Study of GSLT in Curvature-Matter Coupling Gravity

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Outline

- Classical and BH Thermodynamics
- Modified Gravity
- Thermodynamics and Modified Gravity
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Classical and BH Thermodynamics

Classical Thermodynamics

Thermodynamics is one of the fundamental branches of science.

Interrelationship among energy, heat and work.

Zeroth Law:

If system A is brought into contact with system B and there is no change,
when system B is brought into contact with C there is no change, then
when system A is brought into contact with C there will be no change.

If $T_A > T_B$ and $T_B > T_C$ then $T_A > T_C$ and heat flows from A to B to C.

First Law:

If in an isolated system heat is allowed in, dQ, there will be a rise in temperature, dT, which is directly proportional to dQ.

If heat is pumped out, $dQ < 0$, then $dT < 0$.

law of conservation of energy.

the total amount of energy in a physical system remains constant, satisfy the relation

$$dE = TdS + dW.$$

Second Law:

In any process the net change of entropy of an isolated system is always non-negative, $\Delta S > 0$ commonly known as law of increased entropy.

This law states that the entropy of an isolated/closed system always increases or remains constant.

Third Law:

Infinitely many steps are required to reach absolute zero of temperature by heat pump.

First Version: Not possible for a physical system to have an absolute zero temperature in a finite number of physical processes.

Second Version: If temperature approaches to absolute zero, then entropy approaches to zero.

Black hole is a region with gravity so strong not even light escapes.

- Black hole thermodynamics provides the interrelationship among laws of thermodynamics and BH mechanics.
- In GR, BHs obey certain laws which have mathematical resemblance with ordinary laws of thermodynamics.

- In classical theory, they are perfect absorbers and do not emit anything yielding absolute zero temperature.
- In quantum theory, they emit Hawking radiation with a perfect thermal spectrum.
- First law of thermodynamics is energy conservation and hence carries over to black holes.

Black Hole Thermodynamics

- Bekenstein proposed that

$$S_{bh} \sim A_{bh} \sim m_{bh}^2.$$

- If entropy is the black hole area, temperature will have to be the surface gravity so that the product contributes to the total internal energy.

- Now surface gravity $\sim m_{bh}/r_{bh}^2$ and $r_{bh} \sim m_{bh}$, so surface gravity $\sim 1/m_{bh}$.

Thus, the larger the black hole the lower the canonical temperature

$$T_{bh} \sim 1/m_{bh}.$$

Zeroth Law: Surface gravity κ of a stationary BH is constant over the event horizon.

$$T \propto \kappa$$

The temperature T is constant for a system in thermal equilibrium.

First Law: Relates the energy difference of two nearby stationary BH equilibrium states to that in the area of event horizon A, in the angular momentum J and in the charge Q.

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \phi dQ.$$

Ω ↓ Angular velocity

ϕ ↓ Electric potenti[al] of the horizon

This is for the rotating charged BH.

$\underline{\Omega dJ + \phi dQ}$ - the work done on BH by an external agent which increases BH angular momentum and charge by dJ and dQ.

If stationary matter (other than magnetic field) is present outside the BH then there are additional terms on the right side.

$$dE = TdS + \Omega dJ + \phi dQ$$
$$= TdS + \text{"Work Term"}.$$

Second Law: The surface area of BH can never decrease in any process

$$dA \geq 0$$

A quite analogous statement is "total entropy of a closed system never decreases in any process"

$$dS \geq 0$$

This statement is not valid once quantum mechanics comes into play, because the Hawking radiation carries off the mass and therefore reduces the surface of a BH

Generalized Second Law: The total entropy being the sum of BH entropy and entropy of whatever may be present outside the BH is increasing.

$$dS_{tot} = dS_{BH} + dS_{Outside} \geq 0$$

Third Law: The most acceptable statement for third law in BHs is

"The BH temperature cannot be reduced to zero by a finite sequence of operations."

However, there was no equivalent of the third law to say that it would take an infinite number of steps to get to zero temperature.

The discovery of black hole thermodynamics sets up a significant connection between gravity and thermodynamics.

[Bardeen, Carter and Hawking: Commun. Math. Phys. **31**(1973)161]
[Hawking, S.W.: Commun. Math. Phys.**43**(1975)199]
[Bekenstein, J.D.: Phys. Rev. D **7**(1973)2333]

Correspondence

Laws of thermodynamics	Laws of BH mechanics
$T =$ constant	$\kappa =$ constant
$dE = TdS + dW$	$dM = \frac{\kappa}{8\pi}dA + \Omega dJ + \phi dQ$
$dS \geq 0$	$dA \geq 0$
$T \geq 0$	$\kappa \geq 0$

Modified Gravity

Modified Theories of Gravity:

There exist a family of modified theories of gravity that have been the subject of great interest in cosmology and provide a convincing way for settling the issue of late-time acceleration, like

- $f(R)$ theory
- $f(R, T)$ theory
- Gauss-Bonnet theory
- $f(\mathcal{G})$ theory

- $f(\mathcal{G}, T)$ theory

Minimal and Non-minimal Coupling:

MGTs are constructed by incorporating the geometric part whereas matter contribution is considered as additional term in Lagrangian. One can put further modification by introducing direct coupling between matter and curvature components; such theory is named as non-minimally coupled gravity.

$f(\mathcal{G},T)$ Theory of Gravity

Action:

$$\mathcal{I}=\int\sqrt{-g}\left(\frac{R+f(\mathcal{G},T)}{16\pi G}+\mathcal{L}_m\right)d^4x.$$

$f(\mathcal{G},T)$ Field Equations:

Variation of the action with respect to $g_{\mu\nu}$ gives the fourth-order field equations as

$$
\begin{aligned}
R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R &= \frac{1}{2}g_{\mu\nu}f(\mathcal{G},T)-2RR_{\mu\nu}f_{\mathcal{G}}(\mathcal{G},T)+4R^{\xi}_{\mu}R_{\xi\nu}f_{\mathcal{G}}(\mathcal{G},T)\\
&+ 4R_{\mu\xi\nu\eta}R^{\xi\eta}f_{\mathcal{G}}(\mathcal{G},T)-2R^{\xi\eta\delta}_{\mu}R_{\nu\xi\eta\delta}f_{\mathcal{G}}(\mathcal{G},T)-2Rg_{\mu\nu}\\
&\times \nabla^{2}f_{\mathcal{G}}(\mathcal{G},T)+4R_{\mu\nu}\nabla^{2}f_{\mathcal{G}}(\mathcal{G},T)+2R\nabla_{\mu}\nabla_{\nu}f_{\mathcal{G}}(\mathcal{G},T)\\
&- 4R^{\xi}_{\nu}\nabla_{\mu}\nabla_{\xi}f_{\mathcal{G}}(\mathcal{G},T)-4R^{\xi}_{\mu}\nabla_{\nu}\nabla_{\xi}f_{\mathcal{G}}(\mathcal{G},T)+4g_{\mu\nu}R^{\xi\eta}\\
&\times \nabla_{\xi}\nabla_{\eta}f_{\mathcal{G}}(\mathcal{G},T)-4R_{\mu\xi\nu\eta}\nabla^{\xi}\nabla^{\eta}f_{\mathcal{G}}(\mathcal{G},T)-(T_{\mu\nu}+\Theta_{\mu\nu})\\
&\times f_{T}(\mathcal{G},T)+8\pi GT_{\mu\nu},
\end{aligned}
$$

$$f_{\mathcal{G}}(\mathcal{G},T) = \frac{\partial f(\mathcal{G},T)}{\partial \mathcal{G}},$$

$$f_{T}(\mathcal{G},T) = \frac{\partial f(\mathcal{G},T)}{\partial T}, \quad \nabla^2 = \nabla_\mu \nabla^\mu$$

$$\Theta_{\mu\nu} = -2T_{\mu\nu} + g_{\mu\nu}\mathcal{L}_m - 2g^{\xi\eta}\frac{\partial^2 \mathcal{L}_m}{\partial g^{\mu\nu}\partial g^{\xi\eta}}.$$

