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Tunneling decay of false vortices with gravitation

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based on
E. Dupuis, J. Gobel, B.-H. Lee, WL, R. MacKenzie, M. B. Paranjape, U. A. Yajnik, and D.-h. Yeom,
PRD88 085031 (2013), PRD88 105008 (2013) & in preparation

The plan of this talk

1. Motivations
2. Tunneling decay of false vortices
3. Tunneling decay of false vortices with gravitation
4. Israel junction condition
5. Summary and discussions

1. Motivations

Are the soliton solutions quantum mechanically stable ?

What is the role of the topological number for the quantum stability of the vortex solution?

Tunneling decay of false vortices, PRD88, 085031 (2013)

Battle of the bulge: Decay of the thin, false cosmic string PRD88, 105008 (2013)

Tunneling decay of false vortices with gravitation the present work

Decay of the thin, false cosmic string with gravitation the next work

2. Tunneling decay of false vortices

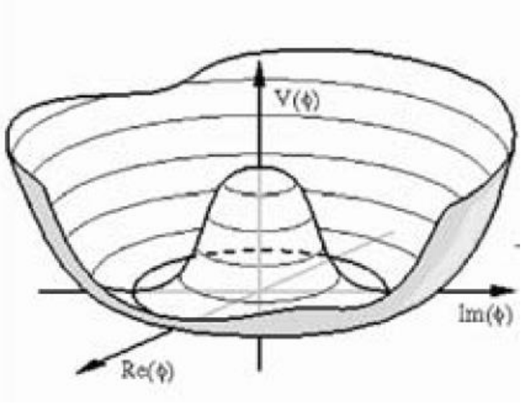
The vortex is the simplest soliton in gauge theory with scalars (Abrikosov 1966, Nielsen and Olesen 1973). It arises in a model with gauge group U(1) and the Higgs mechanism in (2+1)-dimensional spacetime.

The model is described by the Lagrangian

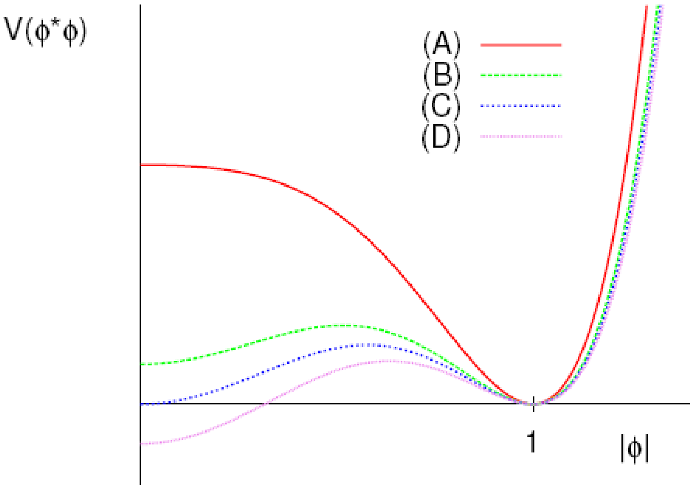
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - V(\phi^*\phi),$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

and $D_\mu\phi = (\partial_\mu - ieA_\mu)\phi$



$$V(\phi^*\phi) = \lambda(|\phi|^2 - \epsilon v^2)(|\phi|^2 - v^2)^2$$



In Figure, we have drawn four possible examples of the scalar potential, starting with (A) which is the usual quartic symmetry-breaking potential with a maximum at φ = 0 and with vacuum at |φ| = 1 after a suitable rescaling, (B) where a metastable local minimum (false vacuum) is formed at φ = 0, (C) where this minimum becomes degenerate with the symmetry-breaking minimum, and (D) where the roles of the two vacua are reversed, with φ = 0 and | φ | = 1 becoming the true and false vacua, respectively.

After rescaling by appropriate powers of v and λ

$$V(\phi^*\phi) = (|\phi|^2 - \epsilon)(|\phi|^2 - 1)^2$$

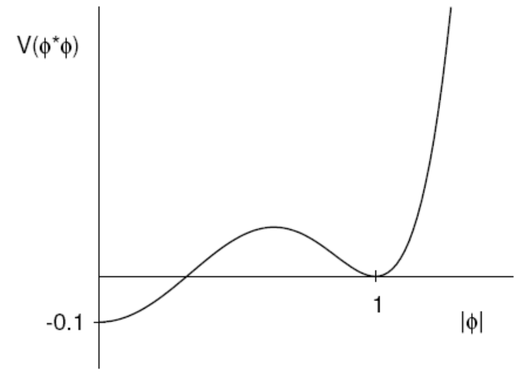


FIG. 2. The rescaled potential (3) with $\epsilon = 0.1$.

