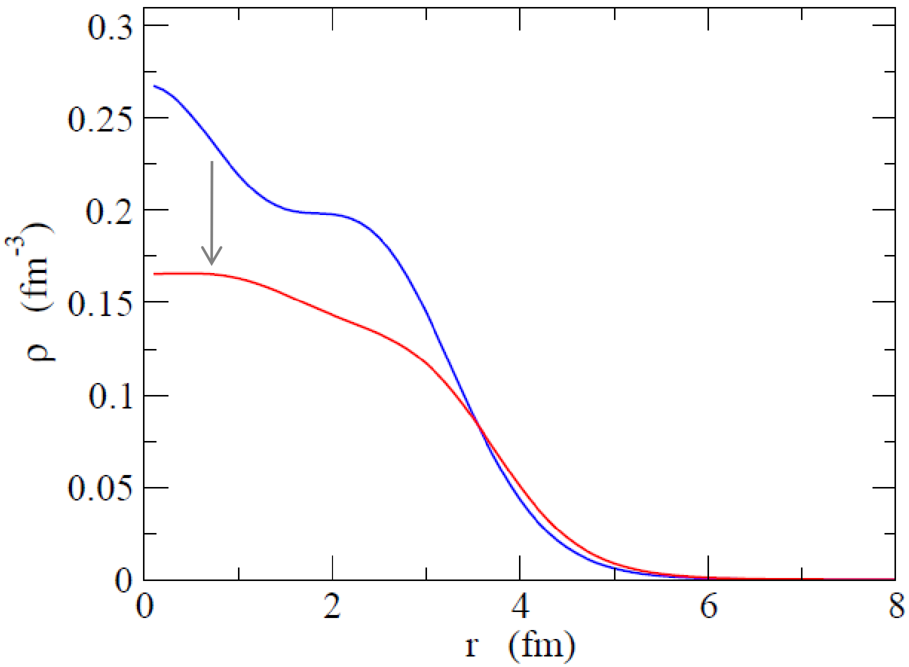
"self-consistent solutions"

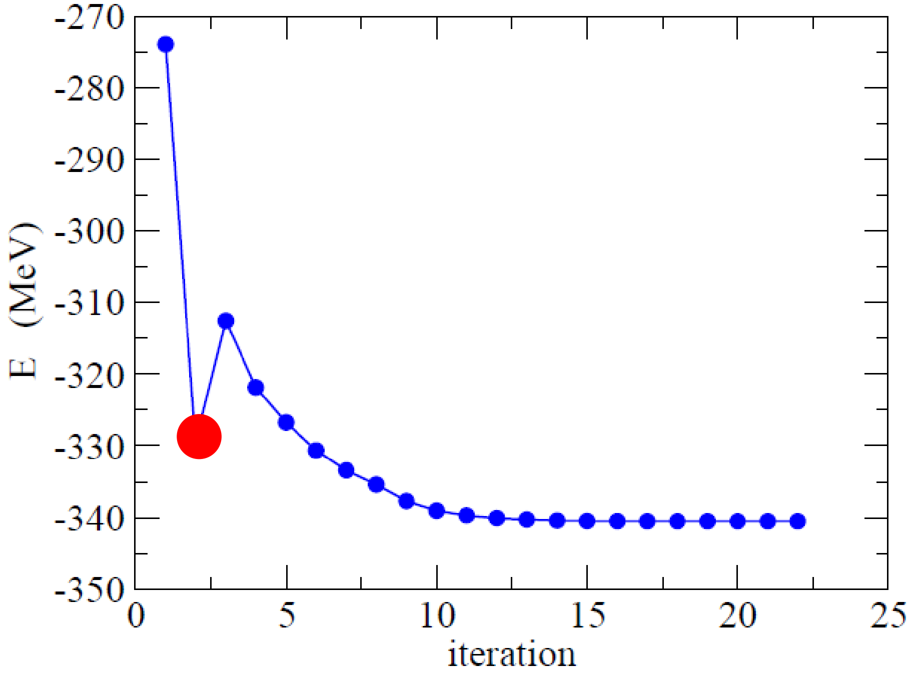
Skyrme-Hartree-Fock calculations for ^{40}Ca



optimize the density by taking into account the nucleon-nucleon interaction

Skyrme-Hartree-Fock calculations for ^{40}Ca





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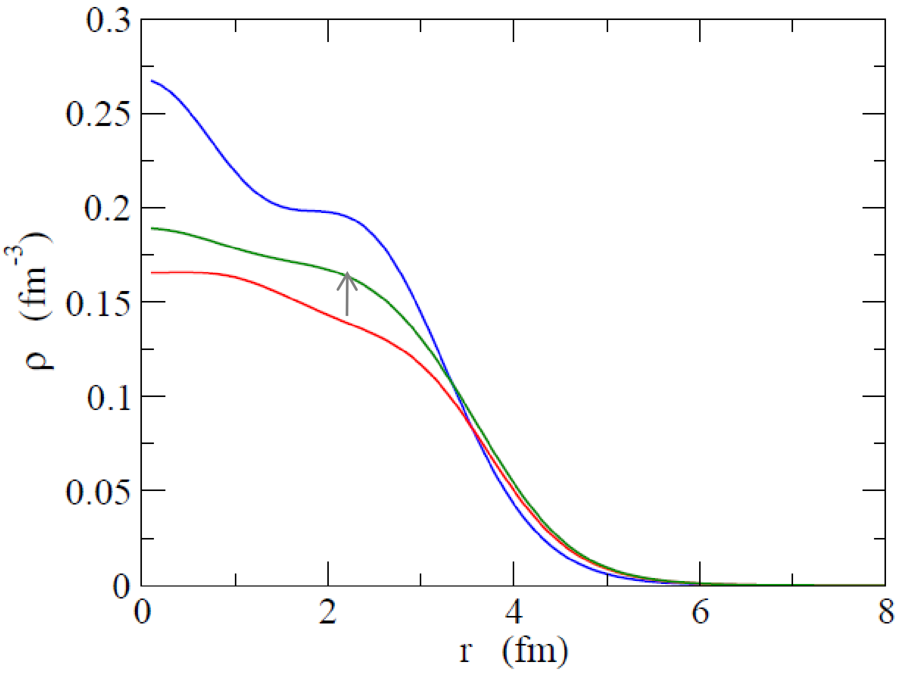
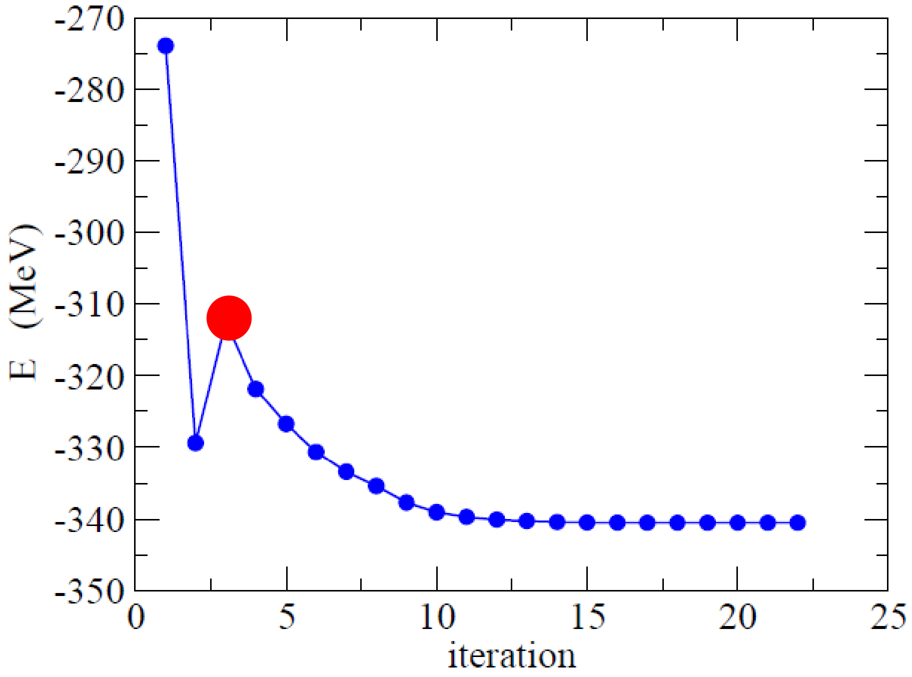
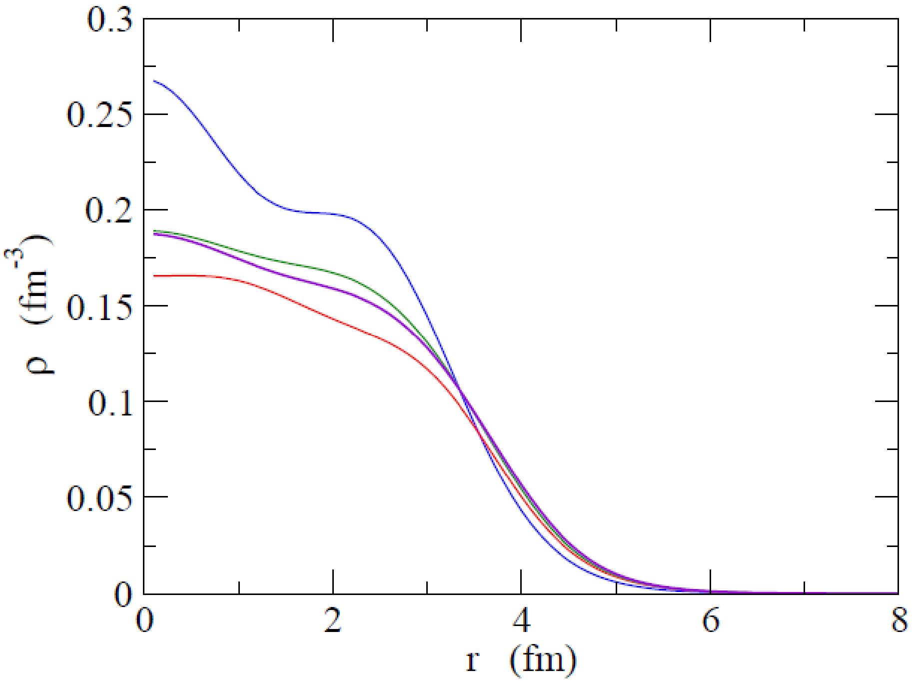
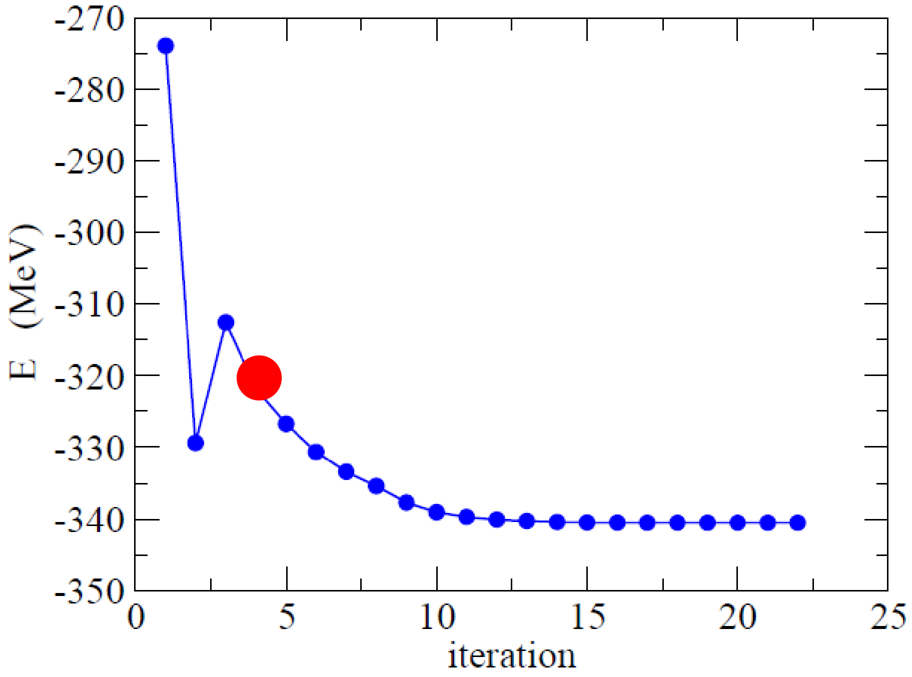