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_m - 2\frac{\partial \mathcal{L}_m}{\partial g^{\mu\nu}}.$$

Covariant derivative of field equations gives

$$\nabla^\mu T_{\mu\nu} = -\frac{f_T(\mathcal{G},T)}{8\pi G - f_T(\mathcal{G},T)}\left[\frac{1}{2}g_{\mu\nu}\nabla^\mu T - (\Theta_{\mu\nu} + T_{\mu\nu})\nabla^\mu \ln f_T(\mathcal{G},T) - \nabla^\mu \Theta_{\mu\nu}\right].$$

Conservation law does not hold in this gravity due to curvature-matter coupling.

The particular forms of $f(\mathcal{G},T)$ are

$$f(\mathcal{G},T) = f_1(\mathcal{G}) + f_2(T),$$
$$f(\mathcal{G},T) = f_1(\mathcal{G}) + f_2(\mathcal{G})f_3(T).$$

First choice is considered as correction to $f(\mathcal{G})$

gravity as it does not involve direct non-minimal curvature-matter coupling.

Second form implies direct coupling.

Energy-momentum tensor for perfect fluid

$$T_{\mu\nu} = (\rho + P)v_\mu v_\nu - P g_{\mu\nu},$$

$P,\ \rho,\ v_\mu$ - pressure, energy density and 4-velocity, satisfying $v^\xi v_\xi = 1$ and $v^\xi \nabla_\nu v_\xi = 0$.

For $\mathcal{L}_m = -P$, we have

$$\Theta_{\mu\nu} = -P g_{\mu\nu} - 2T_{\mu\nu}.$$

Field equations for dust case $(P=0)$ are

$$G_{\mu\nu}=8\pi\widetilde{G}T^{\rm eff}_{\mu\nu}=8\pi\widetilde{G}T_{\mu\nu}+T^{\rm (D)}_{\mu\nu},$$

$$\begin{aligned}
T^{\rm (D)}_{\mu\nu} &= \frac{1}{2}g_{\mu\nu}f(\mathcal{G},T)-2RR_{\mu\nu}f_{\mathcal{G}}(\mathcal{G},T)+4R^{\xi}_{\mu}R_{\xi\nu}f_{\mathcal{G}}(\mathcal{G},T)\\
&+ 4R_{\mu\xi\nu\eta}R^{\xi\eta}f_{\mathcal{G}}(\mathcal{G},T)-2R^{\xi\eta\delta}_{\mu}R_{\nu\xi\eta\delta}f_{\mathcal{G}}(\mathcal{G},T)-2Rg_{\mu\nu}\nabla^2f_{\mathcal{G}}(\mathcal{G},T)\\
&+ 4R_{\mu\nu}\nabla^2f_{\mathcal{G}}(\mathcal{G},T)+2R\nabla_{\mu}\nabla_{\nu}f_{\mathcal{G}}(\mathcal{G},T)-4R^{\xi}_{\nu}\nabla_{\mu}\nabla_{\xi}f_{\mathcal{G}}(\mathcal{G},T)\\
&- 4R^{\xi}_{\mu}\nabla_{\nu}\nabla_{\xi}f_{\mathcal{G}}(\mathcal{G},T)+4g_{\mu\nu}R^{\xi\eta}\nabla_{\xi}\nabla_{\eta}f_{\mathcal{G}}(\mathcal{G},T)\\
&- 4R_{\mu\xi\nu\eta}\nabla^{\xi}\nabla^{\eta}f_{\mathcal{G}}(\mathcal{G},T),\\
\widetilde{G} &= GF,\quad F=1+\frac{f_T(\mathcal{G},T)}{8\pi G}.
\end{aligned}$$

FRW universe model

$$ds^2 = dt^2 - \frac{a^2(t)}{1-kr^2}dr^2 - \hat{r}^2 d\theta^2 - \hat{r}^2 \sin^2\theta d\phi^2,$$

$\hat{r} = a(t)r, \quad a(t)$ - scale factor

k - spatial curvature parameter:

open $(k=-1)$, closed $(k=1)$ and flat $(k=0)$.

GB invariant takes the form

$$\mathcal{G} = 24(H^2 + \dot{H})\left(H^2 + \frac{k}{a^2}\right).$$

Field equations

$$
\begin{aligned}
3\left(H^2+\frac{k}{a^2}\right) &= 8\pi\widetilde{G}\rho+\frac{1}{2}f(\mathcal{G},T)-12(H^2+\dot{H})\left(H^2+\frac{k}{a^2}\right. \\
&\times\ f_{\mathcal{G}}(\mathcal{G},T)+12H\left(H^2+\frac{k}{a^2}\right)\big(f_{\mathcal{G}\mathcal{G}}(\mathcal{G},T)\dot{\mathcal{G}} \\
&+\ f_{\mathcal{G}T}(\mathcal{G},T)\dot{T}\big)\,, \\
-\left(2\dot{H}+3H^2+\frac{k}{a^2}\right) &= -\frac{1}{2}f(\mathcal{G},T)+12(H^2+\dot{H})\left(H^2+\frac{k}{a^2}\right)f_{\mathcal{G}}(\mathcal{G}, \\
&-\ 8H(H^2+\dot{H})\left(f_{\mathcal{G}\mathcal{G}}(\mathcal{G},T)\dot{\mathcal{G}}+f_{\mathcal{G}T}(\mathcal{G},T)\dot{T}\right) \\
&-\ 4\left(H^2+\frac{k}{a^2}\right)\big(f_{\mathcal{G}\mathcal{G}\mathcal{G}}(\mathcal{G},T)\dot{\mathcal{G}}^2+2f_{\mathcal{G}\mathcal{G}T}(\mathcal{G},T)\dot{\mathcal{G}} \\
&+\ f_{\mathcal{G}TT}(\mathcal{G},T)\dot{T}^2+f_{\mathcal{G}\mathcal{G}}(\mathcal{G},T)\ddot{\mathcal{G}}+f_{\mathcal{G}T}(\mathcal{G},T)\ddot{T}\big)\,,
\end{aligned}
$$

Using $H = \dot{a}/a$, field equations

$$3\left(H^2 + \frac{k}{a^2}\right) = 8\pi\widetilde{G}\rho_{_{\rm tot}} = 8\pi\widetilde{G}(\rho + \rho^{(\rm D)}), \qquad (1)$$

$$-2\left(\dot{H} - \frac{k}{a^2}\right) = 8\pi\widetilde{G}(\rho_{_{\rm tot}} + P_{_{\rm tot}}) = 8\pi\widetilde{G}(\rho + \rho^{(\rm D)} + P^{(\rm D)}), \qquad (2)$$

