# Solution-generating methods of Einstein's equations by (2+2) Hamiltonian reduction

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# Outline

# 2+2 Hamiltonian reduction of GR

# Geometry of the privileged coordinates

- The privileged coordinates $\{\tau, R, Y^1, Y^2\}$ are the coordinates on spacetime whose line element is given by

$$ds^2 = -2h(2dRd\tau + dR^2) + \tau\rho_{ab}dY^a dY^b \ .$$

- $h$ and $\rho_{ab}$ depend on all coordinates.
- The tangent space of the whole spacetime is decomposed into

$$T_p(M) = T_p(L_2) \bigoplus T_p(N_2) \ ,$$

- $T_p(L_2)$ is a subtangent space spanned by $\partial_\tau$ and $\partial_R$ with Lorentzian signature.
- $T_p(N_2)$ is a subtangent space spanned by $\partial_{Y^1}$ and $\partial_{Y^2}$ with Rimaniann signature.

# Main Characters

- The privileged coordinates $\{\tau, R, Y^1, Y^2\}$ are the coordinates on spacetime whose line element is given by

$$ds^2 = -2h(2dRd\tau + dR^2) + \tau\rho_{ab}dY^a dY^b \ .$$

- $h$ : conformal factor of $L_2$, superpotential...
- $\tau$ : a time coordinate, the area element of $N_2$...
- $\rho_{ab}$ : conformal two-metric on $N_2$, $det\rho_{ab} = 1$, it contains the genuine degrees of freedom of gravitational fields...
- $\pi^{ab}$ : The conjugate momentum of $\rho_{ab}$ which is traceless,

$$\rho_{ab}\pi^{ab} = 0 \ .$$

# Constraint equations

- The constraint equations are defining equations of physical Hamiltonian and momentum densities.

$$-\pi_\tau = \mathcal{H} - 2\partial_R \ln(-h) \ , \qquad \pi_R = -\pi^{ab}\partial_R \rho_{ab} \ ,$$
$$\tau^{-1}\pi_a = -\pi^{bc}\partial_a \rho_{bc} + 2\partial_b(\pi^{bc}\rho_{ac}) - \partial_a\{\tau(\mathcal{H} + \pi_R)\} \ ,$$

  where a Hamiltonian functional is

$$\mathcal{H} = \tau^{-1}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} + \frac{1}{4}\tau\rho^{ab}\rho^{cd}(\partial_R\rho_{ac})(\partial_R\rho_{bd}) + \pi^{ac}\partial_R\rho_{ac} + \frac{1}{2\tau} > 0 \, .$$

- With an appropriate boundary conditions, the total energy and momentum are given by,

$$E = \int_{\Sigma_\tau} dRd^2Y \ \mathcal{H} \ , \qquad \Pi_R = -\int_{\Sigma_\tau} dRd^2Y \ \pi^{ab}\partial_R\rho_{ab} \ ,$$
$$\Pi_a = -\int_{\Sigma_\tau} dRd^2Y \ \pi^{bc}\partial_a\rho_{bc} \ .$$

# Hamilton's equations in the privileged coordinates

- Superpotential equations: The gradient of $\ln(-h)$ gives the Hamiltonian and momentum densities of gravitational fields.

$$\partial_\tau \ln(-h) = \mathcal{H} - \tau^{-1}$$
$$\partial_R \ln(-h) = -\pi_R$$
$$\partial_a \ln(-h) = -\tau^{-1}\pi_a$$

- Integrability conditions of superpotential $\ln(-h)$:

$$\partial_\tau \pi_R = -\partial_R \mathcal{H},$$
$$\partial_\tau(\tau^{-1}\pi_a) = -\partial_a \mathcal{H},$$
$$\partial_R(\tau^{-1}\pi_a) = \partial_a \pi_R.$$

