Multiphoton amplitudes in a constant background field

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Outline

- Introduction to the worldline formalism and the String-Inspired methods
- Master formulas for the dressed scalar propagator in vacuum
- Master formulas for the dressed scalar propagator in a constant field
- Conclusion

Worldline formalism

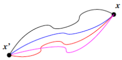
In 1948, Feynman developed the path integral approach to non-relativistic quantum mechanics (based on earlier work by Wentzel and Dirac). Two years later, he started his famous series of papers that laid the foundations of relativistic quantum field theory (essentially quantum electrodynamics at the time) and introduced Feynman diagrams. However, at the same time he also developed a representation of the QED S-matrix in terms of relativistic particle path integrals.

Why worldline formalism?

- No need to compute momentum integrals and Dirac traces.
- Worldline formalism works well for massive particles (on- and off-shell) not even at tree-level but at loop order too.

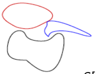
The difference between open line and loop:

- Dirichlet boundary conditions (topology of a line)



$$\langle x|e^{-HT}|x'\rangle = \int_{x(0)=x'}^{x(T)=x} Dx(\tau)\, e^{-S[x,G]}$$

- Periodic boundary conditions (topology of a closed line)



$$\int_{x(0)=x(T)} Dx(\tau) e^{-S[x,G]}$$

String-inspired formalism

Bern-Kosower master formula (Z. Bern and D. Kosower 1991)
In their analysis of the N-gluon amplitude, Bern and Kosower therefore used, instead of the open string, a certain heterotic string model containing $SU(Nc)$ Yang-Mills theory in the infinite string tension limit. This allows for a consistent reduction to four dimensions, at the price of a more complicated representation of this amplitude. By an explicit analysis of the infinite string tension limit, they succeeded in deriving a novel type of parameter integral representation for the on-shell N - gluon amplitude in Yang-Mills theory, at the tree- and one-loop level. Moreover, they established a set of rules which allows one to construct this parameter integral, for any number of gluons and choice of helicities, without referring to string theory any more.

$$\begin{aligned}\Gamma^{a_1\cdots a_N}[p_1,\varepsilon_1;\dots;p_N,\varepsilon_N] \quad = \quad & (-ig)^N \mathrm{tr}(T^{a_1}\dots T^{a_N})\int_0^\infty dT(4\pi T)^{-D/2}e^{-m^2T}\\ & \times\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\dots\int_0^{\tau_{N-2}}d\tau_{N-1}\\ & \times\exp\left\{\sum_{i,j=1}^N\left[\frac{1}{2}G_{Bij}p_i\cdot p_j - i\dot G_{Bij}\varepsilon_i\cdot p_j+\frac{1}{2}\ddot G_{Bij}\varepsilon_i\cdot\varepsilon_j\right]\right\}\Big|_{\mathrm{lin}(\varepsilon_1\dots\varepsilon_N)}\end{aligned}$$

As it stands, this is a parameter integral representation for the (color-ordered) N - gluon vertex, with momenta p_i and polarizations ε_i, induced by a scalar loop, in D dimensions. Here m and T are the loop mass and proper-time, τ_i the location of the ith gluon.

with

$$
\begin{aligned}
G_{Bij} &= |\tau_i - \tau_j| - \frac{(\tau_i - \tau_j)^2}{T} \\
\dot{G}_B(\tau_1, \tau_2) &= \mathrm{sign}(\tau_1 - \tau_2) - 2\frac{(\tau_1 - \tau_2)}{T} \\
\ddot{G}_B(\tau_1, \tau_2) &= 2\delta(\tau_1 - \tau_2) - \frac{2}{T} \,.
\end{aligned}
$$

$G_B(\tau, \tau')$ is the Green's function for the second derivative operator $\frac{d^2}{d\tau^2}$ adapted to the periodicity, as well as to the "string-inspired" ('SI') boundary conditions

$$
\int_0^T d\tau \; G_B(\tau, \tau') = \int_0^T d\tau' \; G_B(\tau, \tau') = 0 \,, \tag{1}
$$

Let us just mention some advantages of the **Bern-Kosower Rules** as compared to the Feynman rules:

- Superior organization of gauge invariance.
- Absence of loop momentum, which reduces the number of kinematic invariants from the beginning, and allows for a particularly efficient use of the spinor helicity method.
- Calculations of scattering amplitudes with the same external states but particles of different spin circulating in the loop are more closely related than usual.