$\rho_{(\rm D)}$ and $P_{(\rm D)}$ are dark source terms

$$\begin{aligned}
\rho^{(\mathrm{D})} &= \frac{1}{8\pi\widetilde{G}F}\left[\frac{1}{2}f(\mathcal{G},T) - 12(H^2+\dot{H})\left(H^2+\frac{k}{a^2}\right)f_{\mathcal{G}}(\mathcal{G},T)\right.\\
&+ \left. 12H\left(H^2+\frac{k}{a^2}\right)\left(f_{\mathcal{G}\mathcal{G}}(\mathcal{G},T)\dot{\mathcal{G}} + f_{\mathcal{G}T}(\mathcal{G},T)\dot{T}\right)\right],\\
P^{(\mathrm{D})} &= \frac{1}{8\pi\widetilde{G}F}\left[-\frac{1}{2}f(\mathcal{G},T) + 12(H^2+\dot{H})\left(H^2+\frac{k}{a^2}\right)f_{\mathcal{G}}(\mathcal{G},T) - 8H\right.\\
&\times (H^2+\dot{H})\left(f_{\mathcal{G}\mathcal{G}}(\mathcal{G},T)\dot{\mathcal{G}} + f_{\mathcal{G}T}(\mathcal{G},T)\dot{T}\right) - 4\left(H^2+\frac{k}{a^2}\right)\\
&\times \left(f_{\mathcal{G}\mathcal{G}\mathcal{G}}(\mathcal{G},T)\dot{\mathcal{G}}^2 + 2f_{\mathcal{G}\mathcal{G}T}(\mathcal{G},T)\dot{\mathcal{G}}\dot{T} + f_{\mathcal{G}TT}(\mathcal{G},T)\dot{T}^2 + f_{\mathcal{G}\mathcal{G}}(\mathcal{G},T)\right.\\
&+ \left.\left. f_{\mathcal{G}T}(\mathcal{G},T)\ddot{T}\right)\right].
\end{aligned}$$

The corresponding continuity equation becomes

$$\dot{\rho}+3H\rho=\frac{-1}{8\pi G+f_T(\mathcal{G},T)}\left[\frac{1}{2}\dot{\rho}f_T(\mathcal{G},T)+\rho\left(f_{\mathcal{G}T}(\mathcal{G},T)\dot{\mathcal{G}}+f_{TT}(\mathcal{G},T)\dot{T}\right)\right].$$

This holds in the absence of curvature-matter coupling for both $f(\mathcal{G})$ gravity and GR.

Thermodynamics and Modified Gravity

We check the validity of first and second laws of thermodynamics in $f(\mathcal{G},T)$ gravity for FRW universe.

First Law: Radius of apparent horizon

$$h^{\mu\nu}\partial_\mu\hat{r}\partial_\nu\hat{r}=0,$$

$$\Downarrow$$

$$\hat{r}_A=\frac{1}{\sqrt{H^2+\frac{k}{a^2}}}$$

$h_{\mu\nu}=\mathrm{diag}(1,\frac{-a^2}{1-kr^2})$ is a two-dimensional metric.

Wald entropy in $f(\mathcal{G},T)$ gravity is given by

$$\mathcal{S}=\frac{\mathcal{A}}{4GF}\left(1-\frac{4}{\hat{r}_A^2}f_{\mathcal{G}}(\mathcal{G},T)\right).$$

This corresponds to $f(\mathcal{G})$ gravity for $F=1$, corresponds to GR for $f_{\mathcal{G}}=0$.
Its differential yields

$$\frac{1}{2\pi\hat{r}_A}d\mathcal{S}=4\pi\left(\rho_{_{\rm tot}}+P_{_{\rm tot}}\right)\hat{r}_A^3Hdt-\frac{2}{\hat{r}_AGF}df_{\mathcal{G}}+\frac{\hat{r}_A}{2G}\left(1-\frac{4}{\hat{r}_A^2}\;f_{\mathcal{G}}\right)d\left(\frac{1}{F}\right). \quad (3)$$

Temperature on the apparent horizon is

$$\mathcal{T} = \frac{|\kappa_{sg}|}{2\pi}. \tag{4}$$

Surface gravity (κ_{sg}) is

$$\kappa_{sg} = \frac{1}{2\sqrt{-h}}\partial_\mu\left(\sqrt{-h}h^{\mu\nu}\partial_\nu\hat{r}\right) = \frac{1}{2}\hat{r}_A\left(\frac{k}{a^2} + H^2 + \dot{H}\right). \tag{5}$$

Using Eqs.(3)-(5), we have

$$\begin{aligned}\mathcal{T}d\mathcal{S} &= 4\pi\left(\rho_{\rm tot}+P_{\rm tot}\right)\hat{r}_A^3Hdt-2\pi\left(\rho_{\rm tot}+P_{\rm tot}\right)\hat{r}_A^2d\hat{r}_A-\frac{4\pi\mathcal{T}}{GF}df_{\mathcal{G}}\\ &+ \frac{\pi}{G}\hat{r}_A^2\mathcal{T}\left(1-\frac{4}{\hat{r}_A^2}f_{\mathcal{G}}\right)d\left(\frac{1}{F}\right). \qquad (6)\end{aligned}$$

Total energy inside the apparent horizon

$$E=\mathcal{V}\rho_{\rm tot}=\frac{4}{3}\pi\hat{r}_A^3\rho_{\rm tot}=\frac{3\mathcal{V}}{8\pi\widetilde{G}}\left(H^2+\frac{k}{a^2}\right).$$

This shows that E is directly related to $\hat{r}_A$

$$dE=4\pi\rho_{\rm tot}\hat{r}_A^2d\hat{r}_A-4\pi\left(\rho_{\rm tot}+P_{\rm tot}\right)\hat{r}_A^3Hdt+\frac{\hat{r}_A}{2G}d\left(\frac{1}{F}\right). \qquad (7)$$

Using Eqs.(6) and (7), it follows that

$$\mathcal{T}d\mathcal{S} = -dE + Wd\mathcal{V} - \frac{4\pi\mathcal{T}}{GF}df_{\mathcal{G}} + \frac{\hat{r}_A}{2G}\left[1 + 2\pi\hat{r}_A\mathcal{T}\left(1 - \frac{4}{\hat{r}_A^2}f_{\mathcal{G}}\right)\right]d\left(\frac{1}{F}\right.$$

$W = (\rho_{\text{tot}} - P_{\text{tot}})/2$ - work done by the system.

The above equation can be written as

$$\mathcal{T}(d\mathcal{S} + d_i\mathcal{S}) = -dE + Wd\mathcal{V}, \qquad (8)$$

$$d_i\mathcal{S} = \frac{4\pi}{GF}df_{\mathcal{G}} - \frac{\hat{r}_A}{2G\mathcal{T}}\left[1 + 2\pi\hat{r}_A\mathcal{T}\left(1 - \frac{4}{\hat{r}_A^2}f_{\mathcal{G}}\right)\right]d\left(\frac{1}{F}\right),$$

- Non-equilibrium picture implies that there is some energy change inside and outside the apparent horizon.

