Relativistic Spherical Top as a Massive Twistor

Sangmin Lee

Seoul National University

APCTP workshop
"Quantum Geometry and Duality"
30 June 2021

This talk is based on ... [arXiv:2102.07063]

The Relativistic Spherical Top as a Massive Twistor

[Hanson,Regge 1974] $\cong$ [Fedoruk,Lukierski 2014]

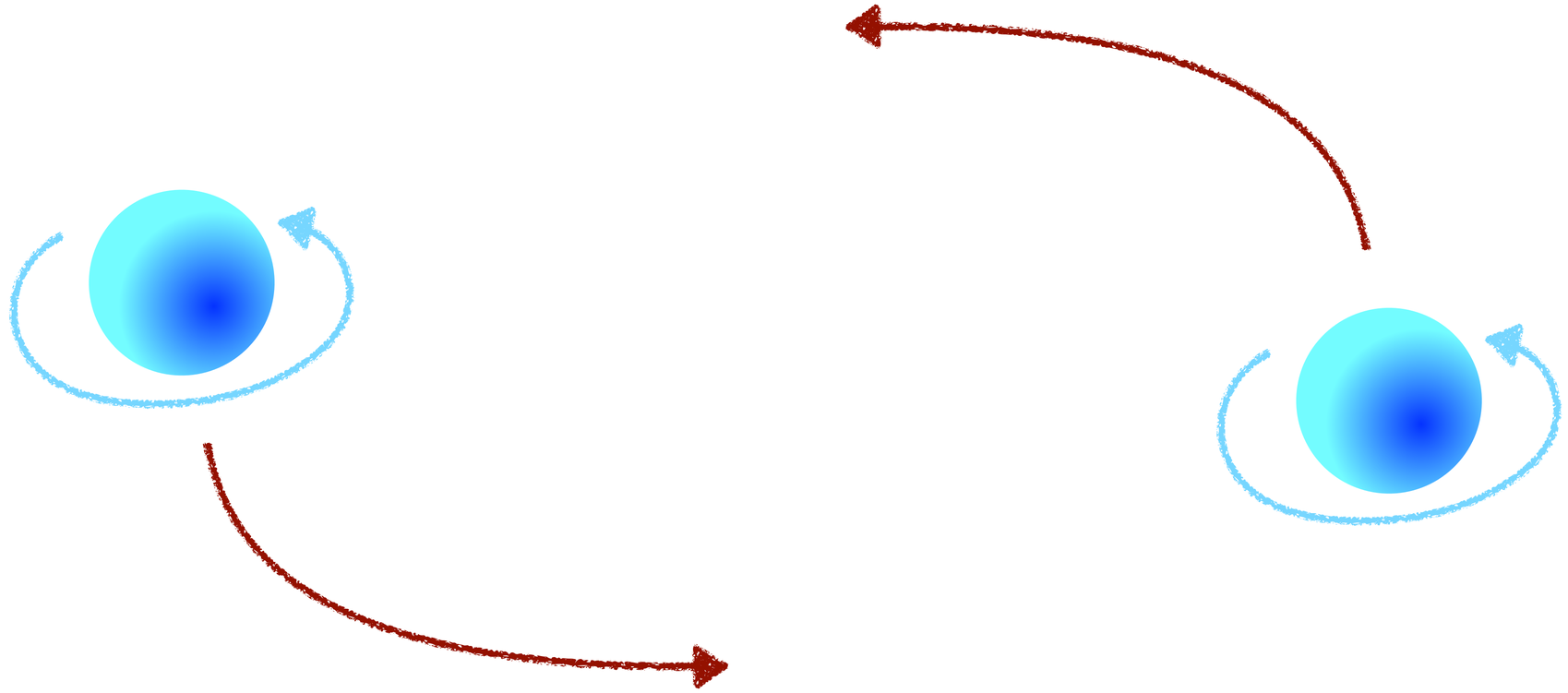
Joon-Hwi Kim[a] **Jung-Wook Kim**[b] **Sangmin Lee**[a,c,d]

2021.02. B.S. SNU
2021.09~, Ph.D. student, Caltech

2020.08. Ph.D. SNU
2020.10~, Queen Mary University of London

Motivation

From QFT scattering amplitudes
to perturbative general relativity



binary system as a source for gravitational wave radiation

Outline

Relativistic Spherical Top

equivalence

Massive Twistor

work in progress:
coupling to background fields

Relativistic Spherical Top

[Hanson,Regge 1974]