We use the following time-dependent ansatz for a vortex of winding number n:

$$\phi(r,\theta,t) = f(r,t)e^{in\theta}, \qquad A_i(r,\theta,t) = -\frac{n}{e}\frac{\varepsilon^{ij}r_j}{r^2}a(r,t),$$

The static vortex solution is the minimum of this functional (without the time-derivative terms); the variational field equations are

$$f'' + \frac{f'}{r} - \frac{n^2}{r^2}(1-a)^2 f - (f^2-1)(3f^2-(1+2\epsilon))f = 0,$$
$$a'' - \frac{a'}{r} + 2e^2(1-a)f^2 = 0.$$

using the following boundary conditions:

$$f(r) \to 0, \quad a(r) \to 0 \quad \text{as} \quad r \to 0,$$
$$f(r) \to 1, \quad a(r) \to 1 \quad \text{as} \quad r \to \infty.$$

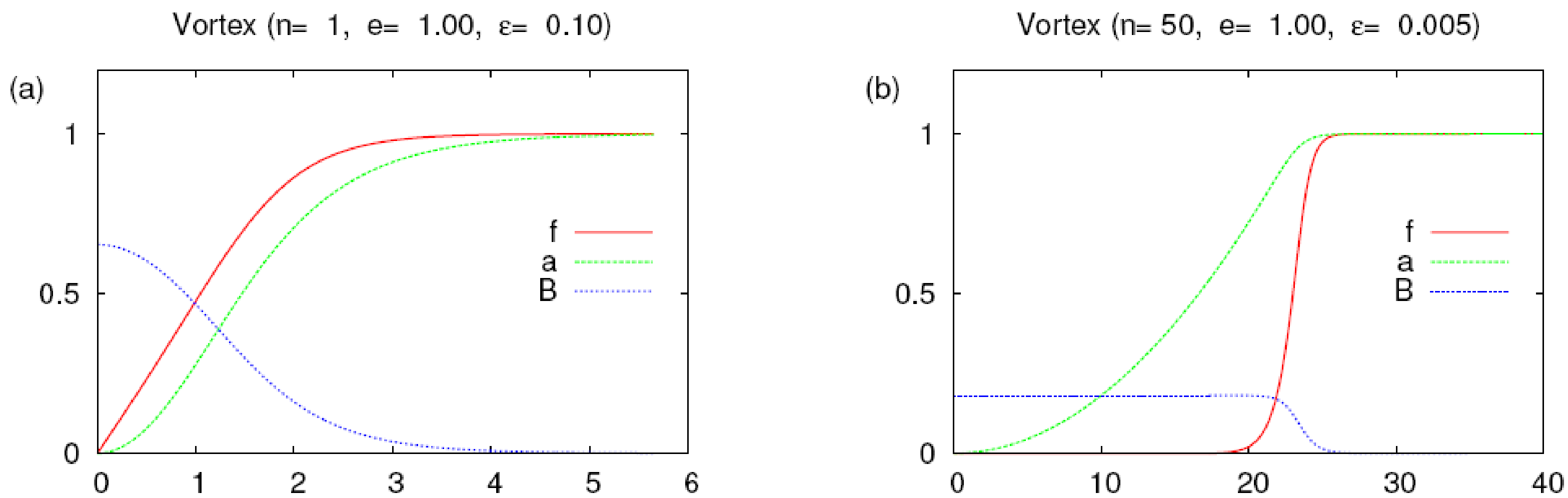


FIG. 3. (Color online) Vortex profile for (a) thick-wall and (b) thin-wall vortices. Displayed are the functions $f(r)$ and $a(r)$ and the magnetic field $B(r) = na'(r)/er$.

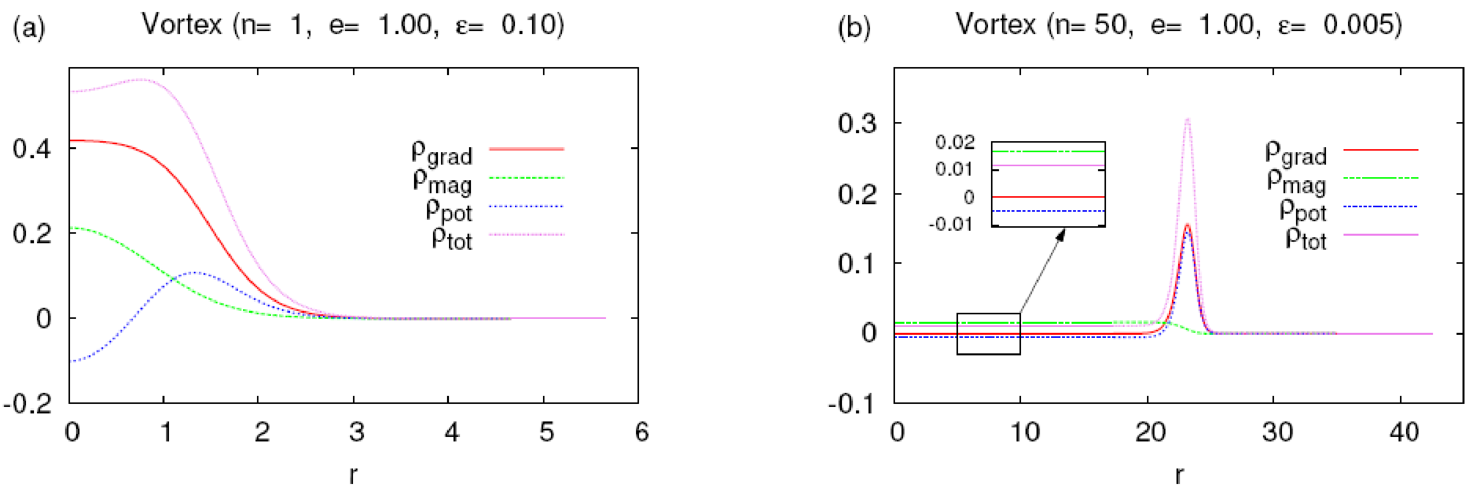


FIG. 4. (Color online) Scalar field gradient energy density ρ_{grad}, magnetic field energy density ρ_{mag}, potential energy density ρ_{pot}, and total energy density ρ_{tot} for (a) thick-wall and (b) thin-wall vortices.