- Non-equilibrium thermodynamical behavior at the apparent horizon leads to entropy production term.

- Due to this extra term, the field equations do not obey the first law of thermodynamics $dE = \mathcal{T}d\mathcal{S} + Wd\mathcal{V}$ in this gravity.

- In modified theories, this auxiliary term usually appears in the first law of thermodynamics while it is absent in GR, GB and Lovelock gravity.

Generalized Second Law:

This states that total entropy of the system is not decreasing in time given by

$$\dot{\mathcal{S}}+\dot{\mathcal{S}}_{\rm tot}+d_i\dot{\mathcal{S}}\geq 0, \tag{9}$$

$\mathcal{S}_{\rm tot}$ - entropy due to energy as well as all matter contents inside the horizon and $d_i\dot{\mathcal{S}}=\partial_t(d_i\mathcal{S})$.

Gibbs equation gives

$$\mathcal{T}_{\text{tot}}d\mathcal{S}_{\text{tot}}=d(\rho_{\text{tot}}\mathcal{V})+P_{\text{tot}}d\mathcal{V}, \tag{10}$$

$\mathcal{T}_{\text{tot}}$ - total temperature corresponding to all matter and energy contents inside the horizon. Not equal to the apparent horizon temperature. We assume

$$\mathcal{T}_{\text{tot}}=\zeta\mathcal{T},\quad 0<\zeta<1.$$

Total temperature inside the horizon is positive - smaller than the temperature at apparent horizon.

Using Eqs.(8) and (10) in (9), we obtain

$$\dot{\mathcal{S}}+\dot{\mathcal{S}}_{\text{tot}}+d_i\dot{\mathcal{S}}=\left(\frac{24+\hat{r}_A^4\mathcal{G}}{96\pi\zeta\hat{r}_A}\right)\Upsilon\geq 0, \qquad (11)$$

$$\Upsilon=(1-\zeta)\mathcal{V}\dot{\rho}_{\text{tot}}+(\rho_{\text{tot}}+P_{\text{tot}})\left(1-\frac{\zeta}{2}\right)\dot{\mathcal{V}}.$$

Using Eqs.(1) and (2), the GSLT condition gives

$$\left(\frac{24+\hat{r}_A^4\mathcal{G}}{192\pi\zeta GF}\right)\hat{r}_A^4\Xi\geq 0, \qquad (12)$$

$$\Xi = (2-\zeta)H\left(\dot{H}-\frac{k}{a^2}\right)^2+\frac{2(1-\zeta)H}{\hat{r}_A}\left(\dot{H}-\frac{k}{a^2}\right)+\frac{(1-\zeta)F}{\hat{r}_A^4}\partial_t\left(\frac{1}{F}\right).$$

GSLT is valid for $\mathcal{G}>0,\ F>0$ and $\Xi>0$.

For flat FRW universe model, the conditions $\mathcal{G}>0,\ F>0,\ H>0,\ \dot{H}>0$ and $\partial_t\left(\frac{1}{F}\right)>0$ must be satisfied to protect GSLT.

Equilibrium description of thermodynamics implies that the temperature inside and at the horizon are same yielding

$$\hat{r}_A^4 H \left(\dot{H} - \frac{k}{a^2}\right)^2 \left(\frac{24 + \hat{r}_A^4 \mathcal{G}}{192\pi\zeta G F}\right) \geq 0, \quad \zeta = 1.$$

Validity of GSLT can be obtained for positive values of H, $\mathcal{G}$ and F.

1. de Sitter Universe:

For this model, the Hubble parameter is constant $(H(t) = H_0)$ and scale factor grows exponentially as $a(t) = a_0 e^{H_0 t}$. For dust fluid,

$$\rho = \rho_0 e^{-3H_0 t}, \quad \mathcal{G} = 24H_0^4.$$

In this case, Eq.(12) takes the form

$$\frac{1 + a_0^4 H_0^4 (a_0^2 H_0^2 + ke^{-2H_0 t})}{\zeta(8\pi G + f_T)^2} \left[2kH_0 e^{-2H_0 t}(b-1)(8\pi G + f_T)\right.$$
$$\times \quad (a_0^2 H_0^4 + ke^{-2H_0 t})^{-1} - 3\rho_0 H_0 e^{-3H_0 t}(b-1) f_{TT} + k^2 H_0 e^{-4H_0 t}(2-b)$$
$$\times \quad \left.(8\pi G + f_T)(a_0^2 H_0^2 + ke^{-2H_0 t})^{-2}\right] \geq 0. \quad (13$$

Corresponding reconstructed $f(\mathcal{G},T)$ model is

$$f(\mathcal{G},T)=c_1c_2e^{c_1\mathcal{G}}T^{-\frac{1}{2}\left(\frac{1-24c_1H_0^4}{1-36c_1H_0^4}\right)}+c_1c_2T^{-\frac{1}{2}}-\frac{16\pi G}{3}T+6H_0^2,$$

Continuity constraint splits this model as

$$f_1(\mathcal{G},T) = \frac{18c_1^2c_2H_0^4(32c_1H_0^4-1)}{(1-36c_1H_0^2)^2}e^{c_1\mathcal{G}}T^{-\frac{1}{2}\left(\frac{1-24c_1H_0^4}{1-36c_1H_0^4}\right)}+6H_0^2, \quad (14)$$

$$f_2(\mathcal{G},T) = \frac{18c_1H_0^4(1-32c_1H_0^4)}{(1-24c_1H_0^4)(1-30c_1H_0^4)}\left(c_1c_2T^{-\frac{1}{2}}-\frac{16\pi G}{3}T\right)+6H_0^2. \quad (15)$$