"Spherical top" in special relativity

$$S = \int dt \left(\frac{1}{2} m \vec{v}^2 \right) \xrightarrow{\text{"spin"}} S = \int dt \left(\frac{1}{2} m \vec{v}^2 + \frac{1}{2} I \vec{\omega}^2 \right)$$

(1+3)-dim space-time

$$S = - \int ds \sqrt{-\eta_{ab} \dot{x}^a \dot{x}^b} \longrightarrow \boxed{S = (?)}$$

[Frenkel 1926][Thomas 1926]
... [Hanson,Regge 1974]

3 translation d.o.f. + 3 rotation d.o.f.

$$\{x^\mu, p_\nu\} = \delta^\mu{}_\nu \,,$$

$$\{\Lambda^\rho{}_A, S_{\mu\nu}\} = -(\delta^\rho{}_\mu \Lambda_{\nu A} - \delta^\rho{}_\nu \Lambda_{\mu A}) \,,$$

$$\{S_{\mu\nu}, S_{\rho\sigma}\} = -(\eta_{\nu\rho} S_{\mu\sigma} - \eta_{\mu\rho} S_{\nu\sigma} - \eta_{\nu\sigma} S_{\mu\rho} + \eta_{\mu\sigma} S_{\nu\rho}) \,.$$

$$x, p \in \mathbb{R}^{1,3}$$
$$\Lambda \in \mathrm{SO}(1,3)$$
$$S \in \mathfrak{so}(1,3)$$

Translation: $4 \rightarrow 3$ $\quad p^2 + m^2 = 0$ "mass shell constraint"

Rotation: $6 \rightarrow 3$ $\quad (?)$ "spin constraints"

Spin constraints

Intuition : remove the "boosts" from $\Lambda^{\mu}{}_{A}$, leave rotations only.

Problem : boosts do not form a closed subalgebra.

Spin Complementary Conditions (SSCs)

$$\Lambda^{\mu}{}_{0} S_{\mu\nu} = 0 \qquad \text{(temporal)}$$

$$\hat{p}^{\mu} S_{\mu\nu} = 0 \qquad \text{(covariant)}$$

$$(\hat{p}^{\mu} + \Lambda^{\mu}{}_{0}) S_{\mu\nu} = 0 \qquad \text{(Pryce-Newton-Wigner)}$$

[Pryce 1948][Newton,Wigner 1949]

Our proposal

(mass shell) (modified PNW) [Steinhoff 2015]

$$\phi_0 = \frac{1}{2}(p^2 + m^2)\,, \qquad \phi_a = \frac{1}{2}(\hat{p}^\mu + \Lambda^\mu{}_0)S_{\mu\nu}\Lambda^\nu{}_a\,,$$

$$\chi^0 = \frac{1}{p^2}x^\mu p_\mu\,, \qquad \chi^a = \hat{p}_\mu \Lambda^{\mu a}\,.$$

Canonically conjugate pairs of ...

gauge generators : ϕ_A

gauge-fixing conditions : χ^A

$$\{\chi^A, \phi_B\} \approx \delta^A{}_B\,, \quad \{\phi_A, \phi_B\} \approx 0 \approx \{\chi^A, \chi^B\}$$

Equations of motion

Lagrange multipliers

$$\mathcal{S} = \int d\sigma \left(p_\mu \dot{x}^\mu + \frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu} - \kappa \phi_0 - \kappa^a \phi_a \right)$$

$$= \int d\sigma \left(p_\mu \dot{x}^\mu + \frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu} - \frac{\kappa}{2} (p^2 + m^2) - \frac{\kappa^a}{2} (\hat{p}^\mu + \Lambda^\mu{}_0) S_{\mu\nu} \Lambda^\nu{}_a \right) .$$

Regge trajectory

$$\frac{dp_\mu}{d\tau} = 0 , \quad p^\mu = m \frac{dx^\mu}{d\tau} ,$$

$$\frac{dS_{\mu\nu}}{d\tau} = 0 , \quad S^{\mu\nu} = \frac{1}{2m'} \bar{\Omega}^{\mu\nu} .$$

$$\left(\bar{\Omega}^{\mu\nu} = \Lambda^{\mu A} \frac{d\Lambda^\nu{}_A}{d\tau} \right)$$