# Hamilton's equations in the privileged coordinates

- Evolution equations of $\rho_{ab}$ and $\pi^{ab}$:

$$\frac{\partial}{\partial\tau}\rho_{ab} = 2\tau^{-1}\rho_{ac}\rho_{bd}\pi^{cd} + \partial_R\rho_{ab},$$
$$\frac{\partial}{\partial\tau}\pi^{ab} = -2\tau^{-1}\rho_{cd}\pi^{ac}\pi^{bd} + \partial_R\pi^{ab} - \frac{\tau}{2}\rho^{ai}\rho^{bj}\rho^{ck}(\partial_R\rho_{ic})(\partial_R\rho_{jk}) + \frac{\tau}{2}\rho^{ac}\rho^{bd}(\partial_R^2\rho_{cd}),$$

- These equations are solely expressed in the terms of $\rho_{ab}$ and $\pi^{ab}$!
- The genuine degrees of freedom of gravitational fields!
- These equations determine the dependences of $\rho_{ab}$ and $\pi^{ab}$ on $\tau$ and $R$.

# Hamilton's equations in the privileged coordinates

- Topological constraint equations

$$\tau R^{(2)}_{ab} - \frac{1}{2}\tau^{-2}\pi_a\pi_b + \tilde{\nabla}_a(\tau^{-1}\pi_b) = 0 \ ,$$

where $\tilde{\nabla}_a$ is an induced covariant derivative on $N_2$.

- This equation restricts the spatial topology of a compact 2-dimensional cross-section of an out-going null hypersurface (either a 2-sphere or a torus)
- This equation determines the dependences of $\rho_{ab}$ and $\pi^{ab}$ on $Y^1$ and $Y^2$.

# Hamilton's equations in the privileged coordinates

- Evolution equations of physical Hamiltonian $-\pi_\tau$ and physical momentum $\tau^{-1}\pi_a$ :

$$
\begin{aligned}
\partial_\tau \pi_a &= 2\tau^{-1}\pi_a + (\pi^{bc} + \frac{1}{2}\rho^{bd}\rho^{ce}\partial_R \rho_{de})\partial_a \rho_{bc} \\
&\quad -\partial_b(2\pi^{bc}\rho_{ac} + \rho^{bc}\partial_R \rho_{ac}) \\
\partial_\tau \pi_\tau &= \frac{1}{2}\tau^{-2} + \tau^{-2}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} - \frac{1}{4}\rho^{ab}\rho^{cd}(\partial_R \rho_{ac})(\partial_R \rho_{bd}) \\
&\quad -2\tau^{-2}\partial_a(h\rho^{ab}\pi_b)
\end{aligned}
$$

- These equations are trivially satisfied if $\rho_{ab}$ and $\pi^{ab}$ satisfy the evolution equations of $\rho_{ab}$ and $\pi^{ab}$.

# A Strategy to solve Einstein's equation

- Step 1. Solve the evolution equations of $\rho_{ab}$ and $\pi^{ab}$. The dependences on $\tau$ and $R$ are determined in this step.

$$\frac{\partial}{\partial\tau}\rho_{ab} = 2\tau^{-1}\rho_{ac}\rho_{bd}\pi^{cd} + \partial_R\rho_{ab},$$

$$\frac{\partial}{\partial\tau}\pi^{ab} = -2\tau^{-1}\rho_{cd}\pi^{ac}\pi^{bd} + \partial_R\pi^{ab} - \frac{\tau}{2}\rho^{ai}\rho^{bj}\rho^{ck}(\partial_R\rho_{ic})(\partial_R\rho_{jk}) + \frac{\tau}{2}\rho^{ac}\rho^{bd}(\partial_R^2\rho_{cd}),$$

# A Strategy to solve Einstein's equation

- Step 2. Solve the topological constraint equation of $\rho_{ab}$ and $\pi^{ab}$. The dependences on $Y^1$ and $Y^2$ are determined in this step.

$$\tau R^{(2)}_{ab} - \frac{1}{2}\tau^{-2}\pi_a\pi_b + \tilde{\nabla}_a(\tau^{-1}\pi_b) = 0 \ .$$

- Step 3. Integrate the superpotential equations. The conformal factor h will be determined in this step.