optimize the density by taking into account the nucleon-nucleon interaction

Skyrme-Hartree-Fock calculations for ^{40}Ca





optimize the density by taking into account the nucleon-nucleon interaction

→ optimized density (and shape) can be determined automatically

Variational Principle (Rayleigh-Ritz method)

optimization ⟷ variational principle

$$\boxed{\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle} \geq E_{\text{g.s.}}}$$

$$|\Psi\rangle = \sum_n C_n \, |\phi_n\rangle$$

$$\longrightarrow \quad \text{lhs} = \frac{\sum_n C_n^2 E_n}{\sum_n C_n^2} \geq E_0$$

H : many-body Hamiltonian

$$\Psi(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots) = \psi_1(\boldsymbol{r}_1) \cdot \psi_2(\boldsymbol{r}_2) \cdot \psi_3(\boldsymbol{r}_3) \cdots$$

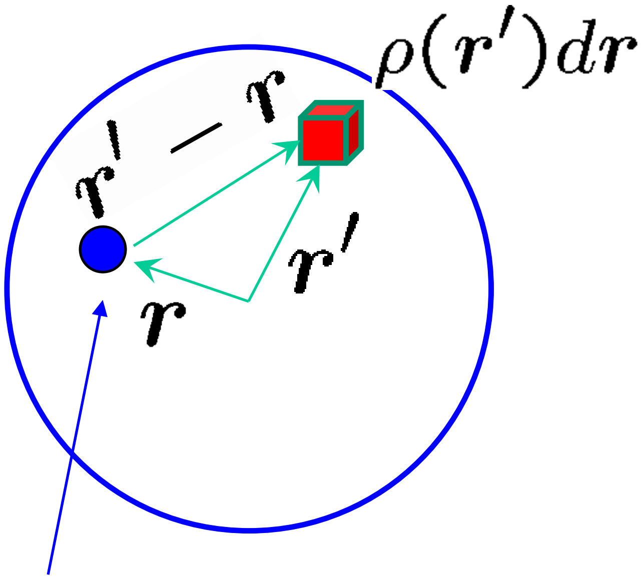
← many-body wave function for independent particles

$$\left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + \int v(\boldsymbol{r} - \boldsymbol{r}')\rho(\boldsymbol{r}')d\boldsymbol{r}' - \epsilon_i\right]\psi_i(\boldsymbol{r}) = 0$$

change gradually the single-particle potential so that the total energy becomes minimum

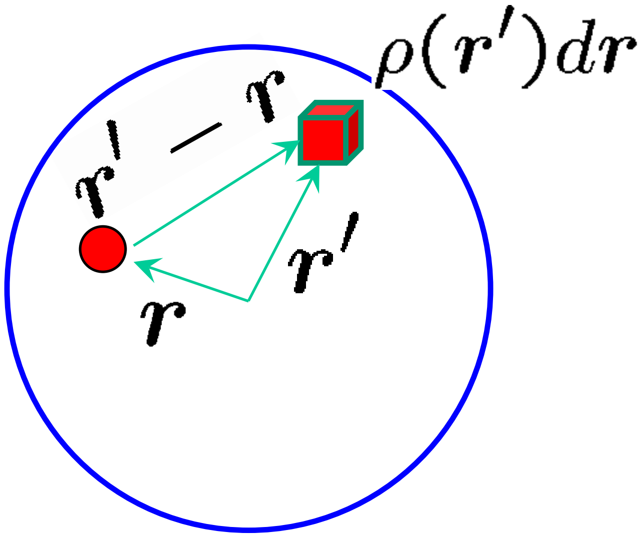
Mean-field (Hartree-Fock) Theory

electro-static potential



test charge

nucleus



interaction between identical particles
→ needs anti-symmetrization

$$\boxed{V(\boldsymbol{r}) \sim \int v(\boldsymbol{r}-\boldsymbol{r}')\rho(\boldsymbol{r}')d\boldsymbol{r}'}$$

anti-symmetrization

nucleon: fermion

$$\Psi(x_1, x_2, x_3 \cdots) = -\Psi(x_2, x_1, x_3 \cdots)$$

$$\psi_1(x_1)\psi_2(x_2) \to [\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)]$$