Physical phase space

(redundant phase space)		(constraints)		(physical phase space)
(x, p, Λ, S)	$\longrightarrow$	(ϕ, χ)	$\longrightarrow$	$(???)$
4 + 4 + 6 + 6 = 20		1 + 1 + 3 + 3 = 8		3 + 3 + 3 + 3 = 12

Two approaches

(1) Dirac bracket

(2) Explicit embedding

Dirac bracket

$$\{f,g\}_* = \{f,g\} - \{f,\phi_A\}\{\chi^A,g\} + \{f,\chi^A\}\{\phi_A,g\}$$

$$\begin{aligned}
\{x^\mu, x^\nu\}_* &= \frac{1}{|p|^2} J^{\mu\nu} \,, \\
\{x^\mu, p_\nu\}_* &= \hat{\delta}^\mu{}_\nu \,, \\
\{x^\mu, \Lambda_\nu{}^a\}_* &= \frac{1}{|p|}\left(\hat{p}_\nu \Lambda^{\mu a} - 2m' x^\mu \tilde{S}_{\nu\rho}\Lambda^{\rho a}\right) \,, \\
\{p_\mu, \Lambda_\nu{}^a\}_* &= 2m' \hat{p}_\mu \tilde{S}_{\nu\rho}\Lambda^{\rho a} \,, \\
\{x^\rho, S_{\mu\nu}\}_* &= -\frac{1}{|p|}\left(\hat{p}_\mu S_\nu{}^\rho - \hat{p}_\nu S_\mu{}^\rho\right) \,, \\
\{\Lambda_\rho{}^a, S_{\mu\nu}\}_* &= -\left(\hat{\eta}_{\rho\mu}\Lambda_\nu{}^a - \hat{\eta}_{\rho\nu}\Lambda_\mu{}^a\right) \,, \\
\{S_{\mu\nu}, S_{\rho\sigma}\}_* &= -\left(\hat{\eta}_{\nu\rho}S_{\mu\sigma} - \hat{\eta}_{\mu\rho}S_{\nu\sigma} + \hat{\eta}_{\nu\sigma}S_{\rho\mu} - \hat{\eta}_{\mu\sigma}S_{\rho\nu}\right) \,.
\end{aligned}$$

The only dynamical information **not** governed by symmetry:

"Regge trajectory"

$$p^2 + m^2(\tilde{S}^2) = 0$$

[Hanson,Regge 1974]

Embedding

Goal : find good coordinates for the physical phase space.

Hint :

$$\{\Lambda_\mu{}^A, \Lambda_\nu{}^B\}_* = 0\,,$$
$$\{\Lambda_\rho{}^A, J_{\mu\nu}\}_* = -(\eta_{\rho\mu}\Lambda_\nu{}^A - \eta_{\rho\nu}\Lambda_\mu{}^A)\,,$$
$$\{J_{\mu\nu}, J_{\rho\sigma}\}_* = -(\eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\rho\mu} - \eta_{\mu\sigma}J_{\rho\nu})\,.$$

Λ

momentum direction : $\Lambda^\mu{}_0 = \hat{p}^\mu$

transverse rotation : $\Lambda^\mu{}_a$

Embedding

(normalized) Pauli-Lubanski

J

$$V_\mu = J_{\mu\nu}\hat{p}^\nu\,, \quad W_\mu = -(*J)_{\mu\nu}\hat{p}^\nu$$

$$V_A = V_\mu \Lambda^\mu{}_A\,, \quad W_A = W_\mu \Lambda^\mu{}_A \qquad (V_0 = 0 = W_0)$$

$$J_{a0} = L_{a0} = V_a = -m x_a\,, \quad J_{ab} = S_{ab} = \varepsilon^c{}_{ab} W_c$$

$\Lambda_\mu{}^a$: (transverse rotation) , $\quad J_{ab} = S_{ab}$: (spin angular momentum) ,

$\Lambda_\mu{}^0$: (momentum direction) , $\quad J_{0a} = L_{0a}$: (transverse position) .

