Hamiltonian reduction of Einstein's equations in (2+2) formalism without isometry assumptions

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Outline

References

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What is Hamiltonian reduction? (P.A.M. Dirac)

- A singular Hamiltonian system

$$S = \int \sum_{I=1}^{n+1} p_I dq^I$$

with a constraint

$$C(p_I, q^I) := p_{n+1} + H(p_i, q^i) = 0 \quad (i = 1, \cdots, n)$$

becomes a regular Hamiltonian system if we solve the constraint

$$p_{n+1} = -H(p_i, q^i)$$

and identify time t as $q^{n+1} = t$:

$$S = \int \sum_{i=1}^{n} p_i dq^i - Hdt$$

- General relavity is a singular Hamiltonian system

$$S = \int d^4 x \sum \ p_{ij} \frac{\partial g^{ij}}{\partial t}$$

subject to 4 constraints

$$C(p_{ij}, g^{ij}) = 0, \quad C_i(p_{ij}, g^{ij}) = 0$$

- Hamiltonian reduction (Arnowitt, Deser, Misner):
Identify suitables functions on the gravitational phase space (p_{ij}, g^{ij}) as spacetime coordinates such that 4 constraints are solved to define non-trivial Hamiltonian and momentum.
- GR as a standard Hamiltonian system with non-zero true Hamiltonian $H_{\rm true}$, with the constraints associated with spacetime diffeomorphism removed
- A sure road to Quantum Gravity: Schrodinger quantization

$$i \frac{\partial \Psi}{\partial \tau} = \hat{H}_{\rm true} \ \Psi$$

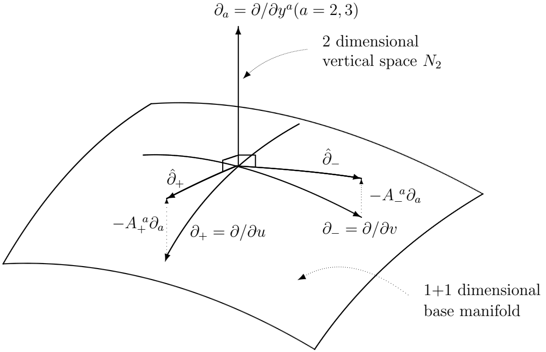
- Partial success:
  ADM: Successul isolation of (p_{TT}, g^{TT}) in asymptotically flat zone
  K. Kuchar: Complete analysis for spacetimes with 2 Killing symmetries
  V. Moncrief and A. Fischer: Beyond 2 Killing symmetries, but not very successful

2+2 fibration of 4-dimensional spacetime

- Kinematics of 2+2 fibration

$$ds^2 = 2dudv - 2hdu^2 + \tau\rho_{ab}\left(dy^a + A_+{}^a du + A_-{}^a dv\right)\left(dy^b + A_+{}^b du + A_-{}^b dv\right)$$

cf. $A_-{}^a = 0 \Rightarrow$ Newman-Unti coordinates where v is an affine parameter for out-going nulls



2+2 fibration of 4-dimensional spacetime

- Horizontal basis $\hat{\partial}_{\pm} = \partial_{\pm} - A_{\pm}{}^{a}\partial_{a}$ ($\partial_{+} = \partial_{u}, \partial_{-} = \partial_{v}$)
- Vertical basis ∂_{a} ($a = 2, 3$) for a (compact) spacelike 2-surface N_2
- Metric on Σ_3 defined by $v =$ constant is

$$ds^2|_{\Sigma_3} = -2hdu^2 + \tau\rho_{ab}\left(dy^a + A_{+}{}^{a}du\right)\left(dy^b + A_{+}{}^{b}du\right)$$

  $\Rightarrow$ Choose Σ_3 spacelike ($h < 0$)
  cf. Timelike ($h > 0$) or null ($h = 0$) cases are also possible
- $\det \rho_{ab} = 1 \Rightarrow \rho_{ab}$ is a conformal 2-metric
- τ is the area element of N_2 (defined by $u, v =$ constant)