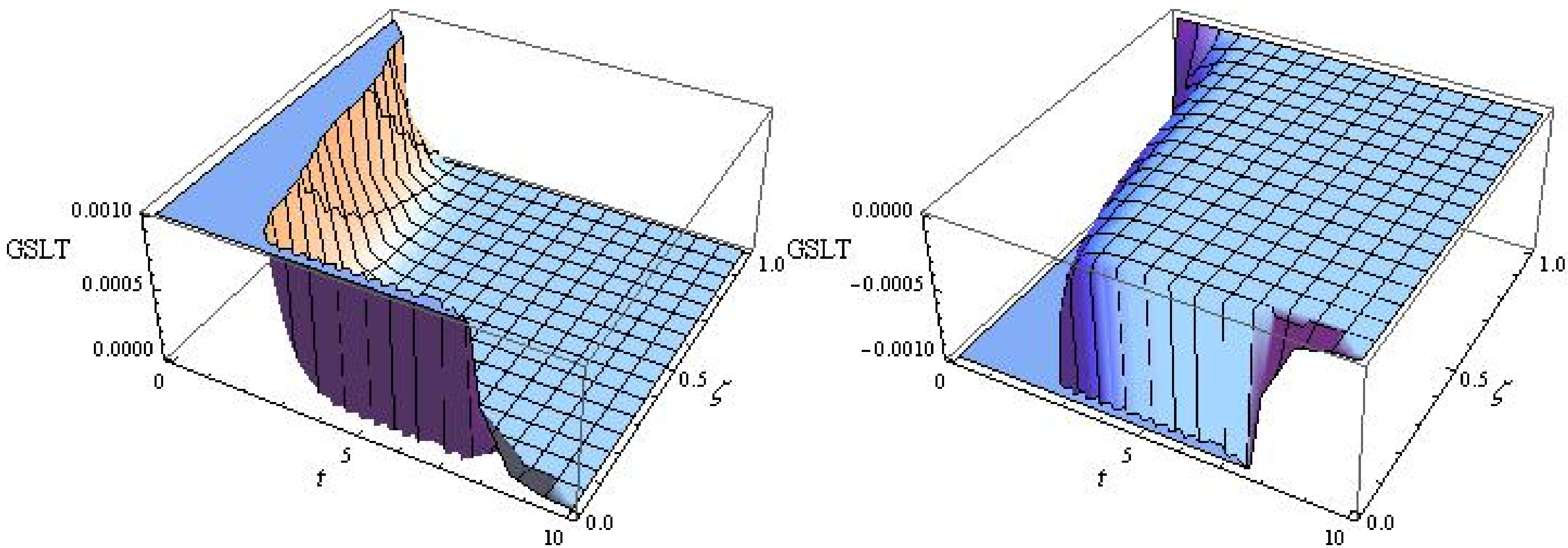

Figure 1: Validity of GSLT for the model (14). The left plot is for $c_1 = c_2 = 1$ and right for $c_1 = 1$ with $c_2 = -1$.

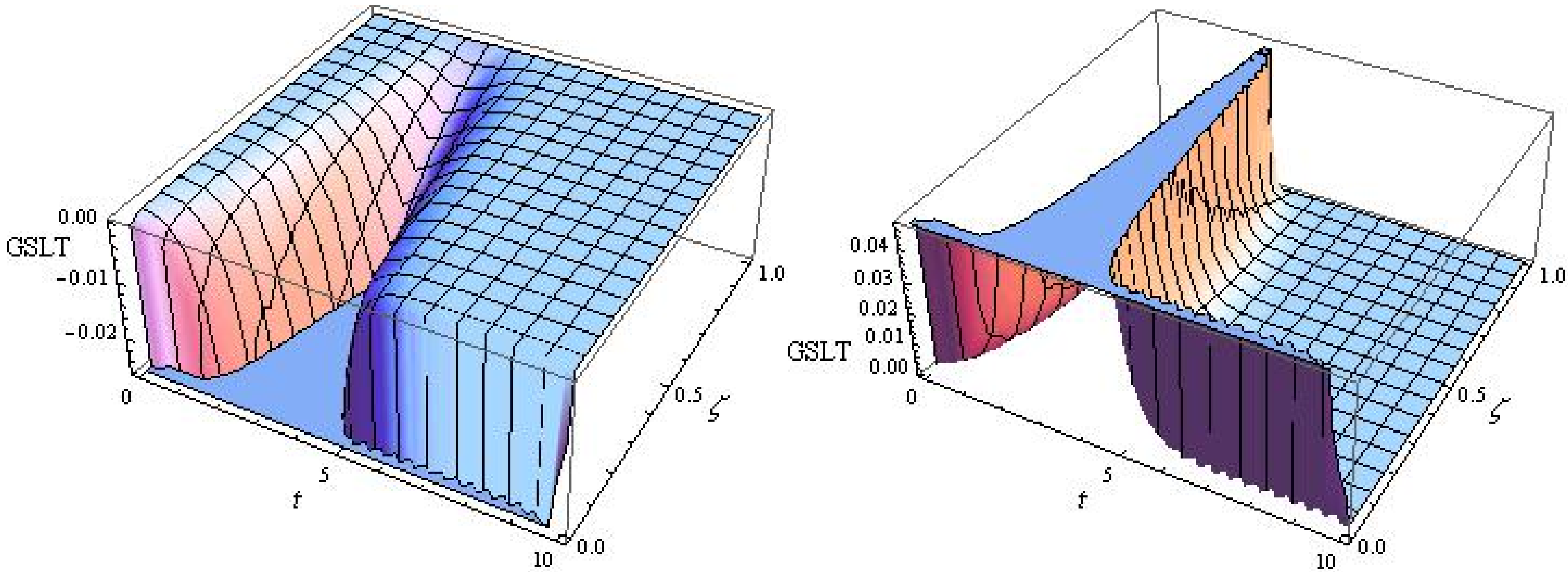

Figure 2: Validity of GSLT for the model (14). The left plot is for $c_1 = -1$ with $c_2 = 1$ and right for $c_1 = c_2 = -1$.

Figures **1** and **2** show the validity of GSLT for the model (14). We have four parameters $c_1,\ c_2,\ \zeta$ and t with $H_0 = 0.67,\ a_0 = 1$ and $\rho_0 = 0.3$.

GSLT holds at all times only for the same signatures of integration constants.

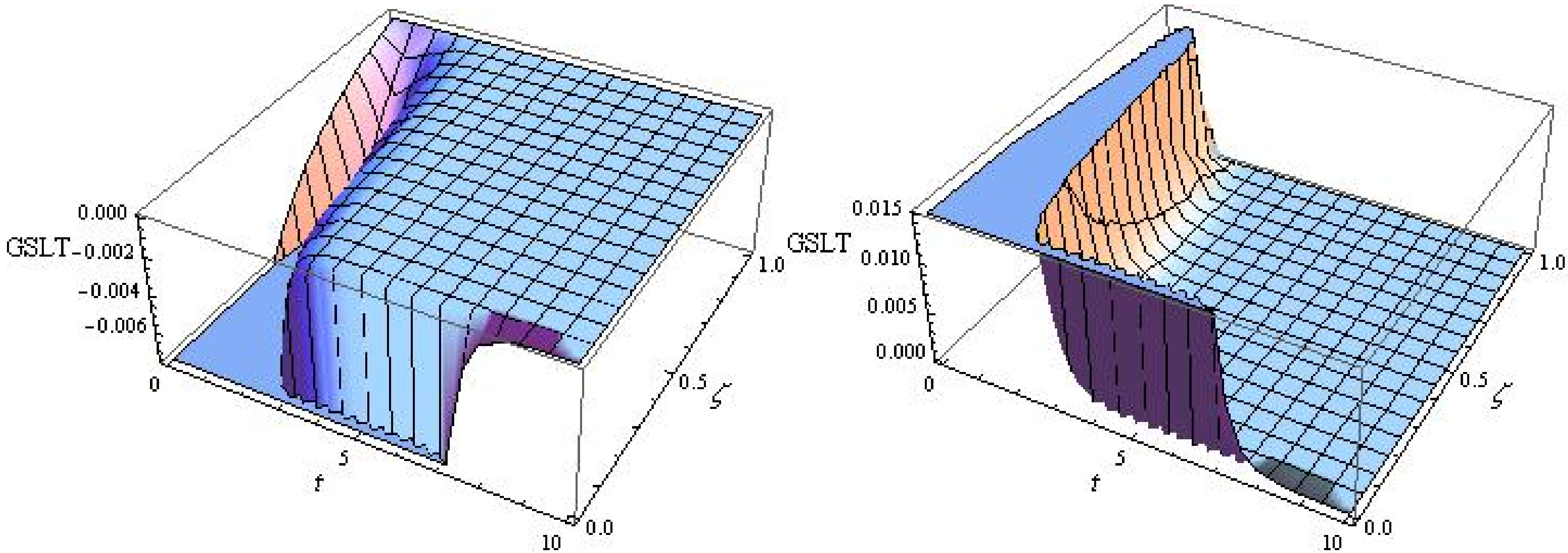

Figure 3: Validity of GSLT for the model (15). The left plot is for $c_1 = c_2 = 1$ and right for $c_1 = 1$ with $c_2 = -1$.