Massive Twistor

[Shirafuji 1983] → massless

[Fedoruk,Lukierski 2014] → massive

Massive twistor

Twistor coordinates :

$(8_{\mathbb{C}} = 16_{\mathbb{R}})$

$$Z_{\mathrm{A}}{}^{I} = \begin{pmatrix} \lambda_{\alpha}{}^{I} \\ i\mu^{\dot{\alpha}I} \end{pmatrix} , \quad \bar{Z}_{I}{}^{\mathrm{B}} = (Z_{\mathrm{A}}{}^{I})^{\dagger} A^{\bar{\mathrm{A}}\mathrm{B}} = \begin{pmatrix} -i\bar{\mu}_{I}{}^{\beta} & \bar{\lambda}_{I\dot{\beta}} \end{pmatrix} .$$

conformal symmetry
to be broken by
the mass shell condition

$$\boxed{\mathrm{SU}(2,2) \times \mathrm{SU}(2)}$$

interpreted as the
"little group" symmetry
of a massive particle

Poisson bracket algebra :

$$\{Z_{\mathrm{A}}{}^{I}, \bar{Z}_{J}{}^{\mathrm{B}}\} = -i\,\delta_{\mathrm{A}}{}^{\mathrm{B}}\delta^{I}{}_{J} \implies \{\lambda_{\alpha}{}^{I}, \bar{\mu}_{J}{}^{\beta}\} = \delta_{\alpha}{}^{\beta}\delta^{I}{}_{J} , \quad \{\bar{\lambda}_{I\dot{\alpha}}, \mu^{\dot{\beta}J}\} = \delta_{\dot{\alpha}}{}^{\dot{\beta}}\delta_{I}{}^{J} .$$

Constraints

cf) $\phi_0 = \frac{1}{2}(p^2 + m^2)\,, \quad \chi^0 = \frac{1}{p^2} x^\mu p_\mu\,.$

Reduction from 16 to 12.

Spin constraints are **not** needed.

Mass-shell constraint should be "complexified".

$$\phi = -\frac{1}{2}\left(\det(\lambda) - m(\tilde{S}^2)\right)\,, \quad \bar{\chi} = \frac{1}{\det(\lambda)}\langle\bar{\mu}\lambda\rangle\,,$$

$$\bar{\phi} = -\frac{1}{2}\left(\det(\bar{\lambda}) - m(\tilde{S}^2)\right)\,, \quad \chi = \frac{1}{\det(\bar{\lambda})}[\bar{\lambda}\mu]\,.$$

Twistor model "solves" the spin constraints.

Map to the spherical top

$$\boxed{p_\mu \qquad \Lambda^\mu{}_A}$$

$\Longleftrightarrow$ bi-linear in $(\lambda, \bar{\lambda})$

$\updownarrow$ conj $\updownarrow$

$x^\mu \qquad S_{\mu\nu}$

$\Longleftrightarrow$ function of $(\lambda, \bar{\lambda}, \mu, \bar{\mu})$

Map to the spherical top

$$p_{\alpha\dot\alpha} = (\sigma^\mu)_{\alpha\dot\alpha} p_\mu = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 - ip_2 & p_0 - p_3 \end{pmatrix}$$

The map (assuming constraints)

$$p_{\alpha\dot\alpha} = -\lambda_\alpha{}^I \bar\lambda_{I\dot\alpha}\,, \quad x^{\dot\alpha\alpha} = \frac{1}{2}\left(\frac{1}{\det(\lambda)}\mu^{\dot\alpha I}\lambda^{\alpha J}\epsilon_{IJ} + \frac{1}{\det(\bar\lambda)}\epsilon^{IJ}\bar\lambda_I{}^{\dot\alpha}\bar\mu_J{}^\alpha\right)\,,$$

$$(\Lambda_{\alpha\dot\alpha})^I{}_J = \frac{2\lambda_\alpha{}^I\bar\lambda_{J\dot\alpha}}{|\det(\lambda)|}\,, \quad S_{\alpha\beta} = \frac{1}{2}\left(\bar\mu_{I(\alpha}\lambda_{\beta)}{}^I - \frac{1}{\det(\lambda)}\lambda_{(\alpha}{}^I\lambda_{\beta)}{}^J(\mu^{\dot\gamma}{}_I\bar\lambda_{J\dot\gamma})\right)\,.$$