Action integral in the first-order form in 2+2 formalism

- The action integral in the first-order form is given by

$$S = \int dudvd^2y \left\{ \pi_\tau \partial_- \tau + \pi_h \partial_- h + \pi_a \partial_- A_+^{\ a} + \pi^{ab} \partial_- \rho_{ab} - C \right\}$$

$$C := \text{``1''} \cdot C_- + \text{``0''} \cdot C_+ + A_-^{\ a} C_a = 0,$$

- "1", "0", $A_-^{\ a}$ are Lagrange multipliers enforcing the 4 Einstein constraints $C_- = C_+ = C_a = 0$.
- $\{C_\pm, C_a\}$ satisfy the first-class algebra (JHY, CQG **31** (2014) 045005)
$\Longleftrightarrow$ residual symmetry of Newman-Unti metric

Classification of fields by diffN_2 transformation properties

- diffN_2-covariant derivative $D_\pm f_{ab\cdots}$ of tensor density with weight w is defined as

$$D_\pm f_{ab\cdots} = \partial_\pm f_{ab\cdots} - [A_\pm,\ f]_{ab\cdots},$$

- Lie derivative $f_{ab\cdots}$ of along $A_\pm^{\ c}\partial_c$ is given by

$$[A_\pm,\ f]_{ab\cdots} := A_\pm^{\ c}\partial_c f_{ab\cdots} + f_{cb\cdots}\partial_a A_\pm^{\ c} + f_{ac\cdots}\partial_b A_\pm^{\ c} \cdots + w(\partial_c A_\pm^{\ c}) f_{ab\cdots}.$$

- Classification into scalar, scalar density, tensor density, field strength:

$$\begin{aligned}
&D_\pm h = \partial_\pm h - A_\pm^{\ a}\partial_a h,\\
&D_\pm \tau = \partial_\pm \tau - A_\pm^{\ a}\partial_a \tau - (\partial_a A_\pm^{\ a})\tau,\\
&D_\pm \rho_{ab} = \partial_\pm \rho_{ab} - A_\pm^{\ c}\partial_c \rho_{ab} - \rho_{cb}\partial_a A_\pm^{\ c} - \rho_{ac}\partial_b A_\pm^{\ c} + (\partial_c A_\pm^{\ c})\rho_{ab},\\
&F_{+-}^{\ \ a} = \partial_+ A_-^{\ a} - \partial_- A_+^{\ a} - A_+^{\ b}\partial_b A_-^{\ a} + A_-^{\ b}\partial_b A_+^{\ a}.
\end{aligned}$$

Einstein constraints

- Let $R^{(2)}$ be the scalar curvature of N_2
- Conjugate momenta $\pi_I = \{\pi_\tau, \pi_h, \pi_a, \pi^{ab}\}$ of $q^I = \{\tau, h, A_+^{\ a}, \rho_{ab}\}$ are defined w.r.t. the v time
- The 4 Einstein's constraints on Σ_3 (defined by $v =$ constant)

$$
\begin{aligned}
C_- &:= \frac{1}{2}\pi_h\pi_\tau - \frac{h}{4\tau}\pi_h^2 - \frac{1}{2\tau}\pi_h D_+\tau + \frac{1}{2\tau^2}\rho^{ab}\pi_a\pi_b - \frac{\tau}{8h}\rho^{ab}\rho^{cd}(D_+\rho_{ac})(D_+\rho_{bd}) \\
&\quad - \frac{1}{2h\tau}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} - \frac{1}{2h}\pi^{ac}D_+\rho_{ac} - \tau R^{(2)} + D_+\pi_h - \partial_a(\tau^{-1}\rho^{ab}\pi_b) = 0, \\
C_+ &:= \pi_\tau D_+\tau + \pi_h D_+ h + \pi^{ab}D_+\rho_{ab} - 2D_+(h\pi_h + D_+\tau) \\
&\quad + 2\partial_a(h\tau^{-1}\rho^{ab}\pi_b + \rho^{ab}\partial_b h) = 0, \\
C_a &:= \pi_\tau\partial_a\tau + \pi_h\partial_a h + \pi^{bc}\partial_a\rho_{bc} - 2\partial_b(\rho_{ac}\pi^{bc}) - D_+\pi_a - \partial_a(\tau\pi_\tau) = 0
\end{aligned}
$$

form the first class algebra.