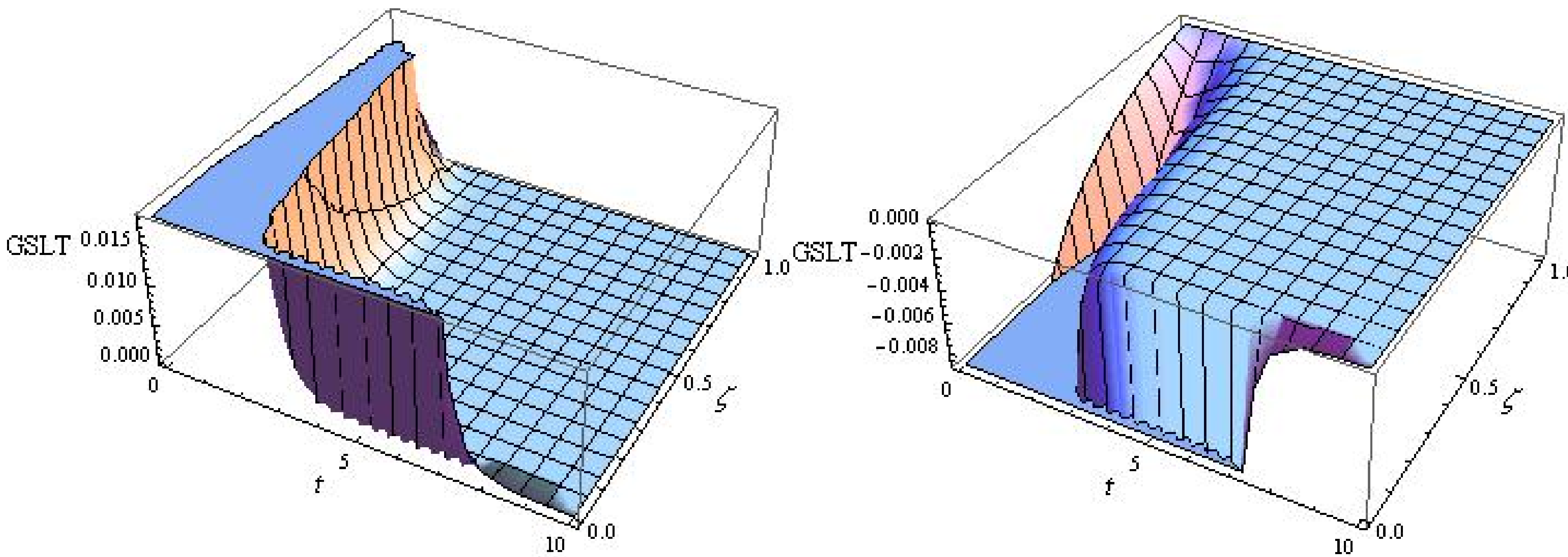

Figure 4: Validity of GSLT for the model (15). The left plot is for $c_1 = -1$ with $c_2 = 1$ and right for $c_1 = c_2 = -1$.

Figure **3** (right) and Figure **4** (left) show that GSLT is true for all considered values of ζ and t with opposite signatures of (c_1, c_2).

2. Power-law Solution:

Power-law solution is characterized by

$$a(t) = a_0 t^{\beta}, \quad H = \frac{\beta}{t}, \quad \beta > 0. \tag{16}$$

Accelerated phase is observed for $\beta > 1$ while $0 < \beta < 1$ covers the decelerated phase including dust ($\beta = \frac{2}{3}$) as well as radiation ($\beta = \frac{1}{2}$) dominated cosmic epochs.

The energy density and GB invariant become

$$\rho=\rho_0 t^{-3\beta},\quad \mathcal{G}=24\frac{\beta^3}{t^4}(\beta-1). \qquad (17)$$

Validity condition for GSLT takes the form

$$\begin{aligned}
&\frac{1+a_0^4\beta^3(\beta-1)t^{-4}(a_0^2\beta^2t^{-2}+kt^{-2\beta})^{-2}}{\zeta(8\pi G+f_T)}\left[\frac{\beta}{t}(2-\zeta)\left(\frac{\beta a_0^2t^{-2}+kt^{-2\beta}}{\beta^2a_0^2t^{-2}+kt^{-2\beta}}\right)^2\right.\\
&-\ 2\frac{\beta}{t}(1-\zeta)\left(\frac{\beta a_0^2t^{-2}+kt^{-2\beta}}{\beta^2a_0^2t^{-2}+kt^{-2\beta}}\right)-\frac{\zeta-1}{8\pi G+f_T}\left(96\frac{\beta^3}{t^5}(\beta-1)f_{\mathcal{G}T}\right.\\
&+\ \left.\left.3\beta\rho_0t^{-4}f_{TT}\right)\right]\geq 0.
\end{aligned}$$

Reconstructed $f(\mathcal{G},T)$ model for dust fluid

$f(\mathcal{G},T) = d_1 d_3 T^{d_2}\mathcal{G}^{\frac{1}{4}(\alpha_1+\alpha_2)} + d_2 d_3 T^{d_2}\mathcal{G}^{\frac{1}{4}(\alpha_1-\alpha_2)} + d_1 d_2 T^{\alpha_3} + \alpha_4 T + \alpha_5 T$

$$\alpha_1 = \frac{1}{2}\left[5-\beta(1+3d_2)\right],$$

$$\alpha_2 = \Bigg[\frac{3}{4}\beta d_2\{3d_2\beta+2(\beta-1)-8\}+\frac{1}{4}(\beta-1)(\beta+7)+4+8d_2(\beta-$$

$$\alpha_3 = -\frac{1}{2}, \quad \alpha_4 = -\frac{16\pi G}{3}, \quad \alpha_5 = \left(\frac{18\beta^3}{3\beta+4}\right)\rho_0^{-\frac{2}{3\beta}}, \quad \alpha_6 = \frac{2}{3\beta},$$

implies that the conservation law holds.