Slater determinat

$$0 = \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + \int v(\boldsymbol{r}-\boldsymbol{r}')\left(\sum_j |\psi_j(\boldsymbol{r}')|^2\right)d\boldsymbol{r}' - \epsilon_i\right]\psi_i(\boldsymbol{r})$$

$$\to \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + \int v(\boldsymbol{r}-\boldsymbol{r}')\left(\sum_j |\psi_j(\boldsymbol{r}')|^2\right)d\boldsymbol{r}' - \epsilon_i\right]\psi_i(\boldsymbol{r})$$

$$-\int v(\boldsymbol{r}-\boldsymbol{r}')\left(\sum_j \psi_j^*(\boldsymbol{r}')\underline{\underline{\psi_i}}(\boldsymbol{r}')\right)d\boldsymbol{r}'\underline{\underline{\psi_j}}(\boldsymbol{r})$$

exchange term

Hartree-Fock theory

Slater determinant

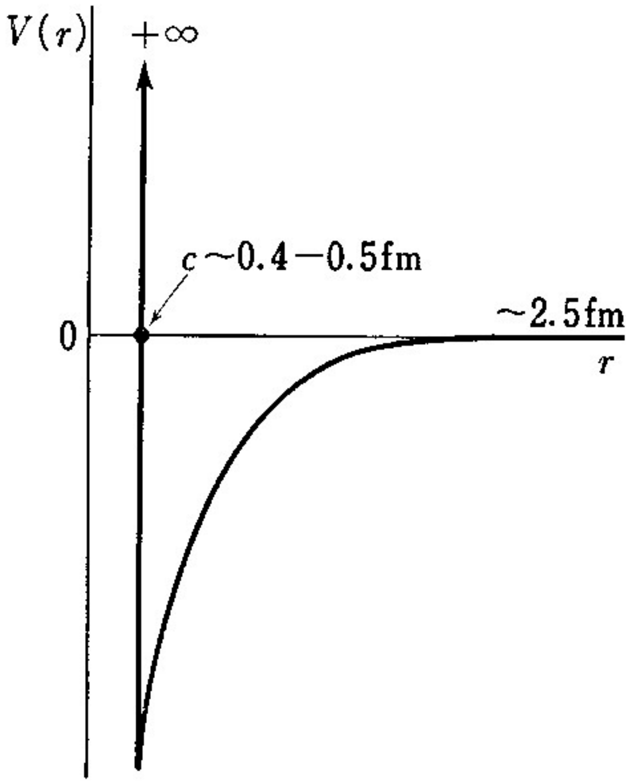
$$\psi_{MF}(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_A) = \frac{1}{\sqrt{A!}} \begin{vmatrix} \phi_1(\vec{r}_1) & \phi_1(\vec{r}_2) & \cdots & \phi_1(\vec{r}_A) \\ \phi_2(\vec{r}_1) & \phi_2(\vec{r}_2) & \cdots & \phi_2(\vec{r}_A) \\ & \vdots & & \\ \phi_A(\vec{r}_1) & \phi_A(\vec{r}_2) & \cdots & \phi_A(\vec{r}_A) \end{vmatrix}$$

anti-symmetrization

$$
\begin{aligned}
0 &= \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + \int v(\boldsymbol{r}-\boldsymbol{r}')\left(\sum_j |\psi_j(\boldsymbol{r}')|^2\right) d\boldsymbol{r}' - \epsilon_i\right]\psi_i(\boldsymbol{r}) \\
&\to \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + \int v(\boldsymbol{r}-\boldsymbol{r}')\left(\sum_j |\psi_j(\boldsymbol{r}')|^2\right) d\boldsymbol{r}' - \epsilon_i\right]\psi_i(\boldsymbol{r}) \\
&\quad - \int v(\boldsymbol{r}-\boldsymbol{r}')\left(\sum_j \psi_j^*(\boldsymbol{r}')\psi_i(\boldsymbol{r}')\right) d\boldsymbol{r}'\psi_j(\boldsymbol{r}) \\
&= \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V(\boldsymbol{r}) - \epsilon_i\right]\psi_i(\boldsymbol{r}) + \int d\boldsymbol{r}' V_{\mathrm{NL}}(\boldsymbol{r},\boldsymbol{r}')\psi_i(\boldsymbol{r}')
\end{aligned}
$$

non-local potential

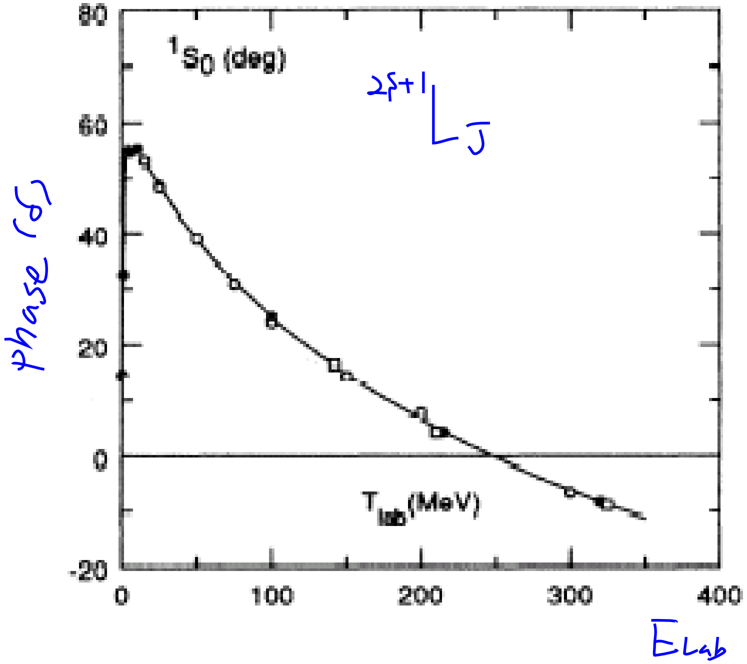
Bare nucleon-nucleon interaction

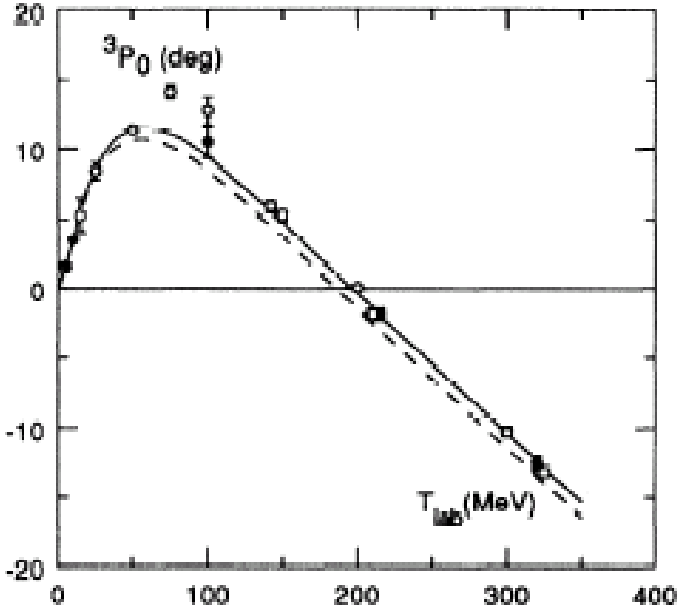


Existence of short range
repulsive core

Bare nucleon-nucleon interaction

Phase shift for p-p scattering





(V.G.J. Stoks et al., PRC48('93)792)

$\delta > 0$: attraction

$\delta < 0$: repulsion

Phase shift:

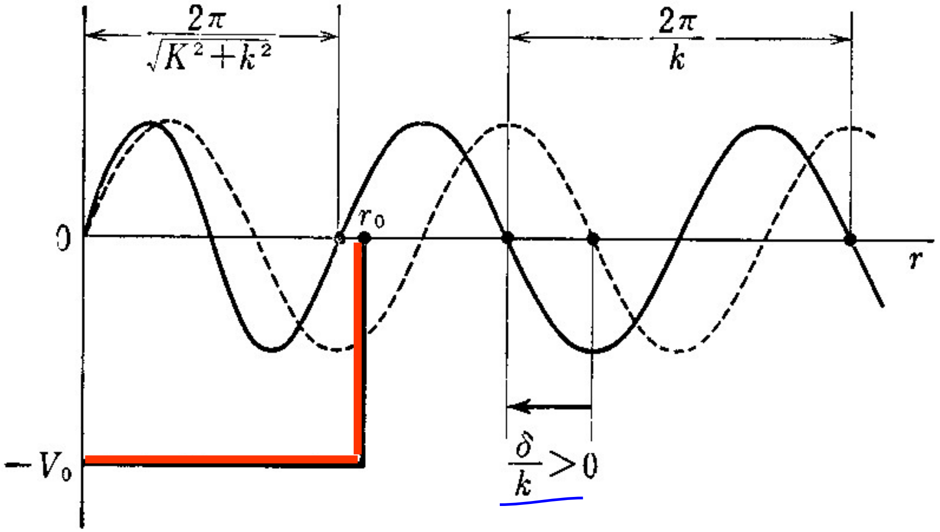
Radial wave function

$$\Psi_l(\boldsymbol{r}) = \frac{u_l(r)}{r} Y_{lm}(\hat{\boldsymbol{r}})$$

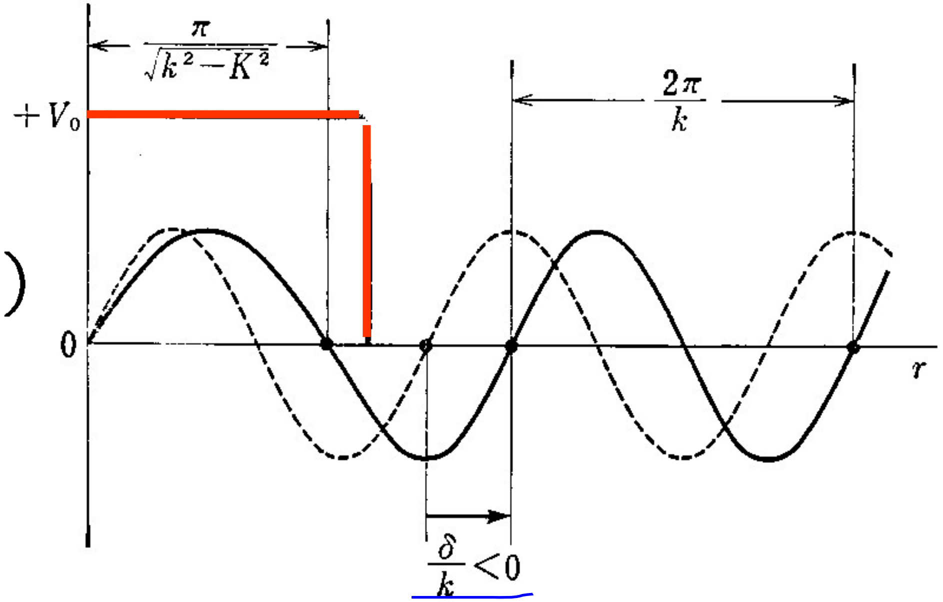
$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V(r) + \frac{l(l+1)\hbar^2}{2mr^2} - E\right] u_l(r) = 0$$

Asymptotic form:

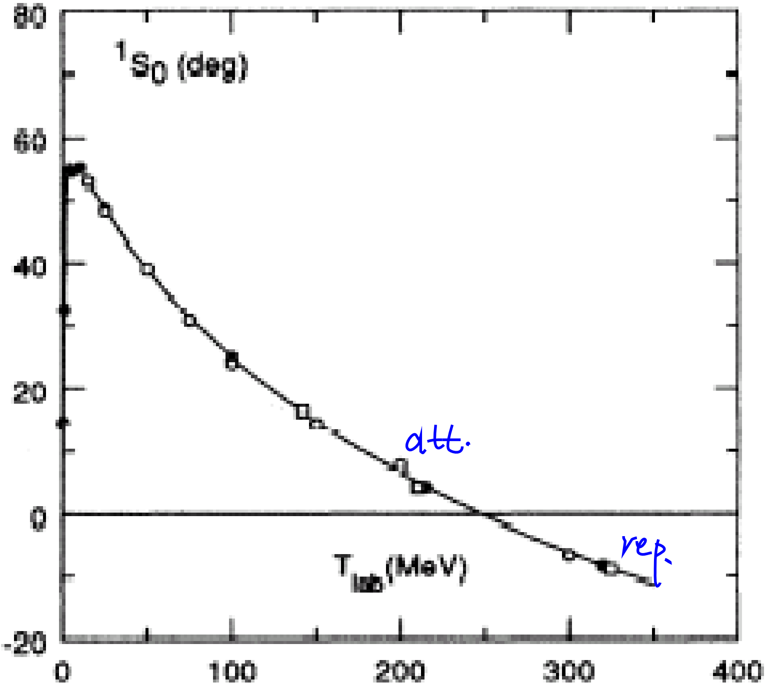
$$u_l(r) \;\rightarrow\; \sin(kr - l\pi/2 + \delta_l) \qquad (r \rightarrow \infty)$$



（a） 引　力　attraction

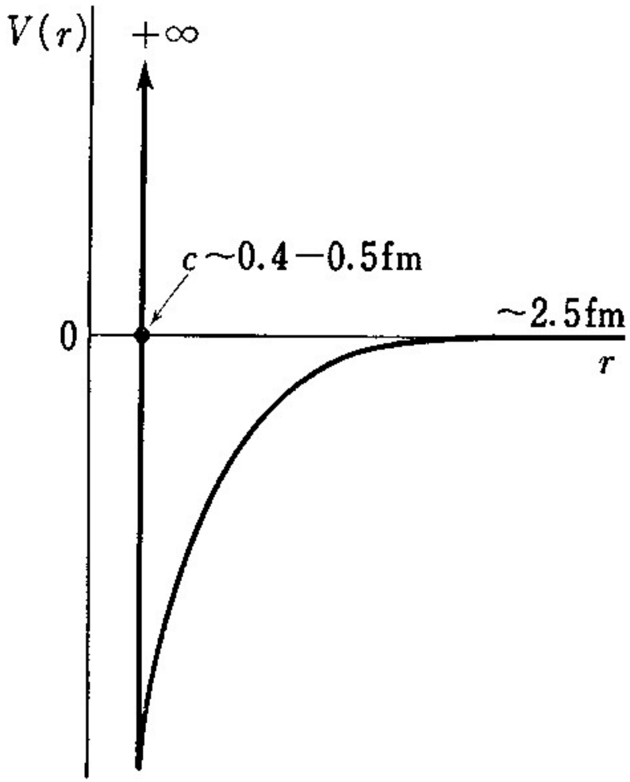


（b） 斥　力　repulsion



Phase shift: +ve ⟶ -ve

at high energies



Existence of short range

repulsive core

Bruckner’s G-matrix

Nucleon-nucleon interaction *in medium*

Nucleon-nucleon interaction with a hard core

⇒ HF method: does not work

← Matrix elements: diverge

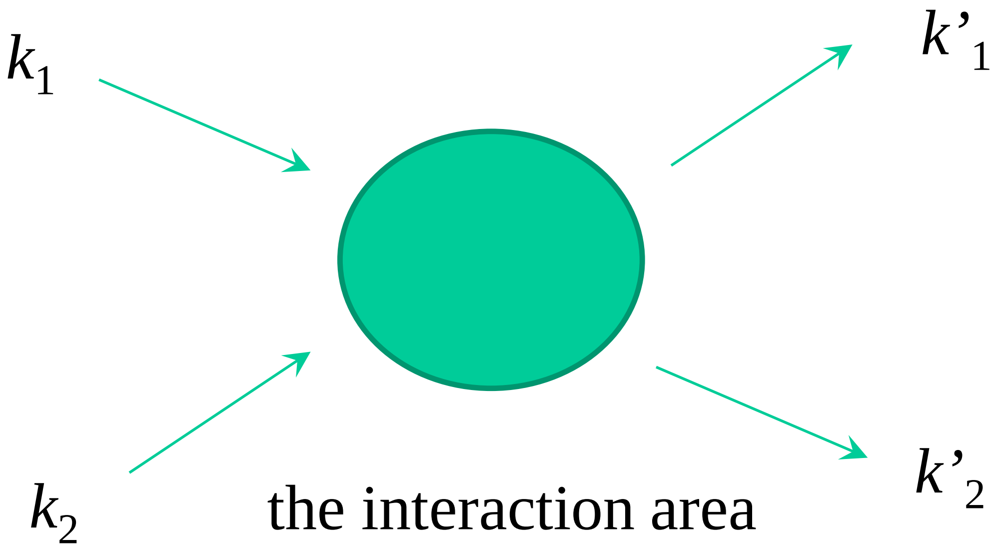
…..but the HF picture seems to work in nuclear systems

Solution: a nucleon-nucleon interaction *in medium* (effective interaction) rather than a bare interaction