2-1. Fate of the false vortices via tunneling

A. Thick-wall ansatz (n=1)

The action of the ansatz

$$S = \int dt\,(T - E)$$

where T is the kinetic energy

$$T = 2\pi \int_0^\infty dr\, r\left(\dot{f}^2 + \frac{\dot{a}^2}{2e^2r^2}\right)$$

and E is the energy of a static configuration

$$E = 2\pi \int_0^\infty dr\, r\left(f'^2 + \frac{(1-a)^2}{r^2}f^2 + \frac{a'^2}{2e^2r^2} + (f^2-\epsilon)(f^2-1)^2\right)$$

We begin by determining the minimum-energy configuration within a family of configurations representing a vortex of width R, treating R as a variational parameter.

$$f(r) = \begin{cases} r/R & r < R \\ 1 & r > R \end{cases}, \qquad a(r) = \begin{cases} (r/R)^2 & r < R \\ 1 & r > R \end{cases}$$

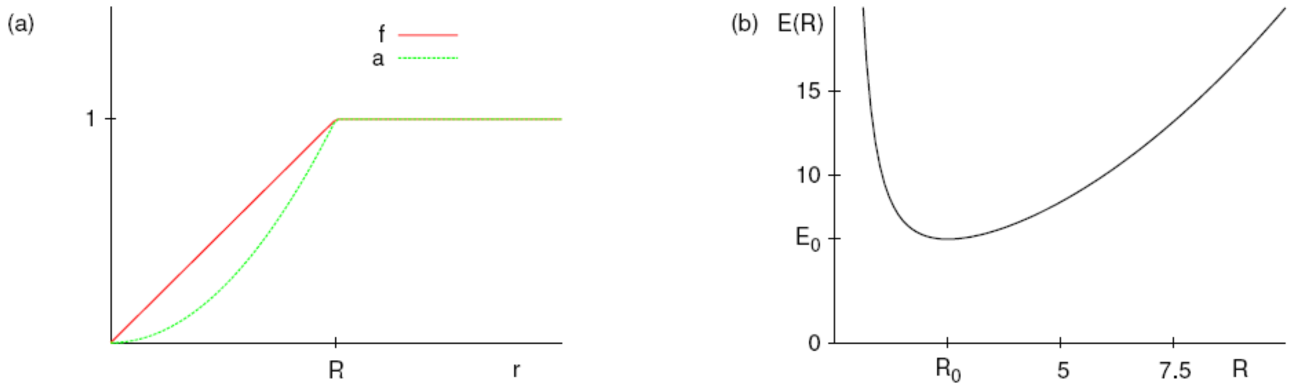


FIG. 6. (Color online) (a) Piecewise vortex ansatz. (b) Vortex energy E(R) for $e = 1$, $\epsilon = 0.1$.

B. Thin-wall ansatz (n>>1)

We introduce n into the action

$$S = \int dt\,(T - E)$$

where T is the kinetic energy

$$T = 2\pi \int_0^\infty dr\, r \left(\dot{f}^2 + \frac{n^2 \dot{a}^2}{2e^2 r^2} \right)$$

and E is the energy of a static configuration

$$E = 2\pi \int_0^\infty dr\, r \left(f'^2 + \frac{n^2 (1-a)^2}{r^2} f^2 + \frac{n^2 a'^2}{2e^2 r^2} + (f^2 - \epsilon)(f^2 - 1)^2 \right)$$

We can divide the energy integral into three regions:

$$E(R) = E_{\text{int}} + E_{\text{wall}} + E_{\text{ext}}$$

Summing the three contributions

$$E(R) = \frac{2\pi n^2}{e^2 R^2} + \pi R - \epsilon \pi R^2,$$

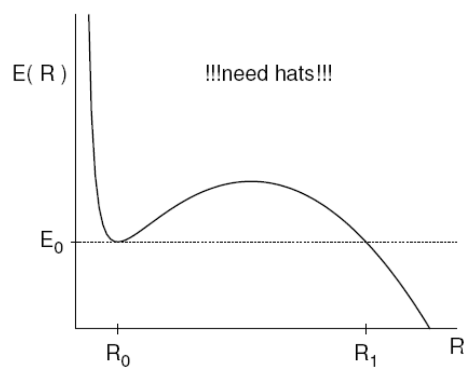


FIG. 8. The rescaled energy $\hat{E}(\hat{R})$ with $\hat{\epsilon} = 0.108$ Numerical values for the three parameters shown are: $\hat{E}_0 = 1.38$, $\hat{R}_0 = 1.09$, $\hat{R}_1 = 7.61$.