Since the Bern-Kosower rules do not refer to string theory any more, the question naturally arises whether it should not be possible to re-derive them completely inside field theory. Obviously, such a re-derivation should be attempted starting from a first-quantized formulation of ordinary field theory, rather than from standard quantum field theory.

We have in mind four main purposes:

1. First, in on-shell amplitudes, the multi-photon generalizations of Compton scattering are becoming important these days for laser physics, for a review on high-intensity laser QED, see: A. Di Piazza et.al Rev. Mod. Phys **84**, 1177 (2012).

2. Second, for off-shell amplitudes, computation of form factors in Scalar QED where the main interest is in QCD and spinor QED, but it is always good to study scalar QED as the simplest nontrivial gauge theory in four dimensions.

3. Incorporating external classical fields non-perturbatively.

4. The scattering of photons by electrons in strong magnetic fields has been a subject of great interest since the discovery of the radio pulsars and cyclotron lines in X-ray pulsars provide evidence that the magnetic fields or order 10^{12} G are present in nature. It has been recognized for some time, while the presence of an external magnetic field cause significant deviations from classical Thomson scattering (e.g,Canuto et al 1971, Ventura 1979) relativistic effects can be at least as important in fields approaching the critical value, $B_{\rm cr} = 4.413 \times 10^{13}$ G.

5. Extension to real QED.

Worldline formalism for scalar propagator

In the following, we discuss our method which is based on the worldline formalism, initially developed by Feynman for scalar QED in Phys. Rev. **80**, 440 (1950) and spinor QED in Phys. Rev. **84**, 108 (1951). If we consider a scalar propagator in the presence of a background gauge filed which propagates from point x' to point x

$$\Gamma[x'; x] \quad = \quad \int dT e^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-S_0 - S_e - S_i}$$

where

$$\begin{aligned} S_0 &= \int_0^T d\tau \frac{1}{4} \dot{x}^2 \rightarrow \textit{describes the free propagation} \,, \\ S_e &= ie \int_0^T d\tau\, \dot{x} \cdot A(x(\tau)) \rightarrow \textit{the interaction of the scalar with the external field} \,, \\ S_i &= \frac{e^2}{2} \int_0^T d\tau_1 \int_0^T d\tau_2\, \dot{x}_1^\mu D_{\mu\nu}(x_1 - x_2) \dot{x}_2^\nu \rightarrow \textit{virtual photons exchanged along the scalar}'\textit{s trajectory} \,. \end{aligned} \tag{2}$$

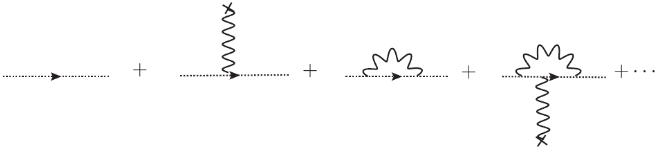
$D_{\mu\nu}$ is the x - space photon propagator in D dimensions. In an arbitrary covariant gauge, it is given by

$$D_{\mu\nu}(x) = \frac{1}{4\pi^{\frac{D}{2}}}\left\{\frac{1+\xi}{2}\Gamma\left(\frac{D}{2}-1\right)\frac{\delta_{\mu\nu}}{{x^2}^{\frac{D}{2}-1}} + (1-\xi)\Gamma\left(\frac{D}{2}\right)\frac{x_\mu x_\nu}{{x^2}^{\frac{D}{2}}}\right\}. \quad (3)$$

where

$$\begin{cases} \xi = 1 & \Rightarrow \quad \text{Feynman gauge} \\ \xi = 0 & \Rightarrow \quad \text{Landau gauge} \end{cases}$$

The expansion of the exponentials of the interaction terms Se and Si generates the Feynman diagrams depicted below



The external legs represent interactions with the field $A(x)$, and are converted into momentum-space photons by choosing $A(x)$ as a sum of plane waves,

$$A^\mu(x) = \sum_{i=1}^{N} \varepsilon_i^\mu \; \mathrm{e}^{ik_i\cdot x} \, . \tag{4}$$

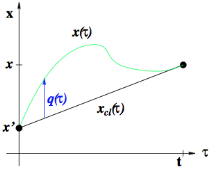
Each external photon then gets represented by a *vertex operator*.

$$V^A_{\rm scal}[k,\varepsilon] \equiv \varepsilon_\mu \int_0^T d\tau \, \dot{x}^\mu(\tau) \, \mathrm{e}^{ik\cdot x(\tau)} \, . \tag{5}$$

According to our convention, external photon momenta are ingoing.