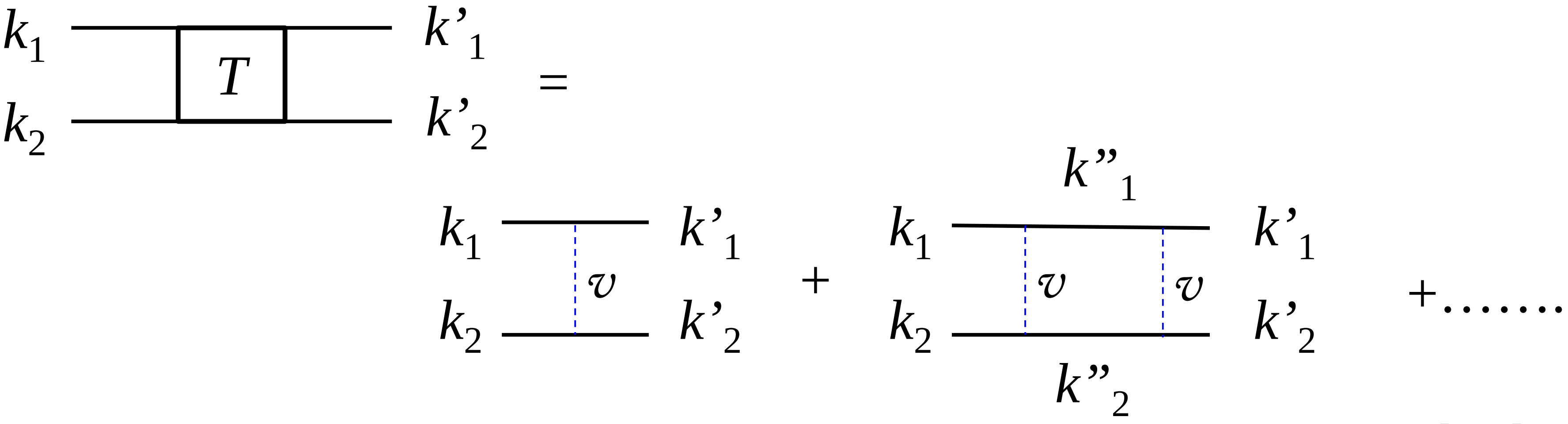
⇑

Bruckner’s G-matrix

➢ two-body (multiple) scattering *in vacuum*



the interaction area

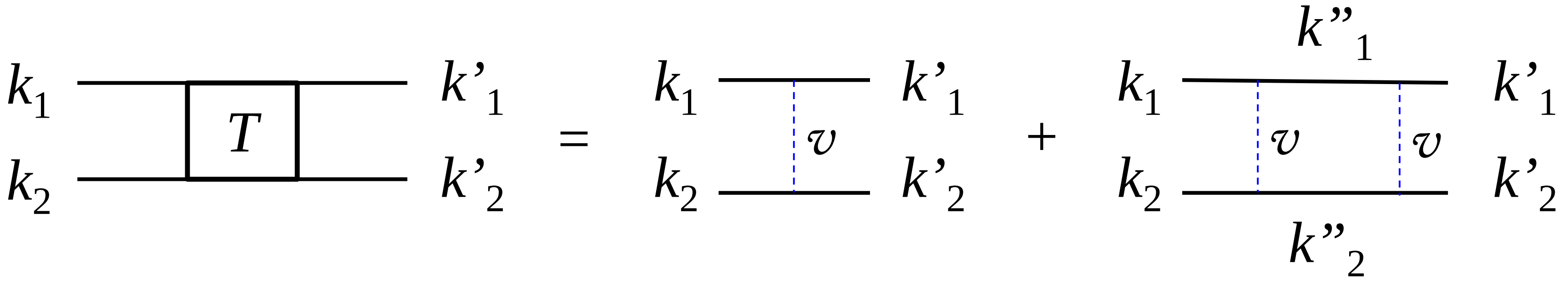


the 1st order

the 2nd order

higher orders

➢ two-body (multiple) scattering *in vacuum*



+…….

Lippmann-Schwinger equation

$$T = v + v\frac{1}{E - H_0}T$$

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = E\psi$$

if $V = 0$,

$$-\frac{\hbar^2}{2m}\nabla^2\phi = E\phi$$

← w.f of free particle

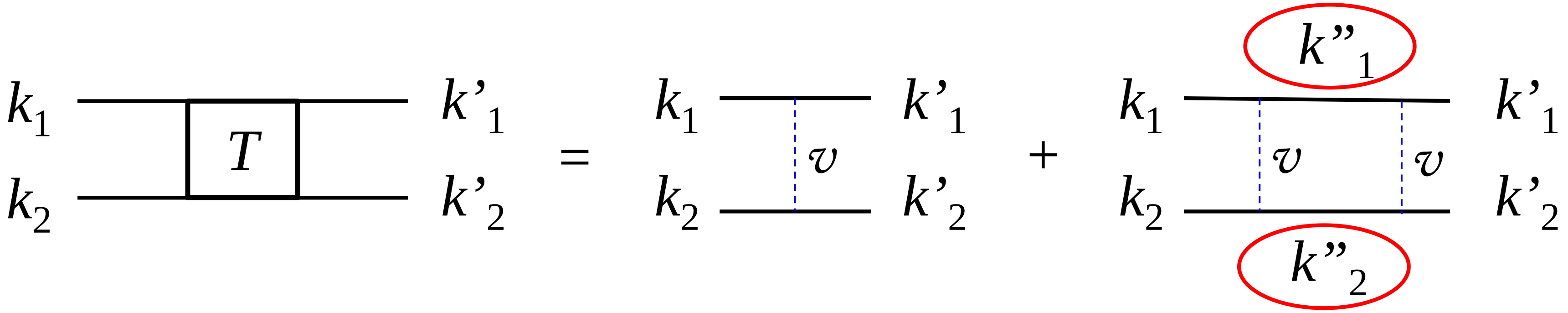
$$(E - H_0)\psi = V\psi$$

$$\psi = \phi + \frac{1}{E - H_0}V\psi,$$

$$\psi = \Omega^+\phi \rightarrow V\psi = \underline{V\overset{\text{Moller op.}}{\Omega^+}}\phi = \overset{\text{transition op}}{T}\phi$$

$$V\psi = V\phi + V\frac{1}{E - H_0}V\psi \xrightarrow{V\psi = T\phi} T = V + V\frac{1}{E - H_0}T$$

- two-body (multiple) scattering *in vacuum*



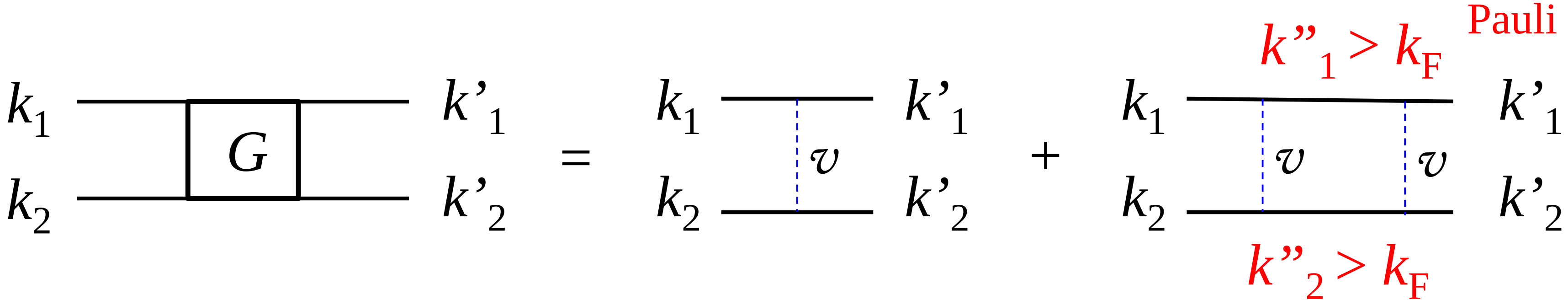
+…….

Lippmann-Schwinger equation

$$T = v + v\frac{1}{E - H_0}T$$

- two-body (multiple) scattering *in medium*

Pauli priciple



+…….

*scattering: suppressed
because intermediate states have to have
$k > k_F$ → independent particle picture

Bethe-Goldstone equation

Pauli op.

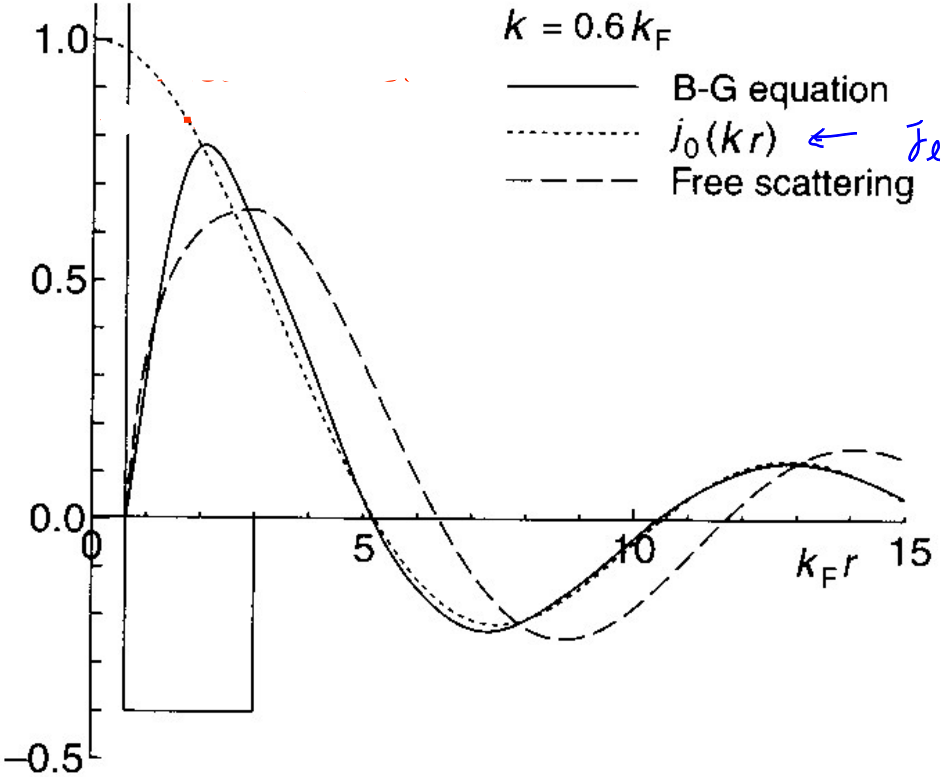
$$G = v + v\frac{\boxed{Q_F}}{E - H_0}G$$