Hints

$$\mu^{\dot\alpha I} = z^{\dot\alpha\beta}\lambda_\beta{}^I\,, \quad z^\mu = x^\mu + iy^\mu\,.$$

incidence relation
complex Minkowski space

$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}\,, \quad S_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}(y_\rho p_\sigma - y_\sigma p_\rho)\,.$$

[Penrose,MacCallum 1972][Newman-Winicour 1974]

Embedding

$$\bar{J}_{\dot{\alpha}\dot{\beta}} = i\,\frac{1}{\det(\bar{\lambda})}\bar{\lambda}^{I}{}_{\dot{\alpha}}\bar{\lambda}^{J}{}_{\dot{\beta}}J_{(IJ)}\,, \quad J_{\alpha\beta} = i\,\frac{1}{\det(\lambda)}\lambda_{\alpha}{}^{I}\lambda_{\beta}{}^{J}\bar{J}_{(IJ)}\,,$$

$$2iJ^{I}{}_{J} = -2\mu^{\dot{\alpha}I}\bar{\lambda}_{J\dot{\alpha}} = V^{I}{}_{J} + iW^{I}{}_{J}\,,$$
$$-2i\bar{J}^{J}{}_{I} = -2\bar{\mu}_{I}{}^{\alpha}\lambda_{\alpha}{}^{J} = V^{I}{}_{J} - iW^{J}{}_{I}\,.$$

$$V_{(IJ)} = -\left(\mu^{\dot{\alpha}}{}_{(I}\bar{\lambda}_{J)\dot{\alpha}} + \lambda_{\alpha(I}\bar{\mu}_{J)}{}^{\alpha}\right), \quad V_0 = \frac{1}{2}\left([\bar{\lambda}\mu] + \langle\bar{\mu}\lambda\rangle\right) = D\,,$$
$$W_{(IJ)} = i\left(\mu^{\dot{\alpha}}{}_{(I}\bar{\lambda}_{J)\dot{\alpha}} - \lambda_{\alpha(I}\bar{\mu}_{J)}{}^{\alpha}\right), \quad W_0 = \frac{1}{2i}\left([\bar{\lambda}\mu] - \langle\bar{\mu}\lambda\rangle\right)\,.$$

$$L_{\alpha\beta} = -\frac{1}{2\det(\lambda)}\lambda_{\alpha}{}^{I}\lambda_{\beta}{}^{J}V_{(IJ)}\,, \quad S_{\alpha\beta} = i\frac{1}{2\det(\lambda)}\lambda_{\alpha}{}^{I}\lambda_{\beta}{}^{J}W_{(IJ)}\,.$$

Dirac bracket

$$\{f,g\}_* = \{f,g\} - \left(\{f,\phi\}\{\bar{\chi},g\} - \{f,\bar{\chi}\}\{\phi,g\}\right) - \left(\{f,\bar{\phi}\}\{\chi,g\} - \{f,\chi\}\{\bar{\phi},g\}\right).$$

$$\{\lambda_\alpha{}^I, \lambda_\beta{}^J\}_* = -\frac{m'}{2im}\left[\left(\lambda_\alpha{}^K W_K{}^I\right)\lambda_\beta{}^J - \lambda_\alpha{}^I\left(\lambda_\beta{}^K W_K{}^J\right)\right],$$

$$\{\lambda_\alpha{}^I, \bar{\lambda}_{J\dot{\beta}}\}_* = -\frac{m'}{2im}\left[\left(\lambda_\alpha{}^K (W_K{}^I)\bar{\lambda}_{J\dot{\beta}} + \lambda_\alpha{}^I\left(W_J{}^K\bar{\lambda}_{K\dot{\beta}}\right)\right]\right.,$$

$$\{\lambda_\alpha{}^I, \bar{\mu}_J{}^\beta\}_* = \left(\delta_\alpha{}^\beta\delta^I{}_J + \frac{1}{2m}\lambda_\alpha{}^I\lambda^\beta{}_J\right) + \frac{m'}{2im}\left[\left(\lambda_\alpha{}^K W_K{}^I\right)\bar{\mu}_J{}^\beta - \lambda_\alpha{}^I\left(W_J{}^K\bar{\mu}_K{}^\beta\right)\right]$$