3. Tunneling decay of false vortices with gravitation

We consider the action for Einstein gravity coupled to a gauge field and complex scalar field

$$S = \int_{\mathcal{M}} \sqrt{-g} d^3x \left[\frac{R}{2\kappa} - \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - (D_\mu \phi)^* (D^\mu \phi) - U(\phi^* \phi) \right] + S^{YGB}$$

The field strength tensor $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$

The covariant derivative $D_\mu \phi = (\nabla_\mu + ieA_\mu)\phi$

The potential

$$U(\phi^* \phi) = \lambda(|\phi|^2 - \epsilon v^2)(|\phi|^2 - v^2)^2$$

We are interested in the case with $0 < \epsilon < 1.$

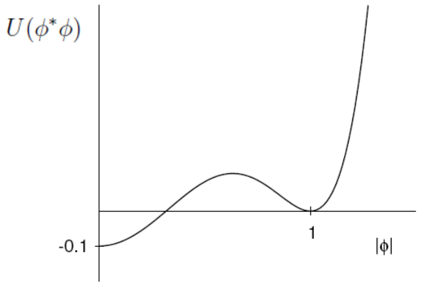


FIG. 2. The rescaled potential (3) with $\epsilon = 0.1$.

The Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} \, ,$$

where the energy-momentum tensor

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{4\pi}(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}) + (D_\mu \phi)^*(D_\nu \phi) + (D_\mu \phi)(D_\nu \phi)^* \\ &- g_{\mu\nu}[(D_\alpha \phi)^*(D^\alpha \phi) + U(\phi^* \phi)] \, . \end{aligned}$$

The gauge field and scalar field equations are respectively

$$\nabla_\nu F^{\nu}{}_{\mu} = 4\pi [ie(\phi \nabla_\mu \phi^* - \phi^* \nabla_\mu \phi) + 2e^2 A_\mu \phi^* \phi] \, ,$$

$$D_\mu D^\mu \phi = \frac{\partial U(\phi^* \phi)}{\partial \phi^*} \, .$$

We take the metric ansatz as to be

$$ds^2 = -A^2(t,r)dt^2 + B^{-2}(t,r)dr^2 + D^2(t,r)d\theta^2 \,,$$

where A(t, r), B(t, r), and D(t, r) are unknown functions of the variable t and r.

We now rewrite the equations in terms of dimensionless variables

$$\frac{\phi}{v} = \tilde{\phi}, \;\; \frac{A_\mu}{v} = \tilde{A}_\mu, \;\; \frac{e}{\sqrt{\lambda}v} = \tilde{e}, \;\; Gv^2 = \tilde{G}, \;\; r\sqrt{\lambda}v^2 = \tilde{r}.$$

The fields ansatz are chosen as

$$\phi(t,r,\theta) = f(t,r)e^{in\theta}, \quad A_\mu(t,r,\theta) = \left[0, 0, \frac{n(a(t,r)-1)}{e}\right]$$

where n is an integer, the winding number.

Using the metric and fields ansatzs, the (tt), (tr), (rr), and (θθ) components of the Einstein equations take the form

$$-\frac{A^2B^2(B'D'+BD'')+\dot{B}\dot{D}}{BD}=\kappa\left[\frac{n^2\dot{a}^2}{8\pi e^2D^2}+\dot{f}^2+\frac{A^2n^2}{8\pi e^2D^2}a'^2B^2+A^2\left(f'^2B^2+\frac{n^2a^2f^2}{D^2}+U\right)\right]$$

$$-\frac{1}{D}\left(\frac{D'\dot{B}}{B}-\frac{A'\dot{D}}{A}+\frac{\dot{D}'}{A}\right)=\kappa\left[\frac{n^2\dot{a}a'}{4\pi e^2D^2}+2\dot{f}f'\right]$$

$$\frac{A^2B^2A'D'+\dot{A}\dot{D}-A\ddot{D}}{A^3B^2D}=\kappa\left[\frac{1}{A^2B^2}\left(\dot{f}^2+\frac{n^2\dot{a}^2}{8\pi e^2D^2}\right)+\frac{n^2a'^2}{8\pi e^2D^2}+f'^2-\frac{n^2f^2a^2}{B^2D^2}-\frac{U}{B^2}\right]$$

$$\frac{D^2[A^2B^3(A'B'+BA'')-B\dot{A}\dot{B}+A(-2\dot{B}^2+B\ddot{B})]}{A^3B^2}$$
$$=\kappa\left[\frac{1}{A^2}\left(\dot{f}^2D^2-\frac{n^2\dot{a}^2}{8\pi e^2}\right)+\frac{n^2a'^2B^2}{8\pi e^2}+n^2a^2f^2-f'^2B^2D^2-UD^2\right]$$

The scalar field equation has the form

$$\frac{1}{A^2}\left[-\ddot{f}+\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}-\frac{\dot{D}}{D}\right)\dot{f}\right]+B^2\left[f''+\left(\frac{A'}{A}+\frac{B'}{B}+\frac{D'}{D}\right)f'\right]-\frac{n^2a^2}{D^2}f=\frac{dU}{df}$$