$$\partial_\tau \ln(-h) = \mathcal{H} - \tau^{-1}$$
$$\partial_R \ln(-h) = -\pi_R$$
$$\partial_a \ln(-h) = -\tau^{-1}\pi_a$$

# Einstein-Rosen Waves

# Ansatz with two commuting spacelike Killing vectors

- let us make the following ansatz for $\rho_{ab}$ and $\pi^{ab}$,

$$\rho_{ab} = \begin{pmatrix} f & 0 \\ 0 & 1/f \end{pmatrix}, \qquad \pi^{ab} = \begin{pmatrix} \pi_F/f & 0 \\ 0 & -f\pi_F \end{pmatrix},$$

  where $f$ and $\pi_F$ are functions of $\tau$ and $R$ to be determined.
- Namely, we are assuming two commuting spacelike Killing vectors $\frac{\partial}{\partial Y^1}$ and $\frac{\partial}{\partial Y^2}$.
- This ansatz satisfies the traceless condition $\rho_{ab}\pi^{ab} = 0$ trivially.

# Evolution eqns and Superpotential eqns

- Substitution of the above ansatz into the evolution equations of $\rho_{ab}$ and $\pi^{ab}$ yields the following two equations,

$$\begin{aligned} \partial_\tau F &= \frac{4}{\tau}\pi_F + \partial_R F, \\ \partial_\tau \pi_F &= \frac{\tau}{4}\partial_R^2 F + \partial_R \pi_F, \end{aligned}$$

  where $F := 2\ln f$.

- Most equations are trivially satisfied due to the Killing condition.
- Non-trivial superpotential equations are,

$$\begin{aligned} \partial_\tau \ln(-h) &= -\frac{1}{2\tau} + \frac{2}{\tau}\pi_F^2 + \frac{\tau}{8}(\partial_R F)^2 + \pi_F(\partial_R F), \\ \partial_R \ln(-h) &= \pi_F(\partial_R F). \end{aligned}$$

# Solution

- $\pi_F$ can be expressed in terms of $F$,

$$\pi_F = \frac{\tau}{4}(\partial_\tau - \partial_R)F.$$

- If we plug $\pi_F$ into the other non-trivial equations, then they become

$$\partial_\tau^2 F + \frac{1}{\tau}(\partial_\tau - \partial_R)F - 2\partial_\tau\partial_R F = 0,$$
$$\partial_\tau \ln(-h) = -\frac{1}{2\tau} + \frac{\tau}{8}(\partial_\tau F)^2,$$
$$\partial_R \ln(-h) = \frac{\tau}{4}(\partial_\tau F)(\partial_R F) - \frac{\tau}{4}(\partial_R F)^2.$$

- The first equation is a linear second-order PDE of $F$, the existence of whose solutions is well-established.
- With a given $F$, the conformal factor $h$ will be determined by the last two equations.

# The field momenta

- The local Hamiltonian and momentum densities are also determined by the constraint equations (1), (2), and (3), which become,

$$-\pi_\tau = \frac{1}{2\tau} + \frac{2}{\tau}(\pi_F - \frac{\tau}{4}\partial_R F)^2,$$
$$\pi_R = -\pi_F(\partial_R F),$$
$$\pi_a = 0.$$

- The last equation is a trivial consequence of the assumption that $\frac{\partial}{\partial Y^1}$ and $\frac{\partial}{\partial Y^2}$ are the Killing vectors.
- Thus, all the components of the metric tensor are completely determined by a single function $F$ for this ansatz.

# The Einstein-Rosen waves

- let us make the following coordinate transformation,

$$t = \tau + R, \quad \rho = \tau, \quad \phi = Y^1, \quad z = Y^2.$$

- Let us introduce two complex functions $\Psi(t,\rho)$ and $\Gamma(t,\rho)$ defined by

$$\Psi = -\ln(f/\rho) + i\pi, \qquad \Gamma = \ln(-2\rho h/f) + i\pi,$$

where the imaginary constants were introduced to keep our sign convention $h < 0$.