Hamilton's equations before Hamiltonian reduction

- Dynamical variables are 6 configuration variables $q^I = \{h, \tau, A_+{}^a, \rho_{ab}\}$ and 6 conjugate momenta $\pi_I = \{\pi_h, \pi_\tau, \pi_a, \pi^{ab}\}$
- Hamilton's equations of motion w.r.t. v time are given by

$$\dot{q}^I = \int dud^2y \frac{\delta C}{\delta \pi_I},$$
$$\dot{\pi}_I = -\int dud^2y \frac{\delta C}{\delta q^I} \quad (\dot{} = \partial_v),$$

where C is the sum of Einstein's constraints

$$C = \text{“1”} \cdot C_- + \text{“0”} \cdot C_+ + A_-{}^a C_a = 0$$

Canonical transformation

- Introduce new variables (R, π_R) such that

$$\partial_+ R := -h\pi_h, \quad \pi_R := -\partial_+ \ln(-h) \tag{1}$$

- It is a canonical transformation of (h, π_h) because

$$\int dvdud^2y\pi_h\dot{h} = \int dvdud^2y\pi_R\dot{R} + \text{surface terms.}$$

- Impose 3 momentum constraints $C_+ = C_a = 0$ and choose the multiplier $A\_^a = 0$

Area element τ as the privileged time coordinate

- $\tau = \tau(v, u, y^a) \Rightarrow v = v(\tau, u, y^a)$ (assume invertibility)
- Canonical variables can be treated as functions of (τ, u, y^a)
  Example: $R(v, u, y^a) = \tilde{R}(\tau, u, y^a)$ (and drop tilde)
  $\Rightarrow \dot{R} = \dot{\tau}\partial_\tau R$, etc
- Use $\dot{\tau} = -\frac{1}{2h}\partial_+ R$ (e.o.m. of τ) and the action becomes

$$
\begin{aligned}
S &= \int dvdud^2y(\pi_\tau\dot{\tau} + \pi_R\dot{R} + \pi_a\dot{A}_+^{\ a} + \pi^{ab}\dot{\rho}_{ab} - C_-) \\
&= \int dvdud^2y\dot{\tau}\{\pi_\tau + \pi_R\partial_\tau R + \pi_a\partial_\tau A_+^{\ a} + \pi^{ab}\partial_\tau\rho_{ab} + (\frac{2h}{D_+R})C_-\} \\
&= \int d\tau dud^2y(\pi_R\partial_\tau R + \pi_a\partial_\tau A_+^{\ a} + \pi^{ab}\partial_\tau\rho_{ab} - \mathcal{K})
\end{aligned}
$$

- with $dv\dot{\tau} = d\tau$
- $\mathcal{K} := -(\frac{2h}{D_+R})C_- - \pi_\tau \approx -\pi_\tau$!!