The gauge field equation has the form

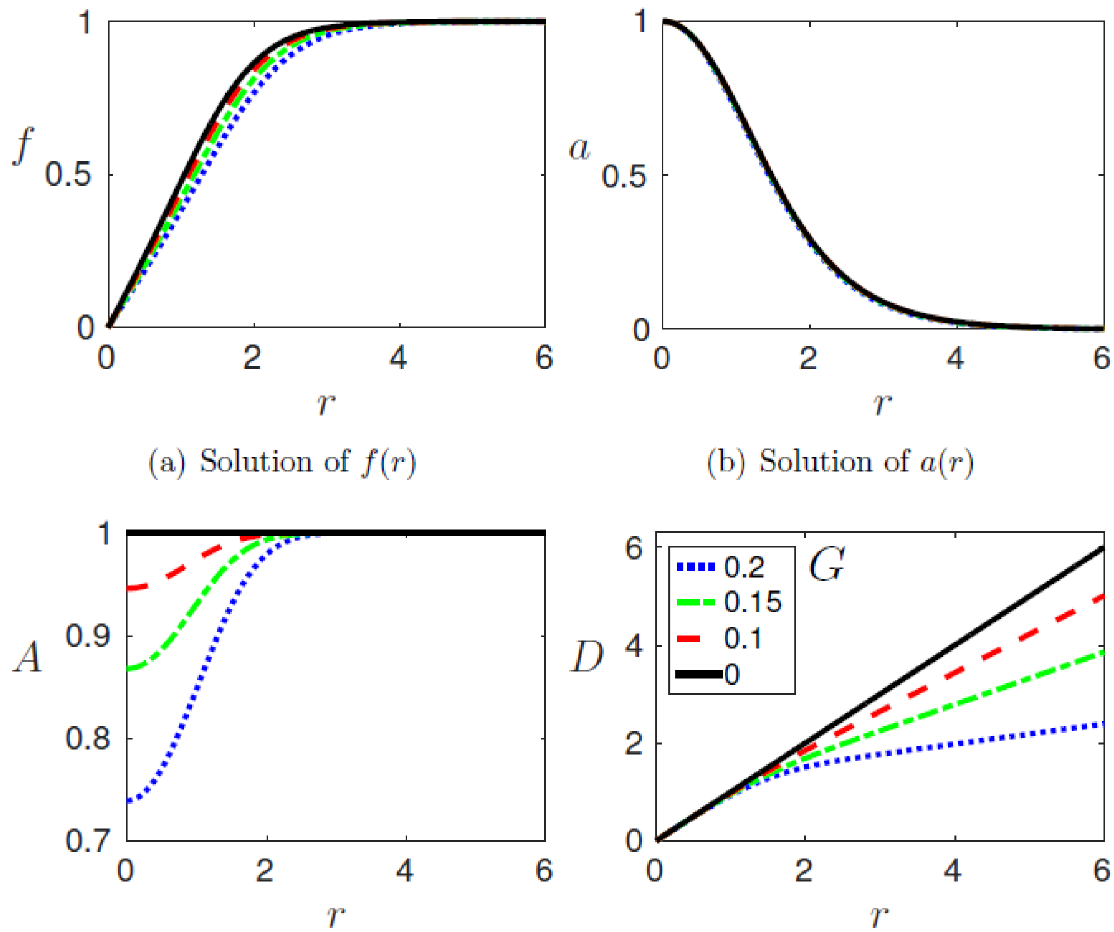
$$\frac{1}{A^2}\left[-\ddot{a}+\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{D}}{D}\right)\dot{a}\right]+B^2\left[a''+\left(\frac{A'}{A}+\frac{B'}{B}-\frac{D'}{D}\right)a'\right]=8\pi e^2f^2a$$

We will only solve the static equation of motions. We will use numerical methods to solve for f(r), a(r), A(r), B(r), and D(r).

Because the metric functions only depend on r, we are free to choose a gauge in which B(r) = 1 everywhere.

We simultaneously solve the coupled Einstein, gauge, and scalar field equations with the following boundary conditions:

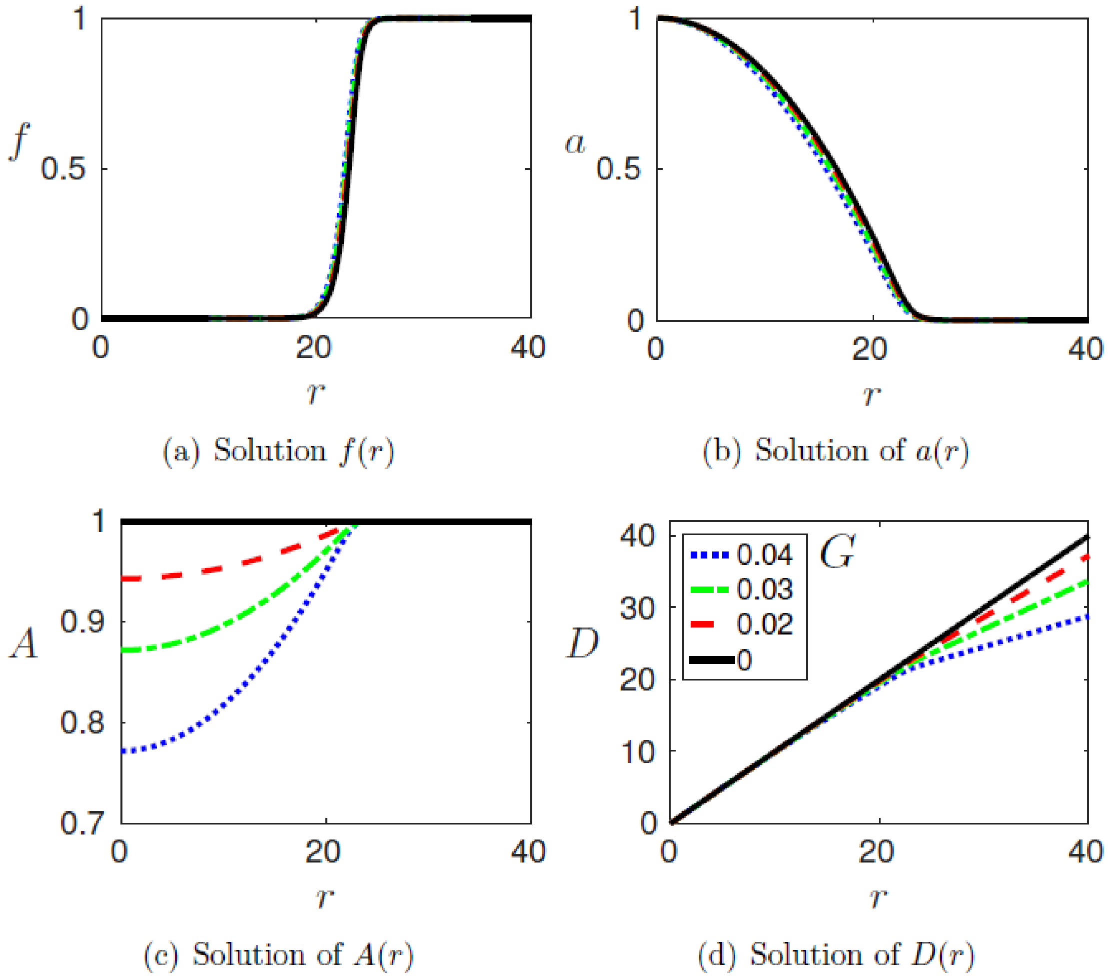
$$f(r) \to 0, \;\; a(r) \to 1, \quad A'(r) \to 0, \; D(r) \to 0, \;\; D'(r) \to 1, \qquad \text{as} \;\; r \to 0,$$
$$f(r) \to 1, \;\; a(r) \to 0, \;\; A(r) \to 1, \qquad \text{as} \;\; r \to \infty.$$