The path integral is computed by splitting $x^\mu(\tau)$ into a "background" part $x^\mu_{\rm bg}(\tau)$, which encodes the boundary conditions, and a fluctuation part $q^\mu(\tau)$, which has **Dirichlet** boundary conditions at the endpoints $\tau = 0, T$:

$$\begin{aligned} x(\tau) &= x_{\rm bg}(\tau) + q(\tau)\,, \\ x_{\rm bg}(\tau) &= x' + \frac{(x-x')\tau}{T}\,, \\ \dot x(\tau) &= \frac{x-x'}{T} + \dot q(\tau)\,, \\ q(0) &= q(T) = 0\,. \end{aligned}$$



The calculation of the path integral then requires only the knowledge of the free path integral normalization, which is

$$\int \mathcal{D}q(\tau)\, {\rm e}^{-\int_0^T d\tau \frac{1}{4}\dot q^2} = (4\pi T)^{-\frac{D}{2}}\,, \qquad (6)$$

and of the two-point correlator, given by

$$\langle q^\mu(\tau_1) q^\nu(\tau_2)\rangle = -2\delta^{\mu\nu}\Delta(\tau_1,\tau_2) \qquad (7)$$

with the worldline Green function $\Delta(\tau_i, \tau_j)$,

$$\Delta(\tau_1, \tau_2) = \frac{\tau_1 \tau_2}{T} + \frac{|\tau_1 - \tau_2|}{2} - \frac{\tau_1 + \tau_2}{2} . \tag{8}$$

We note that this Green function has a nontrivial coincidence limit

$$\Delta(\tau, \tau) = \frac{\tau^2}{T} - \tau , \tag{9}$$

and we will also need its following derivatives:

$$\begin{aligned} {}^{\bullet}\Delta(\tau_1, \tau_2) &= \frac{\tau_2}{T} + \frac{1}{2}\mathrm{sign}(\tau_1 - \tau_2) - \frac{1}{2} , \\ \Delta^{\bullet}(\tau_1, \tau_2) &= \frac{\tau_1}{T} - \frac{1}{2}\mathrm{sign}(\tau_1 - \tau_2) - \frac{1}{2} , \\ {}^{\bullet}\Delta^{\bullet}(\tau_1, \tau_2) &= \frac{1}{T} - \delta(\tau_1 - \tau_2) . \end{aligned} \tag{10}$$

Note that the mixed derivative ${}^{\bullet}\Delta^{\bullet}(\tau_1, \tau_2)$ contains a delta function which brings together two photon legs; this is how the seagull vertex arises in the worldline formalism. In the simplest case, for the free scalar propagator, we thus get the following standard proper-time representation in D dimensions:

$$\Gamma_{\mathrm{free}}[x, x'] = \int_0^\infty dT \, \mathrm{e}^{-m^2 T} \, (4\pi T)^{-\frac{D}{2}} \, \mathrm{e}^{-\frac{1}{4T}(x-x')^2} . \tag{11}$$

Multiphoton amplitude in scalar QED

Now, let's go back to the worldline formula for scalar propagator:

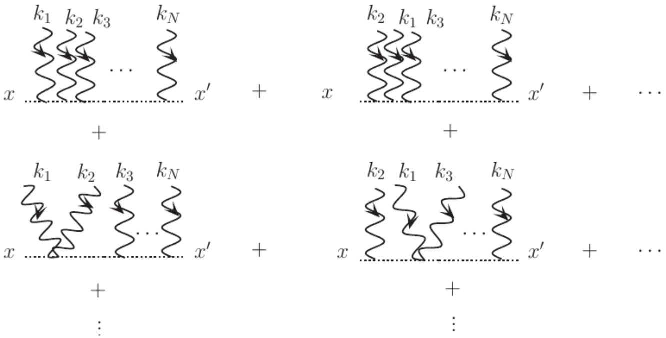
$$\Gamma[x';x] = \int dT e^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-\frac{1}{4}\int_0^T d\tau [\dot{x}^2 + ie\dot{x}\cdot A(x(\tau))]}$$