Privileged spatial coordinates: $u = R, y^a = Y^a$

- Exercise freedoms to choose y^a on N_2 arbitrarily.
  (a) Let $y^a = Y^a$ on N_2 such that N_2 is normal to $Y^a = \mathrm{constant}$
  $\Longleftrightarrow$ the 2 dimensional shift $A_+^{\ a} = 0$
  (b) Choose u as $u = R$
  $\Rightarrow R = \mathrm{constant}$ is an "equipotential" surface (cf. $\partial_u R = -h\pi_h$)
- Stick to the privileged spacetime coordinates $X^A := (\tau, R, Y^a)$ with the following coordinate condition
  $\frac{\partial X^A}{\partial X^B} = \delta^A_B$
- Issues of general covariance are eliminated from consideration
- Spacetime metric in the priviledged coordinates are as simple as

$$ds^2 = -4hdRd\tau - 2hdR^2 + \tau\rho_{ab}dY^a dY^b$$

  where $h = h(\tau, R, Y^a)$ and $\rho_{ab} = \rho_{ab}(\tau, R, Y^a)$

Hamilton's equations in the privileged coordinates

- Solely expressed by physical degrees of freedom ρ_{ab} and π^{ab} ($\det \rho_{ab} = 1, \rho_{ab}\pi^{ab} = 0$)
- Define a function $\mathcal{H}(\tau; \rho_{ab}, \pi^{ab})$ as

$$\mathcal{H} = \tau^{-1}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} + \frac{1}{4}\tau\rho^{ab}\rho^{cd}(\partial_R\rho_{ac})(\partial_R\rho_{bd}) + \pi^{ac}\partial_R\rho_{ac} + \frac{1}{2\tau} > 0$$

- Physical Hamiltonian density $-\pi_\tau$, physical momentum densities π_R and $\tau^{-1}\pi_a$ are given by

1. $C_- = 0 \Rightarrow \pi_\tau = -\mathcal{H} + 2\partial_R \ln(-h)$
2. $C_+ = 0 \Rightarrow \pi_R = -\pi^{ab}\partial_R\rho_{ab}$
3. $C_a = 0 \Rightarrow \tau^{-1}\pi_a = -\pi^{bc}\frac{\partial}{\partial Y^a}\rho_{bc} + 2\frac{\partial}{\partial Y^b}(\pi^{bc}\rho_{ac}) - \frac{\partial}{\partial Y^a}\{\tau(\mathcal{H} + \pi_R)\}$

Hamilton's equations in the privileged coordinates

- The remaining equations are the followings:

  (a) equations that define superpotential ln(-h)
  (b) integrability conditions of superpotential ln(-h)
  (c) evolution equations of physical degrees of freedom ρ_{ab} and π^{ab}
  (d) a topological constraint restricting the topology of N_2
  (e) trivial equations that dictate the evolutions in τ time of physical Hamiltonian $-\pi_\tau$ and physical momentum $\tau^{-1}\pi_a$

Hamilton's equations in the privileged coordinates

(a) Equations that define superpotential $\ln(-h)$:

$$4.\ \partial_\tau \ln(-h) = \mathcal{H} - \tau^{-1}$$

$$5.\ -\partial_R \ln(-h) = \pi_R$$

$$6.\ -\frac{\partial}{\partial Y^a} \ln(-h) = \tau^{-1}\pi_a$$

(b) Integrability conditions of superpotential $\ln(-h)$:

$$7.\ \frac{\partial \pi_R}{\partial \tau} = -\frac{\partial \mathcal{H}}{\partial R},$$

$$8.\ \frac{\partial}{\partial \tau}(\tau^{-1}\pi_a) = -\frac{\partial}{\partial Y^a}\mathcal{H},$$

$$9.\ \frac{\partial}{\partial R}(\tau^{-1}\pi_a) = \frac{\partial}{\partial Y^a}\pi_R.$$

Hamilton's equations in the privileged coordinates

(c) Evolution equations of physical degrees of freedom ρ_{ab} and π^{ab}:

10. $$\frac{\partial}{\partial \tau}\rho_{ab} = \frac{\delta}{\delta \pi^{ab}} \int dR d^2 Y \mathcal{H} = 2\tau^{-1}\rho_{ac}\rho_{bd}\pi^{cd} + \partial_R \rho_{ab},$$
11. $$\frac{\partial}{\partial \tau}\pi^{ab} = -\frac{\delta}{\delta \rho_{ab}} \int dR d^2 Y \mathcal{H}$$
$$= -2\tau^{-1}\rho_{cd}\pi^{ac}\pi^{bd} + \partial_R \pi^{ab} - \frac{\tau}{2}\rho^{ai}\rho^{bj}\rho^{ck}(\partial_R \rho_{ic})(\partial_R \rho_{jk})$$
$$+\frac{\tau}{2}\rho^{ac}\rho^{bd}(\partial_R^2 \rho_{cd}).$$

Hamilton's equations in the privileged coordinates

(d) The transverse degrees of freedom ρ_{ab} and π^{ab} are "almost" free, subject to the following topological constraint:

$$12.\ \mathrm{R}^{(2)}_{ab} - \frac{1}{2}\tau^{-2}\pi_a\pi_b + \nabla^{(2)}_a(\tau^{-1}\pi_b) = 0$$

whose trace is

$$\tau R^{(2)} - \frac{1}{2}\tau^{-2}\rho^{ab}\pi_a\pi_b + \frac{\partial}{\partial Y^a}(\tau^{-1}\rho^{ab}\pi_b) = 0.$$

Integral over a closed N_2 becomes

$$\int_{N_2} d^2Y \tau^{-2}\rho^{ab}\pi_a\pi_b = 16\pi(1-g) \geq 0,$$

where g is the genus of N_2,

$$\int_{N_2} d^2Y \tau R^{(2)} = 8\pi(1-g).$$

Hamilton's equations in the privileged coordinates

$\Rightarrow$ Einstein's equations dictates that the spatial topology of a compact 2-dimensional cross-section of an out-going null hypersurface is either a 2-sphere or a torus (when the 2-dimensional shift $A_{+}^{a} = 0$) cf. Topological Censorship Theorem in black hole physics:

The spatial topology of the event horizon of a four dimensional asymptotically flat black hole spacetime must be a two sphere (or marginally a torus) under physically reasonable conditions
J.L. Friedman, K. Schleich, and D.M. Witt, Phys. Rev. Lett. **71**, 1486 (1993).

Hamilton's equations in the privileged coordinates

(e) Trivial equations that dictate the evolutions in τ time of physical Hamiltonian $-\pi_\tau$ and physical momentum $\tau^{-1}\pi_a$:

$$13. \frac{\partial \pi_a}{\partial \tau} = 2\tau^{-1}\pi_a + (\pi^{bc} + \frac{1}{2}\rho^{bd}\rho^{ce}\partial_R \rho_{de})\frac{\partial}{\partial Y^a}\rho_{bc} - \frac{\partial}{\partial Y^b}(2\pi^{bc}\rho_{ac} + \rho^{bc}\partial_R \rho_{ac})$$

$$14. \frac{\partial \pi_\tau}{\partial \tau} = \frac{1}{2}\tau^{-2} + \tau^{-2}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} - \frac{1}{4}\rho^{ab}\rho^{cd}(\partial_R \rho_{ac})(\partial_R \rho_{bd}) - 2\tau^{-2}\frac{\partial}{\partial Y^a}(h\rho^{ab}\pi_b)$$

Consistency with Einstein's equations $R_{AB} = 0$

- The metric in the privileged coordinates $X^A = (\tau, R, Y^a)$ are given by
$$ds^2 = -4hdRd\tau - 2hdR^2 + \tau\rho_{ab}dY^a dY^b,$$
where $h = h(\tau, R, Y^a)$ and $\rho_{ab} = \rho_{ab}(\tau, R, Y^a)$
- Compute Einstein's equations $R_{AB} = 0$ for the above metric
- Equations $1, \cdots, 14$ are identical to $R_{AB} = 0$!!
$\Rightarrow$ The Hamiltonian reduction is a self-consistent procedure
- A generalization of Hamiltonian reduction of midi-superspace model to spacetimes of 4-dimensions **without isometries**