(a) Solution of $f(r)$ (b) Solution of $a(r)$

(c) Solution of $A(r)$ (d) Solution of $D(r)$

Figure 2: Thick-wall solutions of $f(r)$, $a(r)$, $A(r)$ and $D(r)$ respectively. $n = e/\sqrt{4\pi} = 1$, $\epsilon = 0.1$. The line patterns (——— , — — , — - , - -) respectively represent $G = (0, 0.1, 0.15, 0.2)$.



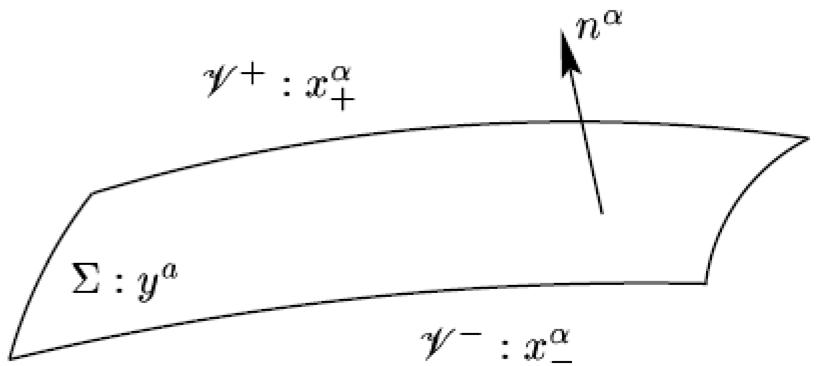
(a) Solution $f(r)$ (b) Solution of $a(r)$

(c) Solution of $A(r)$ (d) Solution of $D(r)$

Figure 3: Thin-wall solutions of $f(r)$, $a(r)$, $A(r)$, and $D(r)$ respectively. $n = 50$, $e/\sqrt{4\pi} = 1$, $\epsilon = 0.005$. The line patterns (——— , — — , — – , – –) respectively represent $G = (0, 0.02, 0.03, 0.04)$.

4. Israel junction conditions

We consider a thin wall or hypersurface Σ partitioning bulk spacetime into two regions $\mathscr{V}^+$ and $\mathscr{V}^-$, with boundaries Σ^+ and Σ^-.



Two regions of spacetime joined at a common boundary.

The bulk spacetime geometry for the inside (−) and outside (+) of the wall have below metric

$$ds_{\pm}^2 = -A_{\pm}^2(r)dt^2 + B_{\pm}^{-2}(r)dr^2 + D_{\pm}^2(r)d\theta^2$$

We take the energy-momentum tensor as the form

$$T^{\mu\nu} = S^{\mu\nu}\delta(\eta) + \text{regular term}$$

where $S^{\mu\nu}(x^i, \eta = \bar{\eta})$ **is the stress-energy tensor of the wall**

$$S_{\mu\nu} = \lim_{\epsilon\to 0}\int_{\bar{\eta}-\epsilon}^{\bar{\eta}+\epsilon} T_{\mu\nu}d\eta.$$

$$h^{\eta\eta} = h^{\eta i} = 0.$$

The extrinsic curvature has only two components, K_θ^θ **and** K_τ^τ**. The form of the stress-energy tensor on the wall could be obtained by the covariant conservation.**