After expanding the interaction term in the exponential we get

$$\Gamma[x';x;k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-ie)^N \int_0^\infty dT e^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau) e^{-\frac{1}{4}\int_0^T d\tau \dot{x}^2} \times \int_0^T \prod_{i=1}^N d\tau_i V^A_{\mathrm{scal}}[k_1,\varepsilon_1]\cdots V^A_{\mathrm{scal}}[k_N,\varepsilon_N]$$

For our purpose, it will be convenient to formally rewrite the vertex operator as

$$V^A_{\mathrm{scal}}[k,\varepsilon] = \int_0^T d\tau \varepsilon \cdot \dot{x}(\tau)\, e^{ik\cdot x(\tau)} = \int_0^T d\tau e^{ik\cdot x(\tau)+\varepsilon\cdot\dot{x}(\tau)}\Big|_{\mathrm{lin}\ \varepsilon}$$


k_1 k_2 k_3 k_N
x x'
k_2 k_1 k_3 k_N
x x'
k_1 k_2 k_3 k_N
x x'
k_2 k_1 k_3 k_N
x x'

Substituting this vertex operator, and applying the split, one gets

$$\Gamma[x,x';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-ie)^N \int_0^\infty dT e^{-m^2 T}\, \mathrm{e}^{-\frac{1}{4T}(x-x')^2} \int_{q(0)=q(T)=0} \mathcal{D}q(\tau)\, e^{-\frac{1}{4}\int_0^T d\tau \dot{q}^2}$$
$$\times \int_0^T \prod_{i=1}^N d\tau_i\, \mathrm{e}^{\sum_{i=1}^N \left(\varepsilon_i\cdot\frac{(x-x')}{T}+\varepsilon_i\cdot\dot{q}(\tau_i)+ik_i\cdot(x-x')\frac{\tau_i}{T}+ik_i\cdot x'+ik_i\cdot q(\tau_i)\right)}\Big|_{\mathrm{lin}(\varepsilon_1\varepsilon_2\cdots\varepsilon_N)} . \tag{12}$$

After completing the square in the exponential, we obtain the following tree-level "Bern-Kosower-type formula" in configuration space:

$$\Gamma[x,x';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-ie)^N \int_0^\infty dT e^{-m^2 T}\, \mathrm{e}^{-\frac{1}{4T}(x-x')^2} (4\pi T)^{-\frac{D}{2}}$$
$$\times \int_0^T \prod_{i=1}^N d\tau_i\, \mathrm{e}^{\sum_{i=1}^N \left(\varepsilon_i\cdot\frac{(x-x')}{T}+ik_i\cdot(x-x')\frac{\tau_i}{T}+ik_i\cdot x'\right)}\, \mathrm{e}^{\sum_{i,j=1}^N \left[\Delta_{ij}p_i\cdot p_j-2i{}^\bullet\Delta_{ij}\varepsilon_i\cdot k_j-{}^\bullet\Delta^\bullet_{ij}\varepsilon_i\cdot\varepsilon_j\right]}\Big|_{\mathrm{lin}(\varepsilon_1\varepsilon_2\cdots\varepsilon_N)} . \tag{13}$$

Now, we also Fourier transform the master formula to momentum space,

$$\Gamma[p;p';k_1,\varepsilon_1;\cdots;k_N,\varepsilon N] = \int d^Dx \int d^Dx' \, \mathrm{e}^{ip\cdot x + ip'\cdot x'} \, \Gamma[x,x';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N]\,. \tag{14}$$

which leads to

$$\Gamma[p;p';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-ie)^N (2\pi)^D \delta^D\Big(p+p'+\sum_{i=1}^N k_i\Big) \int_0^\infty dT\, \mathrm{e}^{-T(m^2+p^2)} \int_0^T \prod_{i=1}^N d\tau_i$$
$$\times \mathrm{e}^{\sum_{i=1}^N(-2k_i\cdot p\tau_i + 2i\varepsilon_i\cdot p) + \sum_{i,j=1}^N \left[(\frac{|\tau_i-\tau_j|}{2} - \frac{\tau_i+\tau_j}{2})p_i\cdot p_j - i(\mathrm{sign}(\tau_i-\tau_j)-1)\varepsilon_i\cdot p_j + \delta(\tau_i-\tau_j)\varepsilon_i\cdot\varepsilon_j\right]}\Big|_{\mathrm{lin}(\varepsilon_1\varepsilon_2\cdots\varepsilon_N)}\,.$$

This is our final representation of the N - propagator in momentum space. It is important to mention that it gives the *untruncated* propagator, including the final scalar propagators on both ends. On-shell it corresponds to multi-photon Compton scattering, while off-shell it can be used for constructing higher-loop amplitudes by sewing.