$$\{\lambda_\alpha{}^I, \mu^{\dot{\beta}J}\}_* = \frac{m'}{2im}\left[\left(\lambda_\alpha{}^K W_K{}^I\right)\mu^{\dot{\beta}J} + \lambda_\alpha{}^I\left(\mu^{\dot{\beta}K}W_K{}^J\right)\right],$$

$$\{\mu^{\dot{\alpha}I}, \mu^{\dot{\beta}J}\}_* = \frac{1}{2m}\left(\mu^{\dot{\alpha}I}\bar{\lambda}^{J\dot{\beta}} - \mu^{\dot{\beta}J}\bar{\lambda}^{I\dot{\alpha}}\right) + \frac{m'}{2im}\left[\left(\mu^{\dot{\alpha}K}W_K{}^I\right)\mu^{\dot{\beta}J} - \mu^{\dot{\alpha}I}\left(\mu^{\dot{\beta}K}W_K{}^I\right)\right]$$

$$\{\mu^{\dot{\alpha}I}, \bar{\mu}_J{}^\beta\}_* = \frac{m'}{2im}\left[\left(\mu^{\dot{\alpha}K}W_K{}^I\right)\bar{\mu}_J{}^\beta + \mu^{\dot{\alpha}I}\left(W_J{}^K\bar{\mu}_K{}^\beta\right)\right],$$

Equations of motion

$$\phi = -\frac{1}{2}\left(\det(\lambda) - m(\tilde{S}^2)\right) ,$$
$$\bar{\phi} = -\frac{1}{2}\left(\det(\bar{\lambda}) - m(\tilde{S}^2)\right) .$$

$$\mathcal{S} = \int d\sigma \left(\frac{i}{2}\left(\bar{Z}_I{}^{\mathrm{A}} \dot{Z}_{\mathrm{A}}{}^I - Z_{\mathrm{A}}{}^I \dot{\bar{Z}}_I{}^{\mathrm{A}}\right) - \bar{\kappa}\phi - \kappa\bar{\phi} \right)$$

$$\Rightarrow \quad \begin{aligned} \dot{\lambda}_\alpha{}^I &= -i(\mathrm{Re}\,\kappa) m' \lambda_\alpha{}^J W_J{}^I , \\ \dot{\mu}^{\dot{\alpha} I} &= -i(\mathrm{Re}\,\kappa) m' \mu^{\dot{\alpha} J} W_J{}^I + \frac{\kappa}{2}\bar{\lambda}^{\dot{\alpha} I} . \end{aligned}$$

Perfect agreement with the spherical top model !

Quantization

"Quantize first, and then impose constraints" $\bar{\mu}_I{}^{\alpha} = -i\frac{\partial}{\partial\lambda_{\alpha}{}^{I}}\,, \quad \mu^{\dot{\alpha}I} = -i\frac{\partial}{\partial\bar{\lambda}_{I\dot{\alpha}}}\,.$

On-shell fields and Fierz-Pauli/Bargmann-Wigner equation:

$$\Phi^{(2s,0)}_{\alpha_1\alpha_2\ldots\alpha_{2s}}(x) = c_{I_1\ldots I_{2s}}\int d[\lambda,\bar{\lambda}]\,\lambda^{I_1}_{\alpha_1}\lambda^{I_2}_{\alpha_2}\cdots\lambda^{I_{2s}}_{\alpha_{2s}}\,\exp\left(ix^{\dot{\beta}\beta}\bar{\lambda}_{\dot{\beta}J}\lambda^{J}_{\beta}\right)\,,$$

$$\Phi^{(2s-1,1)}_{\dot{\alpha}_1|\alpha_2\ldots\alpha_{2s}}(x) = c_{I_1\ldots I_{2s}}\int d[\lambda,\bar{\lambda}]\,\bar{\lambda}^{I_1}_{\dot{\alpha}_1}\lambda^{I_2}_{\alpha_2}\cdots\lambda^{I_{2s}}_{\alpha_{2s}}\,\exp\left(ix^{\dot{\beta}\beta}\bar{\lambda}_{\dot{\beta}J}\lambda^{J}_{\beta}\right)\,,$$