We introduce the Gaussian normal coordinate system near the wall

$$dS^2 = -d\tau^2 + d\eta^2 + \bar{R}^2(\tau,\eta)d\theta^2$$

where $g_{\tau\tau} = -1$ **and** $\bar{R}(\tau,\bar{\eta}) = R(\tau)$**. It must agree with the coordinate *R* of the interior and exterior coordinate systems. In this coordinate system the induced metric on the hyperspace**

$$ds^2_{(\Sigma)} = -d\tau^2 + R(\tau)^2 d\theta^2$$

where τ **is the proper time measured by an observer at rest with respect to the wall and** $R(\tau)$ **is the proper radius of** $\Sigma.$ **The following relation is satisfied**

$$-d\tau^2 = -A^2dt^2 + B^{-2}dr^2 \quad \Rightarrow \quad (-A^2\dot{t}^2 + B^{-2}\dot{r}^2 = -1),$$

where $\cdot$ **denotes the differentiation with respect to** τ **.**

The induced metric of the hypersurface

$$h_{ab} = g_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b}$$

We take the tangent vectors

$$e^{\alpha}_{\tau} = (\dot{T}, \dot{R}, 0), \qquad e^{\alpha}_{\theta} = (0, 0, RD^{-1})$$

The three-velocity of any point on the wall

$$u^{\alpha} = (\dot{T}, \dot{R}, 0), \qquad u_{\alpha} = (-A^2 \dot{T}, B^{-2} \dot{R}, 0),$$

Then the relation $u_{\alpha} u^{\alpha} = -1$ **is satisfied.**

We take the normal vectors

$$n^{\alpha} = B^{-1} A (A^{-2} \dot{R}, B^2 \dot{T}, 0), \qquad n_{\alpha} = B^{-1} A (-\dot{R}, \dot{T}, 0)$$

where we take the factor $B^{-1} A$ **to normalize the vectors, so that** $n_{\alpha} n^{\alpha} = 1.$ **The hypersurface is timelike.**

We take outside geometry as the flat Minkowski spacetime minus a wedge related to the deficit angle

$$ds^2_{(+)} = -dt^2 + dr^2 + (1-\Delta)r^2 d\theta^2.$$

We employ the insider geometry as the magnetic solution in the anti-de Sitter spacetime

Hirshmann & Welch PRD53, 5579 (1996)

$$\begin{aligned} ds^2_{(-)} &= -N(r)dt^2 + \frac{L dr^2}{N(r)J(r)} + \frac{r^2 J(r)}{L} d\theta^2, \\ N(r) &\equiv (1 + L\Lambda r^2), \quad J(r) \equiv [1 + \left(Q_m^2/(Lr^2)\right)\ln\left(1 + L\Lambda r^2\right)], \quad L \equiv \left(1 + Q_m^2\Lambda\right), \end{aligned}$$

where Q_m represents the magnetic charge

The scalar magnetic field measured in an orthonormal basis is $\mathcal{B} = Q_m \Lambda / \sqrt{G(1 + L\Lambda r^2)}$

We change the metric into the following because two geometries do not have the proper circumferential radius. The outside geometry takes the form

$$ds^2_{(+)} = -dt^2 + \frac{d\tilde{r}^2}{(1-\Delta)} + \tilde{r}^2 d\theta^2$$

and the insider geometry takes the form

$$ds^2_{(-)} = -F^2(\tilde{r})dt^2 + \frac{(dr/d\tilde{r})^2}{E^2(\tilde{r})}d\tilde{r}^2 + \tilde{r}^2 d\theta^2$$

$$\tilde{r} = r\sqrt{\frac{J(r)}{L}}, \quad d\tilde{r} = \frac{\sqrt{K(r)}}{\sqrt{L}\sqrt{J(r)}}dr, \quad K(r) \equiv \left(1 + \frac{Q_m^2 \Lambda}{1 + L\Lambda r^2}\right)^2$$

Finally, we obtain

$$\frac{1}{2}\dot{\tilde{R}}^2 + V_{\text{eff}}(\tilde{R}) = 0,$$

$$V_{\text{eff}}(\tilde{R}) = \frac{(1-\Delta)}{2} - \frac{(E^2(d\tilde{R}/dR)^2 - (1-\Delta) - \kappa^2\sigma^2\tilde{R}^2)^2}{8\kappa^2\sigma^2\tilde{R}^2}$$

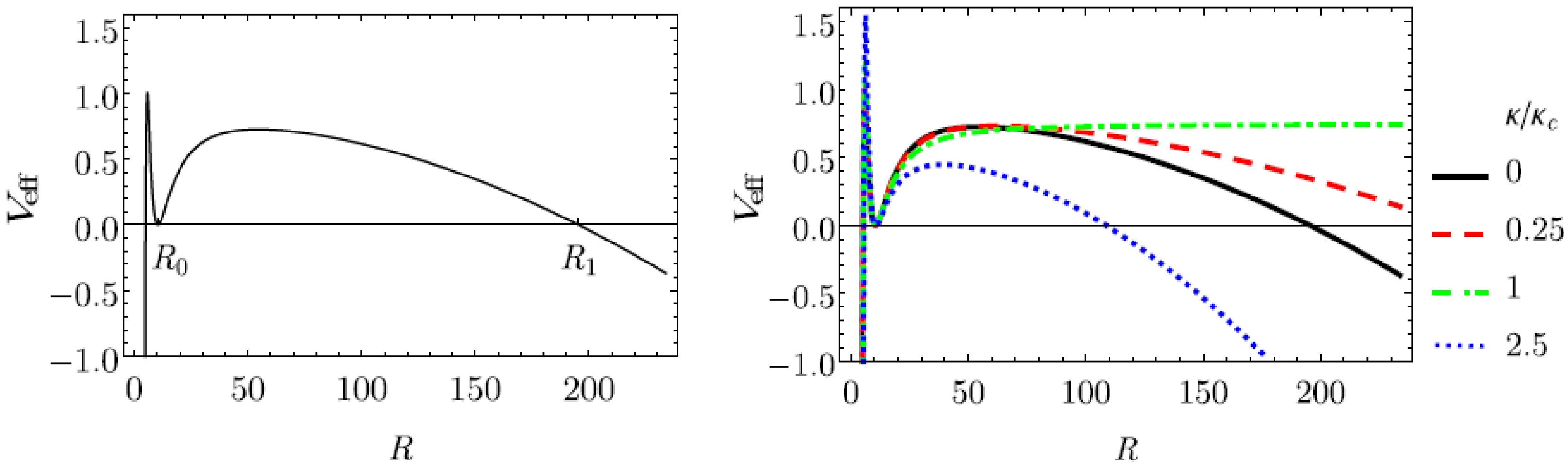
The thin wall is located at position $r = R$ (or $\tilde{R} = \tilde{r}\big|_{r=R}$)