Multiphoton amplitude in constant background fields

Shaisultanov in PLB 378, 354 (1996) generalized both the scalar and spinor QED master formulas to the case of QED in a constant external field $F_{\mu\nu}$. For the scalar case, this generalized master formula can be written as

$$\begin{aligned}\Gamma(k_1,\varepsilon_1;\dots;k_N,\varepsilon_N) &= -(-ie)^N(2\pi)^D\delta(\sum k_i)\\ &\times\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\,\mathrm{e}^{-m^2T}\det{}^{-\frac{1}{2}}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right]\prod_{i=1}^N\int_0^T d\tau_i\\ &\times\exp\left\{\sum_{i,j=1}^N\left[\frac{1}{2}k_i\cdot\mathcal{G}_{Bij}k_j - i\varepsilon_i\cdot\dot{\mathcal{G}}_{Bij}\cdot k_j+\frac{1}{2}\varepsilon_i\cdot\ddot{\mathcal{G}}_{Bij}\cdot\varepsilon_j\right]\right\}\Big|_{\varepsilon_1\varepsilon_2\cdots\varepsilon_N}\end{aligned} \tag{15}$$

where we have introduced the abbreviation $\mathcal{Z}\equiv eFT$. This master formula differs from the vacuum one, only by the additional determinant factor $\det^{-\frac{1}{2}}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right]$, which represents the dependence of the free (photonless) path integral on the external field, and a change of the worldline Green's function G_B to a new one $\mathcal{G}_B$ that holds information on the external field,

$$\mathcal{G}_B(\tau_i,\tau_j) = \frac{T}{2\mathcal{Z}^2}\left(\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\,\mathrm{e}^{-i\mathcal{Z}\dot{G}_{Bij}}+i\mathcal{Z}\dot{G}_{Bij}-1\right). \tag{16}$$

Much less has been done for the analogous amplitudes involving an open line, let us re-express the master formula for multiphoton amplitude in vacuum:

$$
\begin{aligned}
D^{pp'}(k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N) \quad &= \quad (ie)^N(2\pi)^D\delta^D\Big(p+p'+\sum_{i=1}^N k_i\Big)\int_0^\infty dT\,\mathrm{e}^{-(m^2+p^2)T} \\
&\times\prod_{i=1}^N\int_0^T d\tau_i\,\mathrm{e}^{\sum_{i=1}^N[\varepsilon_i\cdot\frac{x_-}{T}+ik_i\cdot\frac{x_-\tau_i}{T}+ik_i\cdot x']+\sum_{i,j=1}^N[\Delta_{ij}k_i\cdot k_j-2i{}^\bullet\Delta_{ij}\varepsilon_i\cdot k_j-{}^\bullet\Delta^\bullet_{ij}\varepsilon_i\cdot\varepsilon_j]}\Big|_{\varepsilon_1\varepsilon_2\cdots\varepsilon_N}
\end{aligned}
$$

Here a different worldline Green's function $\Delta(\tau,\tau')$ appears,

$$
\Delta(\tau,\tau') \quad = \quad \frac{\tau\tau'}{T}+\frac{|\tau-\tau'|}{2}-\frac{\tau+\tau'}{2}\,. \tag{17}
$$

Instead of the SI boundary conditions, it is adapted to Dirichlet boundary conditions ('DBC')

$$
\Delta(0,\tau')=\Delta(T,\tau')=\Delta(\tau,0)=\Delta(\tau,T)=0\,. \tag{18}
$$

The Green's functions G_B and Δ are related by (Fliegner, Schmidt, Schubert, Z. Phys. C 64, 111 (1994))

$$
G_B(\tau,\tau')=2\Delta(\tau,\tau')-\Delta(\tau,\tau)-\Delta(\tau',\tau')\,, \tag{19}
$$