◆Hard core

$$G = v + v\frac{Q_F}{E - H_0}G \quad \Longleftrightarrow \quad G = \frac{v}{1 - v\,Q_F/(E - H_0)}$$

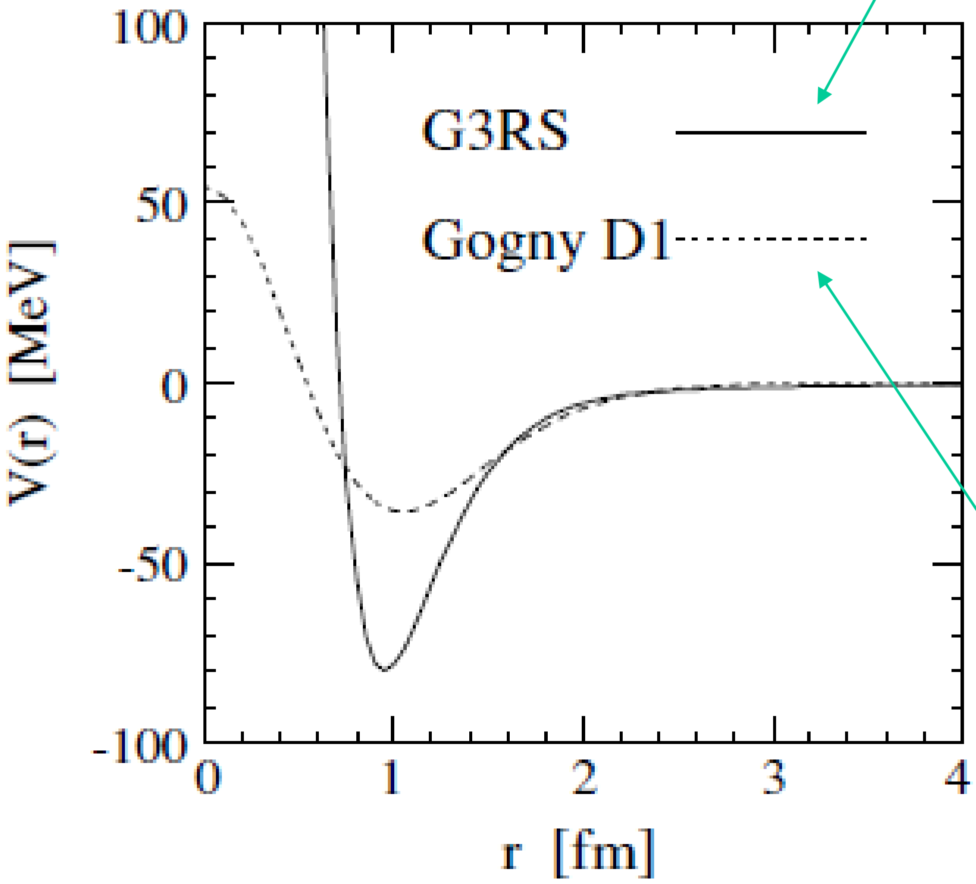
Even if v tends to infinity, G may stay finite.

◆Independent particle motion



← $j_\ell(\rho) = (\quad)^\ell \frac{\sin\rho}{\rho}$ infinite square well

→ use G instead of v in mean-field calculations bare interaction



effective
interaction

M. Matsuo, Phys. Rev. C73(‘06)044309

Phenomenological effective interactions

G-matrix

- ab initio
- but, cumbersome to compute (especially for finite nuclei)
- qualitatively good, but quantitatively not successful

HF calculations with a phenomenological effective interaction

Philosophy: take the functional form of *G*, but determine the parameters phenomenologically

- ➢Skyrme interaction (non-rel., zero range)
- ➢Gogny interaction (non-rel., finite range)
- ➢Relativistic mean-field model (relativistic, “meson exchanges”)

Skyrme interaction density dependent zero-range interaction

$$\boxed{\begin{aligned} v(\boldsymbol{r},\boldsymbol{r}') \;=\;& t_0(1+x_0\hat{P}_\sigma)\delta(\boldsymbol{r}-\boldsymbol{r}') \\ & +\frac{1}{2}t_1(1+x_1\hat{P}_\sigma)(\boldsymbol{k}^2\delta(\boldsymbol{r}-\boldsymbol{r}')+\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{k}^2) \\ & +t_2(1+x_2\hat{P}_\sigma)\boldsymbol{k}\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{k} \\ & +\frac{1}{6}t_3(1+x_3\hat{P}_\sigma)\delta(\boldsymbol{r}-\boldsymbol{r}')\rho^\alpha((\boldsymbol{r}+\boldsymbol{r}')/2) \\ & +iW_0(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}\times\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{k} \end{aligned}}$$

$$\boldsymbol{k}=(\boldsymbol{\nabla}_1-\boldsymbol{\nabla}_2)/2i$$

if $x_i=0$, $t_1=t_2=0$:

$$v(\boldsymbol{r},\boldsymbol{r}') \;=\; \underbrace{t_0\delta(\boldsymbol{r}-\boldsymbol{r}')}_{\text{short-range attraction}}+\underbrace{\frac{1}{6}t_3\delta(\boldsymbol{r}-\boldsymbol{r}')\rho^\alpha(\boldsymbol{r})}_{\text{repulsion to avoid collapse}}$$

$$\underbrace{+iW_0(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}\times\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{k}}_{\text{spin-orbit interaction}}$$

Skyrme interaction   density dependent zero-range interaction

$$\boxed{\begin{aligned} v(\boldsymbol{r},\boldsymbol{r}') &= t_0(1+x_0\widehat{P}_\sigma)\delta(\boldsymbol{r}-\boldsymbol{r}') \\ &+\frac{1}{2}t_1(1+x_1\widehat{P}_\sigma)(\boldsymbol{k}^2\delta(\boldsymbol{r}-\boldsymbol{r}')+\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{k}^2) \\ &+t_2(1+x_2\widehat{P}_\sigma)\boldsymbol{k}\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{k} \\ &+\frac{1}{6}t_3(1+x_3\widehat{P}_\sigma)\delta(\boldsymbol{r}-\boldsymbol{r}')\rho^\alpha((\boldsymbol{r}+\boldsymbol{r}')/2) \\ &+iW_0(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}\times\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{k} \end{aligned}}$$

$$\boldsymbol{k}=(\boldsymbol{\nabla}_1-\boldsymbol{\nabla}_2)/2i$$

(note) finite range effect $\Longleftrightarrow$ momentum dependence

$$\begin{aligned} \langle\boldsymbol{p}|V|\boldsymbol{p}'\rangle &= \frac{1}{(2\pi\hbar)^3}\int d\boldsymbol{r}\, e^{-i(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{r}/\hbar}V(\boldsymbol{r}) \\ &\sim V_0+V_1(\boldsymbol{p}^2+\boldsymbol{p}'^2)+V_2\boldsymbol{p}\boldsymbol{p}'+\cdots \\ &\to V_0\delta(\boldsymbol{r})+V_1(\hat{\boldsymbol{p}}^2\delta(\boldsymbol{r})+\delta(\boldsymbol{r})\hat{\boldsymbol{p}}^2)+V_2\,\hat{\boldsymbol{p}}\delta(\boldsymbol{r})\hat{\boldsymbol{p}} \end{aligned}$$

Skyrme interaction density dependent zero-range interaction

$$
\begin{aligned}
v(\boldsymbol{r},\boldsymbol{r}') &= t_0(1+x_0\hat{P}_\sigma)\delta(\boldsymbol{r}-\boldsymbol{r}') \\
&+\frac{1}{2}t_1(1+x_1\hat{P}_\sigma)(\boldsymbol{k}^2\delta(\boldsymbol{r}-\boldsymbol{r}')+\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{k}^2) \\
&+t_2(1+x_2\hat{P}_\sigma)\boldsymbol{k}\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{k} \\
&+\frac{1}{6}t_3(1+x_3\hat{P}_\sigma)\delta(\boldsymbol{r}-\boldsymbol{r}')\rho^\alpha((\boldsymbol{r}+\boldsymbol{r}')/2) \\
&+iW_0(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}\times\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{k}
\end{aligned}
$$

$$\boldsymbol{k}=(\boldsymbol{\nabla}_1-\boldsymbol{\nabla}_2)/2i$$

the exchange potential ⟶ local

$$
\begin{aligned}
0 &= \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2+\int v(\boldsymbol{r}-\boldsymbol{r}')\left(\sum_j|\psi_j(\boldsymbol{r}')|^2\right)d\boldsymbol{r}'-\epsilon_i\right]\psi_i(\boldsymbol{r}) \\
&\quad -\int v(\boldsymbol{r}-\boldsymbol{r}')\left(\sum_j\psi_j^*(\boldsymbol{r}')\psi_i(\boldsymbol{r}')\right)d\boldsymbol{r}'\psi_j(\boldsymbol{r})
\end{aligned}
$$