We express everything as a function of r

$$d\tilde{R} = \frac{\sqrt{K(R)}dR}{\sqrt{L(R)}\sqrt{J(R)}}, \quad K(R) \equiv \left(1 + \frac{Q_m^2\Lambda}{1 + L\Lambda R^2}\right)^2$$

$$\frac{1}{2}\dot{R}^2 + V_{\text{eff}}(R) = 0$$

$$V_{\text{eff}}(R) = \frac{L(R)J(R)}{K(R)} \times \left\{ \frac{(1-\Delta)}{2} - \frac{\left[\frac{N(R)K(R)}{L^2(R)} - (1-\Delta) - \frac{\kappa^2\sigma^2R^2J(R)}{L(R)}\right]^2}{8\kappa^2\sigma^2R^2J(R)/L(R)} \right\}$$



(a) Potential turning points R_0 and R_1 for $\kappa = 0$.

(b) The line patterns (——— , — — , — – , – –) respectively represent $\kappa = (0, 0.5, 1, 2.5)\kappa_c$. The critical value of κ is $\kappa_c \equiv \epsilon/\sigma^2$.

Figure 5: Effective potential for $\Phi = 100$, $\epsilon = 0.001$ and multiple gravitational coupling κ.

We now evaluate Δ related to the deficit angle

$$\begin{aligned}\Delta &= 1 + 2\kappa\sigma\tilde{R}(d\tilde{R}/dR)\sqrt{E^2 + \dot{R}^2} - E^2(d\tilde{R}/dR)^2 - \kappa^2\sigma^2\tilde{R}^2 \\ &= 1 + 2\kappa\sigma R\frac{\sqrt{K}}{L}\sqrt{\frac{NJ}{L} + \dot{R}^2} - \frac{NK}{L^2} - \kappa^2\sigma^2 R^2\frac{J}{L}\end{aligned}$$

$$\begin{aligned}\frac{\sqrt{K}}{L} &= 1 - \frac{G\Phi^2}{\pi^2 R^2} + \mathcal{O}(G^2), \\ \frac{NJ}{L} &= 1 + 8\pi G\epsilon R^2 + \frac{G\Phi^2}{2\pi^2 R^2} + \mathcal{O}(G^2), \\ \frac{NK}{L^2} &= 1 + 8\pi G\epsilon R^2 - \frac{G\Phi^2}{\pi^2 R^2} + \mathcal{O}(G^2).\end{aligned}$$

$$\Delta \equiv 8G\mu = 8G\left(\frac{\Phi^2}{8\pi^2 R^2} + 2\pi\sigma R\sqrt{1 + \dot{R}^2} - \epsilon\pi R^2\right) + \mathcal{O}(G^2),$$

In non-relativistic limit $\dot{r} \ll 1$, the energy of the vortex is

$$\mu \simeq \frac{\Phi^2}{8\pi^2 R^2} + 2\pi\sigma R\left(1 + \frac{1}{2}\dot{R}^2\right) - \epsilon\pi R^2,$$

We now examine whether the gravitational effect on the surface tension exists or not. We ignore the contribution from the negligible magnetic flux on the wall. The energy density on the wall is given by

$$\begin{aligned}
\sigma &= \frac{\int_0^{2\pi}\int_{\bar{r}-\varepsilon/2}^{\bar{r}+\varepsilon/2}\sqrt{\det g_{ij}}d^2x[g^{rr}f'^2+(f^2-\epsilon)(f^2-1)^2]}{\int_0^{2\pi}\sqrt{g_{\theta\theta}}d\theta} \\
&= \int_{\bar{r}-\varepsilon/2}^{\bar{r}+\varepsilon/2}\sqrt{g_{rr}}dr[g^{rr}f'^2+(f^2-\epsilon)(f^2-1)^2] \\
&\simeq 1/2+\mathcal{O}(\epsilon)
\end{aligned}$$

Decay rates : in preparation

5. Summary and discussions

We study the decay of vortices trapped in the false vacuum of a theory of scalar electrodynamics in 2+1 dimensions.

When n is large there exists thin-wall vortices solutions. The false vortices are quantum mechanically unstable.

We study the model with gravitation. The dynamics of the vortex with gravitation can be estimated by employing Israel junction condition.