- In the coordinates $(t, \rho, \phi, z)$, the line element becomes

$$ds^2 = e^{\Gamma - \Psi}(dt^2 - d\rho^2) - \rho^2 e^{-\Psi} d\phi^2 - e^{\Psi} dz^2,$$

# The Einstein-Rosen waves

- Under the previous coordinate transformation, the three main equations become

$$\partial_t^2\Psi - \frac{1}{\rho}\partial_\rho\Psi - \partial_\rho^2\Psi = 0,$$
$$\partial_t\Gamma = \rho(\partial_t\Psi)(\partial_\rho\Psi),$$
$$\partial_\rho\Gamma = \frac{\rho}{2}\{(\partial_t\Psi)^2 + (\partial_\rho\Psi)^2\}.$$

- This proves that the spacetime discussed in this section represents the Einstein-Rosen spacetime of cylindrically symmetric graviational waves.

# Schwarzschild spacetime

# The same ansatz again!

- Remark that our time coordinate $\tau$ is the area element of $N_2$. And at the same time $L_2$ must be conformally flat.
- So the Kruskal & Szekeres coordinates cannot be a privileged coordinates. We have to find a new slice of spacetime.
- It is helpful to use the fact that Schwarzschild metric can be expressed in the form of the canonical Weyl metric.
- let us begin with the ansatz for $\rho_{ab}$ and $\pi^{ab}$,

$$\rho_{ab} = \begin{pmatrix} f & 0 \\ 0 & 1/f \end{pmatrix}, \qquad \pi^{ab} = \begin{pmatrix} \pi_F/f & 0 \\ 0 & -f\pi_F \end{pmatrix},$$

where $f$ and $\pi_F$ are functions of $\tau$ and $R$.

# Coordinate transform

- If we make the following *complex* coordinate transformations from $(\tau, R, Y^1, Y^2)$ to $(t, \rho, \phi, z)$ defined by

$$t = Y^1, \quad \rho = \tau, \quad \phi = iY^2, \quad z = i(\tau + R),$$

and introduce two functions $\psi(\rho, z)$ and $\gamma(\rho, z)$ defined by

$$\psi = \frac{1}{2}\ln(\rho f), \quad \gamma = \frac{1}{2}\ln(-2\rho f h),$$

the line element becomes

$$ds^2 = e^{2\psi}dt^2 - e^{2(\gamma-\psi)}(d\rho^2 + dz^2) - \rho^2 e^{-2\psi}d\phi^2,$$

which is just the canonical Weyl metric.

# Schwarzschild solution in Weyl metric

- In the coordinates $(t, \rho, \phi, z)$, the three main equations become

$$
\begin{aligned}
&\partial_\rho^2 \psi + \frac{1}{\rho}\partial_\rho \psi + \partial_z^2 \psi = 0, \\
&\partial_\rho \gamma = \rho\{(\partial_\rho \psi)^2 - (\partial_z \psi)^2\}, \\
&\partial_z \gamma = 2\rho(\partial_\rho \psi)(\partial_z \psi).
\end{aligned}
$$

- All the remaining Einstein's equations hold trivially as before.
- As is well-known, the canonical Weyl metric represents the Schwarzschild solution if $\psi$ and $\gamma$ are given by

$$
\psi = \frac{1}{2}\ln\frac{r_+ + r_- - 2m}{r_+ + r_- + 2m}, \quad \gamma = \frac{1}{2}\ln\frac{(r_+ + r_-)^2 - 4m^2}{4r_+ r_-},
$$

where $r_\pm = [\rho^2 + (z \pm m)^2]^{\frac{1}{2}}$ and $m$ is the Schwarzschild mass .