Continuity constraint splits this model as

$$f_1(\mathcal{G},T) = d_1d_3\gamma_1T^{d_2}\mathcal{G}^{\frac{1}{4}(\alpha_1+\alpha_2)} + d_1d_2\gamma_2T^{\alpha_3} + \gamma_3T + \gamma_4T^{\alpha_6}, \quad (18)$$

$$f_2(\mathcal{G},T) = d_2d_3\gamma_5T^{d_2}\mathcal{G}^{\frac{1}{4}(\alpha_1-\alpha_2)} + d_1d_2\gamma_6T^{\alpha_3} + \gamma_7T + \gamma_8T^{\alpha_6}, \quad (19)$$

$$\gamma_1 = 1-\frac{\alpha_7}{\alpha_8}, \quad \gamma_2 = 1-\frac{\alpha_3^2}{\alpha_8}, \quad \gamma_3 = \alpha_4\left(1-\frac{1}{\alpha_8}\right), \quad \gamma_4 = \alpha_5\left(1-\frac{\alpha}{\alpha}\right.$$

$$\gamma_5 = 1-\frac{\alpha_8}{\alpha_7}, \quad \gamma_6 = 1-\frac{\alpha_3^2}{\alpha_7}, \quad \gamma_7 = \gamma_4\left(1-\frac{1}{\alpha_7}\right), \quad \gamma_8 = \gamma_5\left(1-\frac{\alpha_6^2}{\alpha_7}\right.$$

$$\alpha_7 = \frac{d_2}{6\beta}\left[6d_2\beta-3\beta+2(\alpha_1+\alpha_2)\right], \quad \alpha_8 = \frac{d_2}{6\beta}\left[6d_2\beta-3\beta+2(\alpha_1-\alpha\right.$$

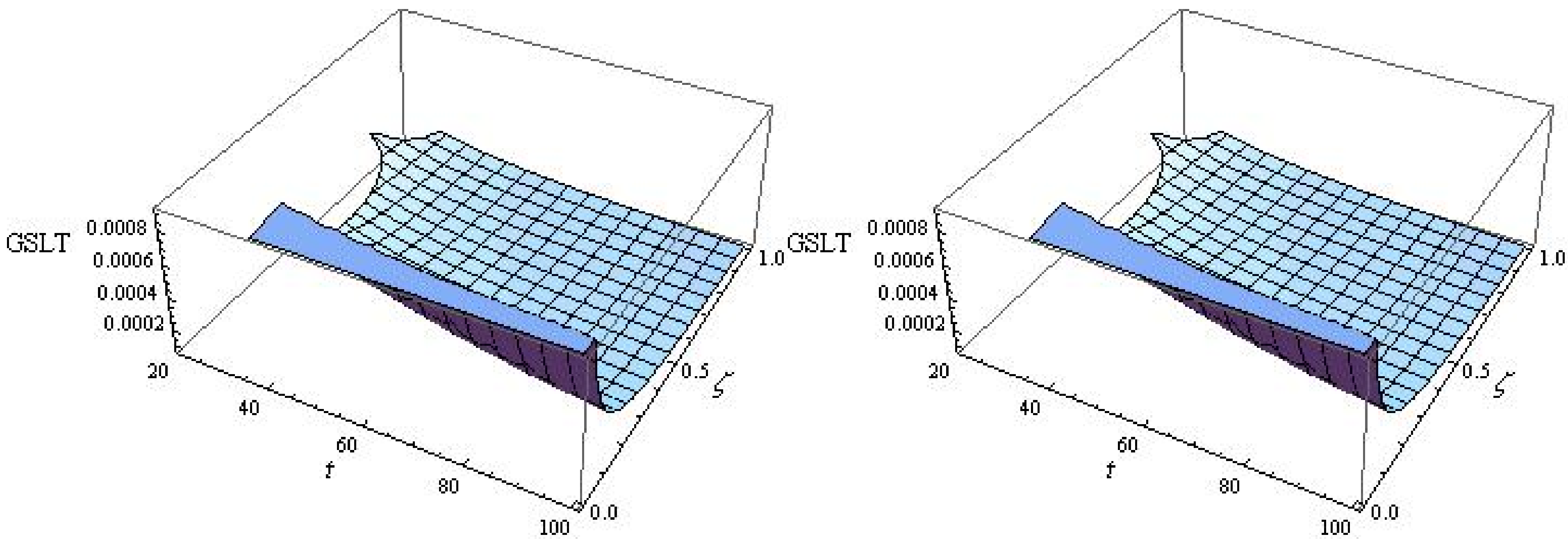

Figure 5: Validity of GSLT for the model (18). The left plot is for $d_1 = d_3 = 1$ and right for $d_1 = 1$ and $d_3 = -1$.

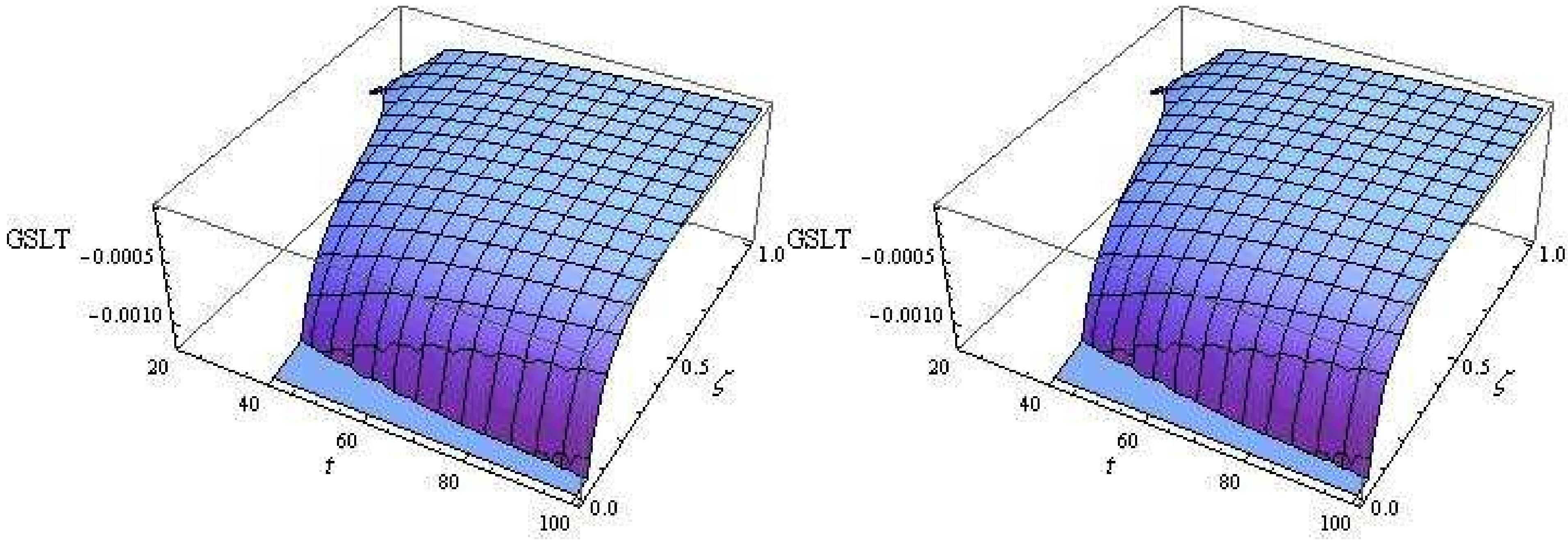

Figure 6: Validity of GSLT for the model (18). The left plot is for $d_1 = -1$ with $d_3 = 1$ and right for $d_1 = d_3 = -1$.

Validity of GSLT for models (18) and (19) depend on five parameters $d_1,\ d_2,\ d_3,\ \zeta$ and t.

We take $a_0 = 1,\ \rho_0 = 0.3,\ H_0 = 0.67,\ n = \frac{2}{3}$ and $d_2 = -1.64285$.

Figure **5** shows that GSLT is satisfied for both cases $d_3 > 0$ and $d_3 < 0$ with $d_1 > 0$.