Skyrme interactions: 10 adjustable parameters

$$
\begin{aligned}
v(\boldsymbol{r},\boldsymbol{r}') \;=\;& t_0(1+x_0\widehat{P}_\sigma)\delta(\boldsymbol{r}-\boldsymbol{r}') \\
&+\frac{1}{2}t_1(1+x_1\widehat{P}_\sigma)(\boldsymbol{k}^2\delta(\boldsymbol{r}-\boldsymbol{r}')+\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{k}^2) \\
&+t_2(1+x_2\widehat{P}_\sigma)\boldsymbol{k}\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{k} \\
&\frac{1}{6}t_3(1+x_3\widehat{P}_\sigma)\delta(\boldsymbol{r}-\boldsymbol{r}')\rho^\alpha((\boldsymbol{r}+\boldsymbol{r}')/2) \\
&+iW_0(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{k}\times\delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{k}
\end{aligned}
$$

A fitting strategy:

B.E. and r_{rms}: ^{16}O, ^{40}Ca, ^{48}Ca, ^{56}Ni, ^{90}Zr, ^{208}Pb,…..

Infinite nuclear matter: E/A, ρ_{eq} ,….

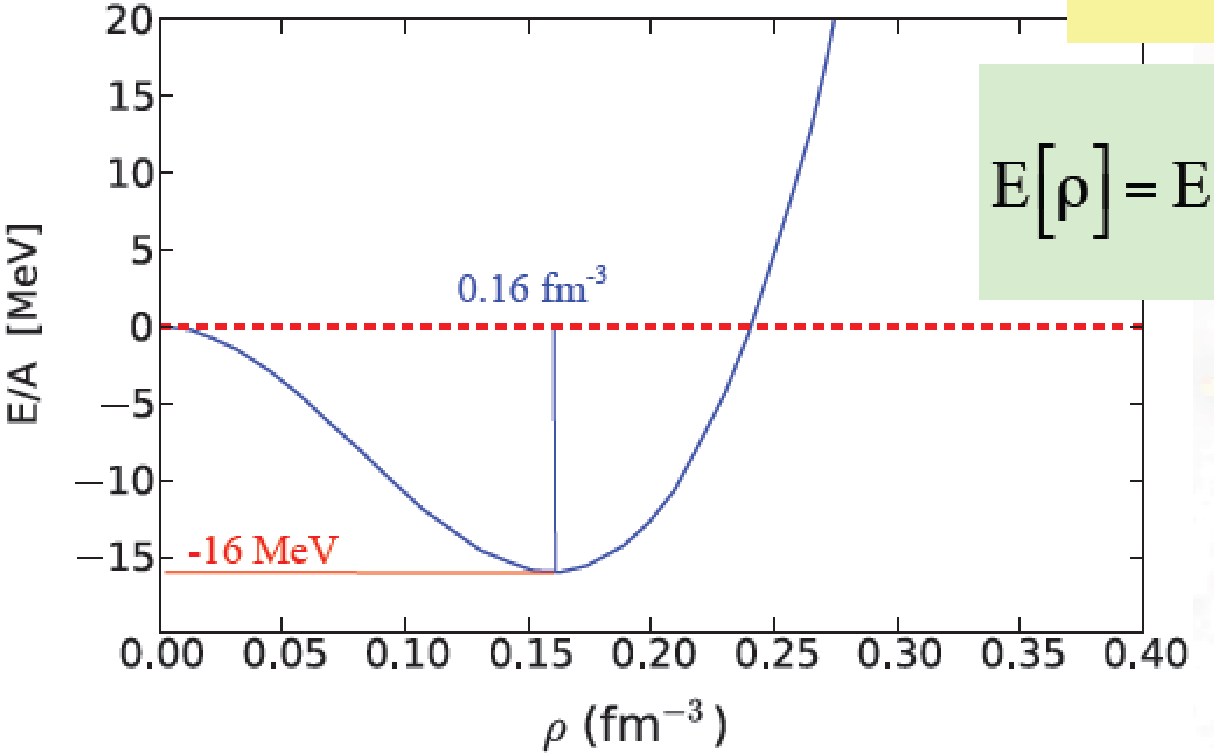
Parameter sets:

SIII, SkM*, SGII, SLy4,…….

EOS of infinite nuclear matter

$$K_\infty = 9\rho^2 \frac{d^2\left[E(\rho)/\rho\right]}{d\rho^2}\bigg|_{\rho_0}$$

$$E[\rho] = E[\rho_0] + \frac{1}{18}K_\infty\left(\frac{\rho-\rho_0}{\rho_0}\right)^2$$



slide: Carlos Bertulani

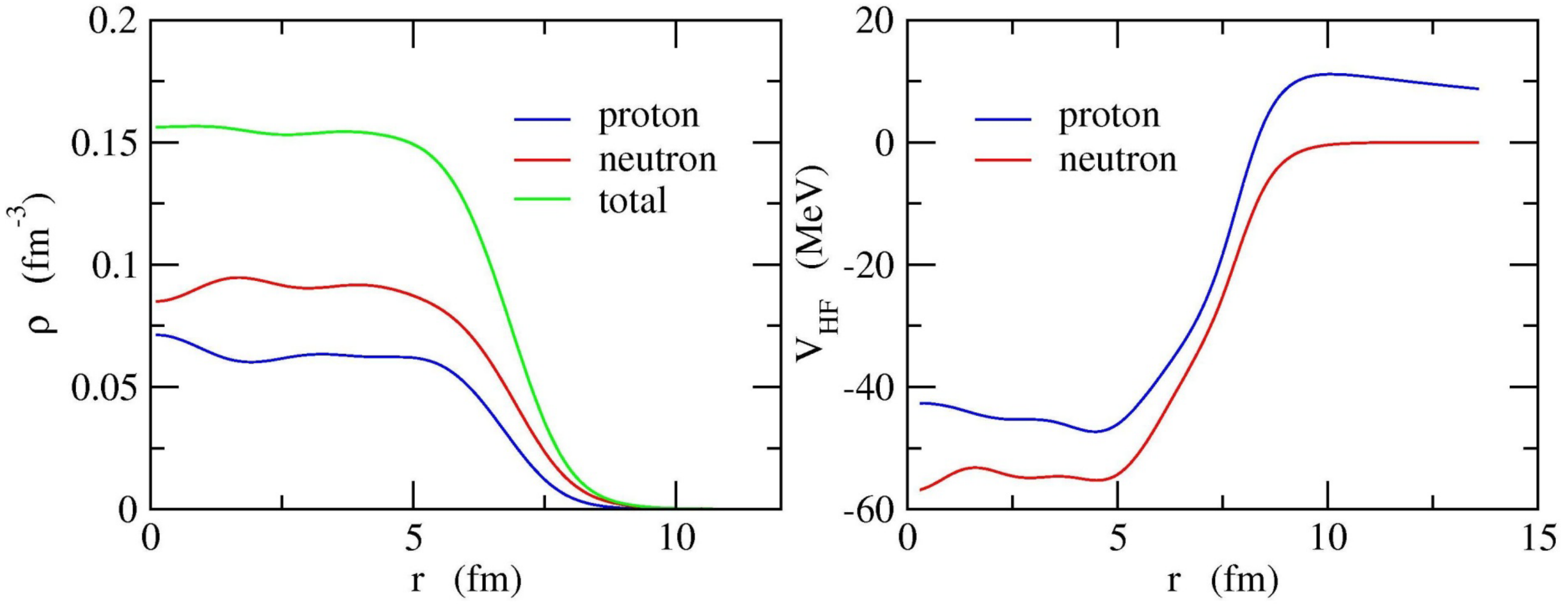
$$\boxed{\begin{aligned}-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi_i(\boldsymbol{r}) + \int v(\boldsymbol{r},\boldsymbol{r}')\rho_{\mathsf{HF}}(\boldsymbol{r}')d\boldsymbol{r}'\psi_i(\boldsymbol{r}) \\ - \int \rho_{\mathsf{HF}}(\boldsymbol{r},\boldsymbol{r}')v(\boldsymbol{r},\boldsymbol{r}')\psi_i(\boldsymbol{r}')\,d\boldsymbol{r}' = \epsilon_i\,\psi_i(\boldsymbol{r})\end{aligned}}$$