# Schwarzschild solution in (2+2) metric

- It is a straightforward exercise to find that the Schwarzschild solution in the privileged coordinates becomes

$$ds^2 = -2h\left(2dRd\tau + dR^2\right) + \tau\rho_{ab}\, dY^a dY^b \ ,$$

where

$$\rho_{ab} = \begin{pmatrix} f & 0 \\ 0 & 1/f \end{pmatrix}, \qquad f = \frac{1}{\tau}\frac{\sqrt{\xi+\sqrt{\xi^2+\eta^2}}-\sqrt{2}m}{\sqrt{\xi+\sqrt{\xi^2+\eta^2}}+\sqrt{2}m} \ ,$$

$$h = -\frac{1}{4\sqrt{\xi^2+\eta^2}}\left(\sqrt{\xi+\sqrt{\xi^2+\eta^2}}+\sqrt{2}m\right)^2 \ ,$$

and $\xi$ and $\eta$ are functions of $\tau$ and $R$ given by

$$\xi = m^2 - 2R\tau - R^2, \quad \eta = 2m(\tau + R) \ ,$$

respectively.

# Field momenta

The local Hamiltonian and momentum densities of the Schwarzschild spacetime are found to be

$$
\begin{aligned}
-\pi_\tau = \quad & \frac{1}{2\tau} + \frac{\tau}{2}\Big[\frac{2\sqrt{2}m\{(m^2+R^2)(R+2\tau) - R\sqrt{\xi^2+\eta^2}\ \}}{\sqrt{\xi^2+\eta^2}\sqrt{\xi+\sqrt{\xi^2+\eta^2}}(\xi+\sqrt{\xi^2+\eta^2}-2m^2)} - \frac{1}{\tau}\Big]^2 \\
& + \frac{4(\tau+R)}{\sqrt{\xi^2+\eta^2}}\Big(\frac{\sqrt{\xi+\sqrt{\xi^2+\eta^2}}-\sqrt{2}m}{\sqrt{\xi+\sqrt{\xi^2+\eta^2}}} + \frac{R^2+2R\tau+m^2}{\sqrt{\xi^2+\eta^2}}\Big), \\
\pi_R = \quad & \frac{2(\tau+R)}{\sqrt{\xi^2+\eta^2}}\Big(\frac{\sqrt{\xi+\sqrt{\xi^2+\eta^2}}-\sqrt{2}m}{\sqrt{\xi+\sqrt{\xi^2+\eta^2}}} + \frac{R^2+2R\tau+m^2}{\sqrt{\xi^2+\eta^2}}\Big), \\
\pi_a = \quad & 0.
\end{aligned}
$$

# Field momenta

- The local Hamiltonian and momentum densities can be also obtained by taking the gradient of the "superpotential" $\ln(-h)$, as are indicated by the superpotential equations.
- It is clear that both $-\pi_\tau$ and $\pi_R$ depend on $\tau$ explicitly.
- This simply reflects the fact that the area time $\tau$ is a geometric object that is quite different from the usual Killing time.

# Discussion

- First of all, locally defined Hamiltonian and momentum densities of gravitational fields can be straightforwardly calculated, provided that the metric of the gravitating system is known in the privileged coordinates.
- The complete deparametrization of the Einstein's theory is achieved by choosing certain functions of the gravitational phase space as the privileged coordinates in which all the constraints become trivial.
- It is clear in this (2+2) Hamiltonian reduction that true physical degrees of freedom reside in the conformal two metric $\rho_{ab}$ (and its conjugate $\pi^{ab}$) and that the time evolutions of $\rho_{ab}$ and $\pi^{ab}$ are completely determined by the Hamilton's equations of motion.

# Discussions

- The (2+2) Hamiltonian reduction can be used to find exact solutions to the Einstein's equations in a minimal way in the sense that, if the solutions to the Hamilton's equations of motion of $\rho_{ab}$ and $\pi^{ab}$ are known, then the rest of the Einstein's equations are either trivial or can be solved by integrating the first-order differential equations.
- In our Hamiltonian reduction, however, this issue of stability of the constraint equations does not appear. One can bypass the *notorious* problem of solving the constraint equations because they are simply the defining equations of the local Hamiltonian and momentum densities of gravitational fields.
- It is a very interesting problem to find the (2+2) Hamiltonian reduction formalism of the Einstein's theory with a non-vanishing two-dimensional shift. This problem is under investigation.

# Thank you for your attention!