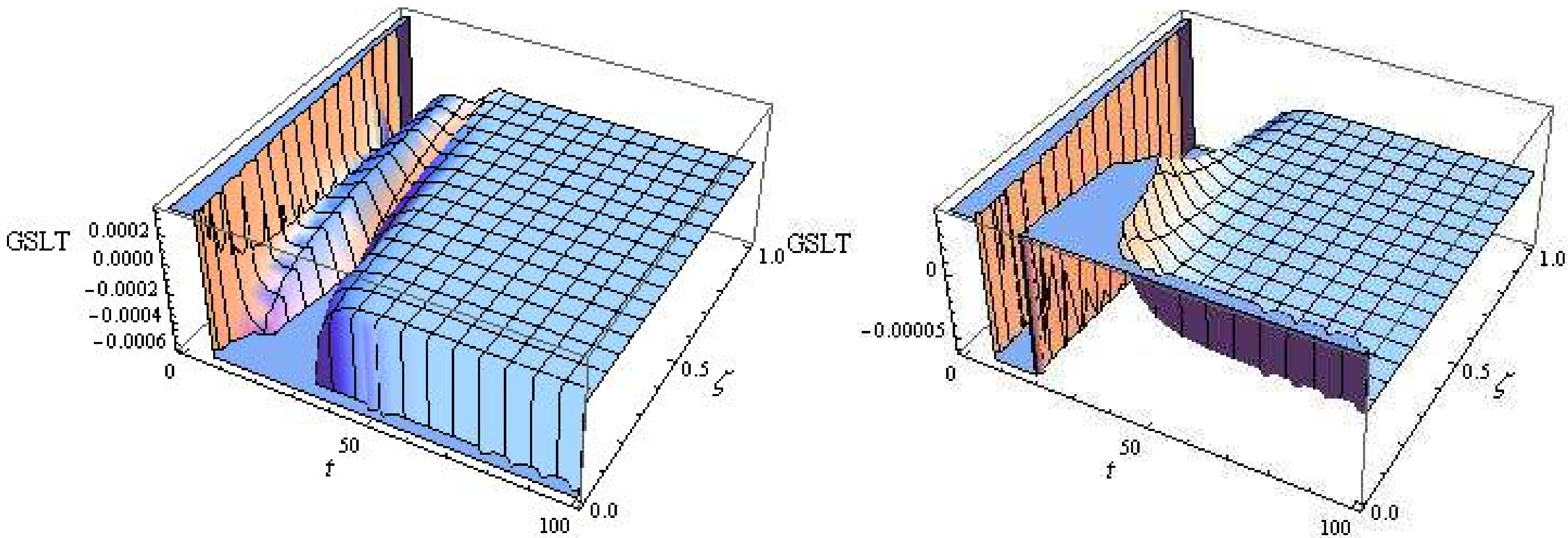

Figure 7: Validity of GSLT for the model (19). The left plot is for $d_1 = d_3 = 1$ and right for $d_1 = 1$ and $d_3 = -1$.

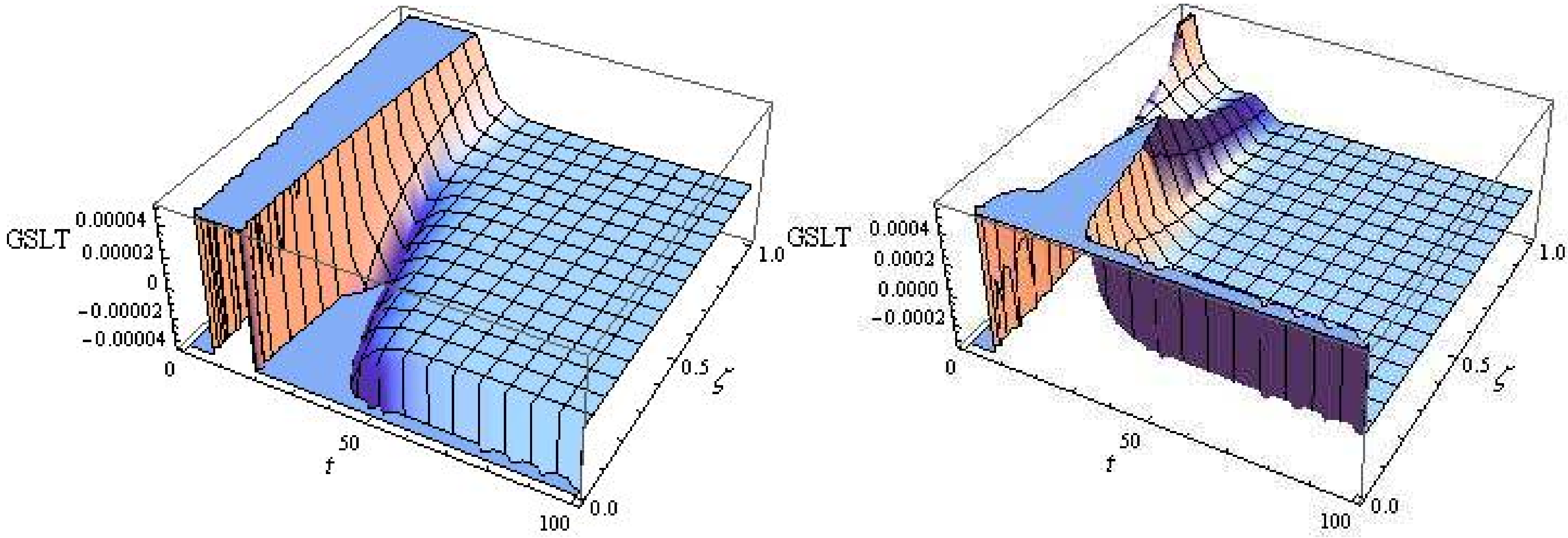

Figure 8: Validity of GSLT for the model (19). The left plot is for $d_1 = -1$ with $d_3 = 1$ and right for $d_1 = d_3 = -1$.

The left panel shows that this law is satisfied for all values of ζ at the initial times as well as when ζ approaches to 1 with $t \geq 27$ for $(d_1, d_3) > 0$ while the feasible region for $d_1 > 0$ and $d_3 < 0$ is shown in the right plot. Similarly, Figure **8** shows the regions where GSLT holds for the remaining two signatures of (d_1, d_3). GSLT is true for all four possible choices of integration constants for the specific ranges of ζ and t.

Concluding Remarks

- First and second laws in non-equilibrium description of thermodynamics are studied at the apparent horizon of FRW universe model. Validity of GSLT for reconstructed models is also checked.

- Total entropy in the first law involves contribution from horizon entropy in terms of area and entropy production term (due to non-equilibrium behavior).

- No such entropy production term appears in GR, GB, Lovelock and braneworld theories of gravity.
- Validity of GSLT in terms of horizon entropy, entropy production term as well as entropy corresponding to all matter and energy contents inside the horizon.

Concluding Remarks

- Viability condition for this law is consistent with the universal condition for its validity in modified theories of gravity.
- Validity of GSLT for the reconstructed models (de Sitter universe and power-law solution) with dust fluid is also studied.
- For de Sitter reconstructed models, GSLT holds for model (14) when the integration

Concluding Remarks

constants (c_1, c_2) have same signatures while for the second model (15), the feasible regions are obtained for the opposite signatures (Figures **1**-**4**).

- For power-law, GSLT holds when d_1 is positive for the model (18) while this holds for all possible choices of d_1 and d_3 for the model (19) (Figures **5**-**8**).

Concluding Remarks