$$\vdots \qquad\qquad \vdots$$

$$\Phi^{(0,2s)}_{\dot{\alpha}_1\dot{\alpha}_2\ldots\dot{\alpha}_{2s}}(x) = c_{I_1\ldots I_{2s}}\int d[\lambda,\bar{\lambda}]\,\bar{\lambda}^{I_1}_{\dot{\alpha}_1}\bar{\lambda}^{I_2}_{\dot{\alpha}_2}\cdots\bar{\lambda}^{I_{2s}}_{\dot{\alpha}_{2s}}\,\exp\left(ix^{\dot{\beta}\beta}\bar{\lambda}_{\dot{\beta}J}\lambda^{J}_{\beta}\right)\,,$$

where $c_{I_1\ldots I_{2s}}$ is a totally symmetrized SU(2) tensor, and the integration measure is

$$d[\lambda,\bar{\lambda}] = d^4\lambda d^4\bar{\lambda}\left[\delta(\det(\lambda)-m)\delta(\det(\bar{\lambda})-m)\right]\,.$$

"massive spinor helicity variables" [Arkani-Hamed,Huang,Huang 2017]

Towards an interacting theory

Bargmann-Michel-Telegdi (BMT) equation

The BMT equation [Bargmann, Michel, Telegdi 1959]

$$\frac{dy_\mu}{d\tau} = \frac{q}{2m}\left(gF_{\mu\nu}y^\nu + (g-2)(y^\rho F_{\rho\sigma}u^\sigma)u_\mu\right) , \quad u^\mu = \frac{dx^\mu}{d\tau}$$

When $g = 2$, the equation take the same form as

$$\frac{du_\mu}{d\tau} = \frac{q}{m}F_{\mu\nu}u^\nu$$

Mathisson-Papapetrou-Dixon (MPD) equation

The MPD equation [Mathisson 1937][Papapetrou 1951][Dixon 1970]

$$\frac{Dp^\mu}{d\tau} + \frac{1}{2} R^\mu{}_{\nu\rho\sigma} S^{\rho\sigma} u^\nu = 0$$

$$\frac{DS_{\mu\nu}}{d\tau} + u_\mu p_\nu - u_\nu p_\mu = 0$$

$$S_{\mu\nu} p^\nu = 0$$

A hybrid viewpoint

[Shirafuji 1983]

Earlier in this talk, while discussing $(x, p, \Lambda, S) \leftrightarrow (\lambda, \bar{\lambda}, \mu, \bar{\mu})$,

we encountered the "incidence relation":

$$\mu^{\dot{\alpha} I} = z^{\dot{\alpha}\beta} \lambda_{\beta}{}^{I}\,, \quad z^{\mu} = x^{\mu} + i y^{\mu}\,.$$

Now, we switch the viewpoint

and take $(z, \bar{z}, \lambda, \bar{\lambda})$ as independent variables.

A hybrid viewpoint

$$e^{\dot\alpha\alpha} \leftrightarrow e^m{}_\mu(x)dx^\mu$$

$$D\lambda_\alpha{}^I = d\lambda_\alpha{}^I + \omega_\alpha{}^\beta \lambda_\beta{}^I\,,$$
$$D\bar\lambda_{I\dot\alpha} = d\bar\lambda_{I\dot\alpha} + \bar\omega_{\dot\alpha}{}^{\dot\beta}\bar\lambda_{I\dot\beta}\,.$$

Free theory revisited :

$$\theta = (-\lambda_\alpha{}^I\bar\lambda_{I\dot\alpha})dx^{\dot\alpha\alpha} - iy^{\dot\alpha\alpha}(d\lambda_\alpha{}^I\bar\lambda_{I\dot\alpha} - \lambda_\alpha{}^I d\bar\lambda_{I\dot\alpha})$$

$$\omega = -(d\lambda_\alpha{}^I\bar\lambda_{I\dot\alpha})\wedge d\bar z^{\dot\alpha\alpha} - (\lambda_\alpha{}^I d\bar\lambda_{I\dot\alpha})\wedge dz^{\dot\alpha\alpha} + 2iy^{\dot\alpha\alpha}(d\lambda_\alpha{}^I\wedge d\bar\lambda_{I\dot\alpha})$$

A "minimal coupling" to gravity?