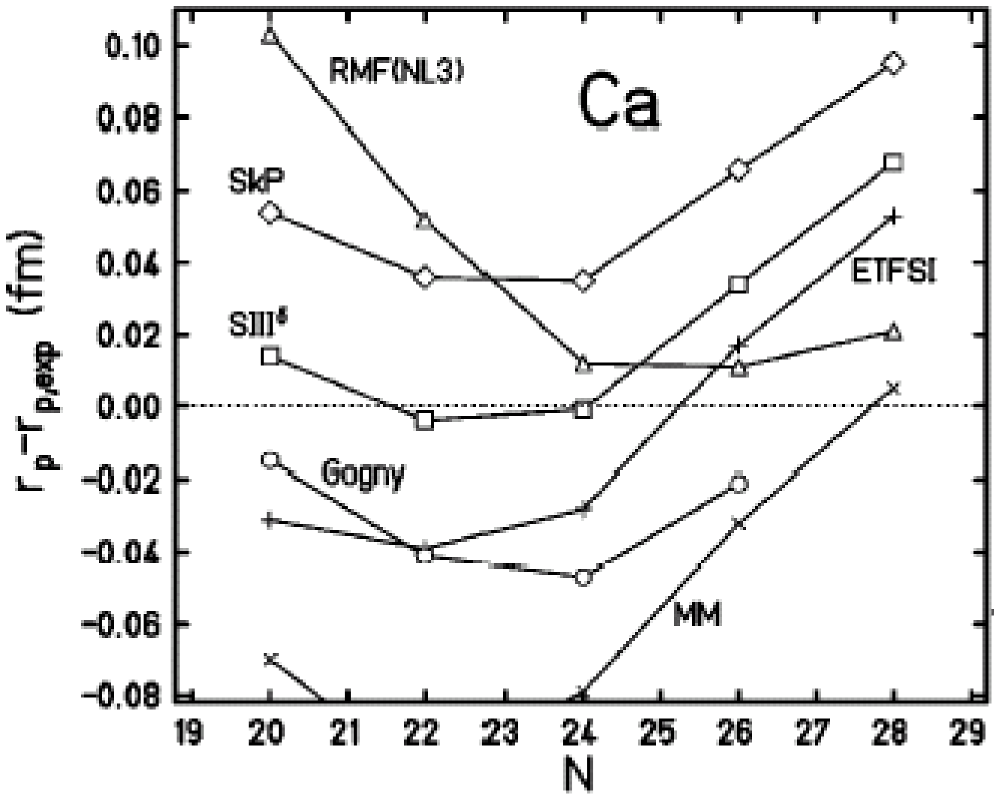
Iteration

V_{HF}: depends on ψ_i ⟵ non-linear problem

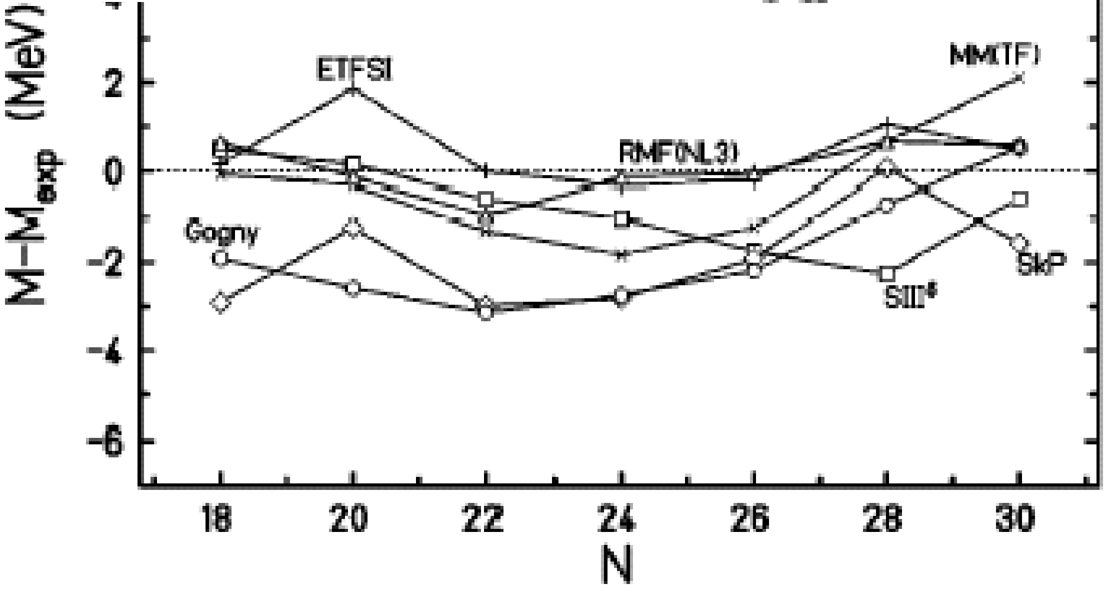
Iteration: $\{\psi_i\} \to \rho_{\mathsf{HF}} \to V_{\mathsf{HF}} \to \{\psi_i\} \to \cdots$

^{208}Pb (Skyrme Hartree-Fock with SKM*)



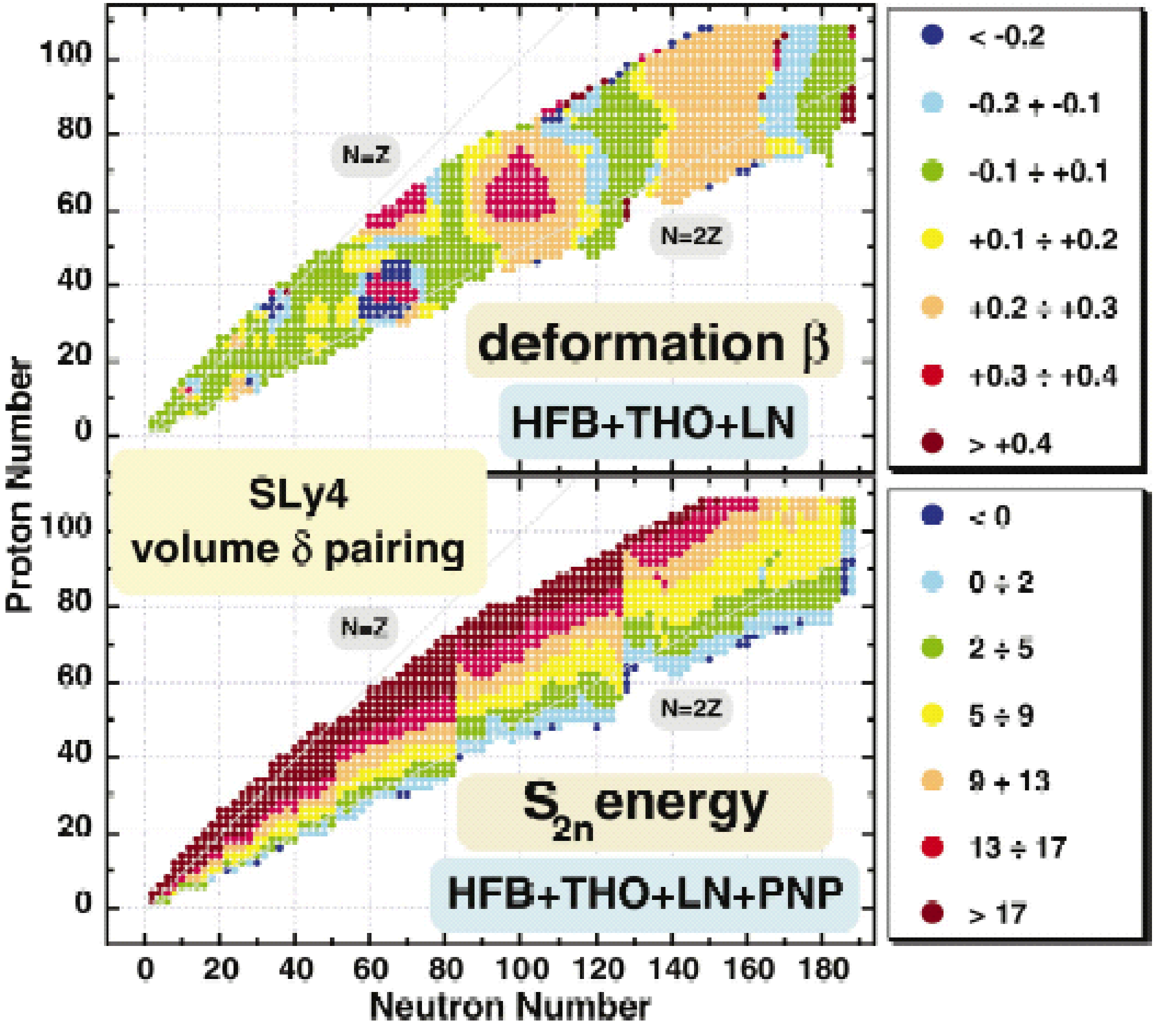


Examples of HF calculations for masses and radii



Z. Patyk et al., PRC59('99)704

deformation and two-neutron separation energy



M.V. Stoitsov et al., PRC68('03)054312