$$\varepsilon^m{}_\mu = e^m{}_\mu dx^\mu + idy^m$$
$$\bar\varepsilon^m{}_\mu = e^m{}_\mu dx^\mu - idy^m$$

$$\theta = -\lambda_\alpha{}^I\bar\lambda_{I\dot\alpha}\underline{e^{\dot\alpha\alpha}} - iy^{\dot\alpha\alpha}(\underline{D\lambda_\alpha}{}^I\bar\lambda_{I\dot\alpha} - \lambda_\alpha{}^I\underline{D\bar\lambda}_{I\dot\alpha})$$

$$\omega = -(D\lambda_\alpha{}^I\bar\lambda_{I\dot\alpha})\wedge\underline{\bar\varepsilon^{\dot\alpha\alpha}} - (\lambda_\alpha{}^I D\bar\lambda_{I\dot\alpha})\wedge\underline{\varepsilon^{\dot\alpha\alpha}} + 2iy^{\dot\alpha\alpha}(D\lambda_\alpha{}^I\wedge D\bar\lambda_{I\dot\alpha})$$

Newman-Janis shift

[Newman, Janis 1965]

Complex Minkowski spacetime with the"spin-length" vector :

$$x^\mu \quad \rightarrow \quad x^\mu \pm i y^\mu \,, \quad y_\mu = \frac{1}{m^2} W_\mu = \frac{1}{m^2} \varepsilon_{\mu\nu\rho\sigma} p^\nu S^{\rho\sigma}$$

Solution generating technique:

Schwarzschild $\longrightarrow$ Kerr

Reissner-Nordström $\longrightarrow$ Kerr-Newman

Kerr BH as a "minimally coupled" particle

"$g = 2$ as the natural value of the tree-level gyromagnetic ratio"

[Ferrara,Porrati,Telegdi, 92][Holstein 06]

Kerr BH solution to all orders in spin [Vines, 1709.06016]

$$\bar{h}^{\mu\nu} = \left(u^\mu u^\nu \cos(y \cdot \partial) + u^{(\mu} \epsilon^{\nu)}{}_{\rho\sigma\gamma} u^\rho y^\sigma \partial^\gamma \frac{\sin(y \cdot \partial)}{(a \cdot \partial)} \right) \frac{4GM}{r}$$

On-shell minimal coupling [Arkani-Hamed,Huang,Huang, 17] [Chung, Huang, Kim, SL, 18]

$$\mathcal{A}_3^{+h} = \kappa_h m x^h \left(\frac{\langle \mathbf{12} \rangle^{2S}}{m^{2S}} + c_1 x \frac{\langle \mathbf{12} \rangle^{2S-1} \langle \mathbf{13} \rangle \langle \mathbf{32} \rangle}{m^{2S+1}} + \cdots \right).$$

$$c_1 = 0 \iff g = 2$$

"Kerr particle"

[Guevara, Maybee, O'Connell, Ochirov, Vines 2012.11570]

NJ shift for an EM coupling of a massive particle : "$\sqrt{\text{Kerr}}$"

$$L_{\text{int}} = qA_\mu(x)\dot{x}^\mu \quad \rightarrow \quad (L_{\text{int}})^{\text{NJ}} = \frac{q}{2}(A^+_\mu(z)\dot{z}^\mu + A^-_\mu(\bar{z})\dot{\bar{z}}^\mu)$$

$$A^\pm_\mu(x) = A_\mu(x) \pm \tilde{A}_\mu(x)\,, \quad \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu = (*F)_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

EOM for the spin-length vector y : BMT equation with $g = 2$ (!)

Similarly, for the Kerr particle" in gravity,

$$\bar{\omega}^{\dot{\alpha}}{}_{\dot{\beta}}(x) \rightarrow \bar{\omega}^{\dot{\alpha}}{}_{\dot{\beta}}(z)$$

$$\omega^\alpha{}_\beta(x) \rightarrow \omega^\alpha{}_\beta(\bar{z})$$

Summary & Outlook

Summary and Outlook

- Optimized the constraints of the spherical top model.
- Incorporated the Regge trajectory in the massive twistor model.
- Established the equivalence (up to double cover) :

$$\text{spherical top} \cong \text{massive twistor}$$

- Interaction with EM field and gravity to be turned on (NJ shift?)

- Applications are likely to be found in
  ... post-Minkowskian gravity
  ... scattering amplitudes of QFT