The propagator in a constant field

The propagator of a scalar particle in the Maxwell background:

$$D^{xx'}[A] \quad = \quad \int_0^\infty dT\ \mathrm{e}^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} Dx\ \mathrm{e}^{-\int_0^T d\tau \left[\frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A(x)\right]}\ . \tag{20}$$

where

$$A = A_{\mathrm{ext}} + A_{\mathrm{phot}} \tag{21}$$

Choosing Fock-Schwinger gauge, the gauge potential for a constant field can be written as

$$A^\mu(y) = -\frac{1}{2}F^{\mu\nu}(y-x')^\nu\,, \tag{22}$$

We decompose the arbitrary trajectory $x(\tau)$ into a straight-line part and a fluctuation part $q(\tau)$ obeying Dirichlet boundary conditions, $q(0) = q(T) = 0$:

$$x(\tau) = x' + \frac{\tau}{T}(x-x') + q(\tau). \tag{23}$$

After plugging this potential to the scalar propagator we get ($Q^\mu \equiv \int_0^T d\tau q^\mu(\tau)$)

$$D^{xx'}(F) \quad = \quad \int_0^\infty dT\,\mathrm{e}^{-m^2 T}\,\mathrm{e}^{-\frac{(x-x')^2}{4T}} \int Dq(\tau)\,\mathrm{e}^{-\int_0^T d\tau\,\frac{1}{4}q\left(-\frac{d^2}{d\tau^2}+2ieF\frac{d}{d\tau}\right)q + \frac{ie}{T}(x-x')FQ}\ . \tag{24}$$

The worldline Green's function does change, but still relates to the one for string-inspired boundary conditions in the same way as in the vacuum case:

$$\underset{\smile}{\Delta}(\tau,\tau') \equiv \langle \tau \mid \Big(\frac{d^2}{d\tau^2} - 2ieF\frac{d}{d\tau}\Big)^{-1} \mid \tau' \rangle_{\rm DBC} = \frac{1}{2}\Big(\mathcal{G}_B(\tau,\tau') - \mathcal{G}_B(\tau,0) - \mathcal{G}_B(0,\tau') + \mathcal{G}_B(0,0)\Big)\,. \tag{25}$$

Finally after using this new Green's function, we obtain the well-known proper-time representation of the constant-field propagator (E.S. Fradkin, D.M. Gitman, S.M. Shvartsman, Quantum Electrodynamics with Unstable Vacuum, Springer 199)

in configuration space:

$$D^{xx'}(F) \quad = \quad \int_0^\infty dT\, {\rm e}^{-m^2 T}(4\pi T)^{-\frac{D}{2}}\,{\rm det}^{\frac{1}{2}}\Big[\frac{\mathcal{Z}}{\sin\mathcal{Z}}\Big]\,{\rm exp}\Big\{-\frac{1}{4T}(x-x')\mathcal{Z}\cot\mathcal{Z}(x-x')\Big\}\,. \tag{26}$$

and in momentum space:

$$D^{pp'}(F) \quad = \quad (2\pi)^D\delta(p+p')\int_0^\infty dT\ {\rm e}^{-m^2 T}\ \frac{{\rm e}^{-Tp(\frac{\tan\mathcal{Z}}{\mathcal{Z}})p}}{{\rm det}^{\frac{1}{2}}\,[\cos\mathcal{Z}]}\,. \tag{27}$$

The dressed propagator in a constant field

We now wish to dress the propagator with N photons in addition to the constant field. As before, we start in configuration space. For this purpose, the potential in (20) has to be chosen as

$$A = A_{\rm ext} + A_{\rm phot}\,, \tag{28}$$

where $A_{\rm ext}$ is the same as in (22), and $A_{\rm phot}$ represents a sum of plane waves:

$$A^{\mu}_{\rm phot}(x) = \sum_{i=1}^{N} \varepsilon_i^{\mu}\ {\rm e}^{ik_i\cdot x}\,. \tag{29}$$

Each photon then effectively gets represented by a vertex operator

$$V^A[k,\varepsilon] = \int_0^T d\tau\,\varepsilon.\dot{x}(\tau)\,{\rm e}^{ik\cdot x(\tau)}\,, \tag{30}$$

integrated along the scalar line. This leads to the following path integral representation of the constant-field propagator dressed with N photons:

$$\begin{aligned} D^{xx'}(F|k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N) \quad = \quad & (-ie)^N\int_0^\infty dT\ {\rm e}^{-m^2T}\int_P Dx\ {\rm e}^{-\int_0^T d\tau[\frac{1}{4}\dot{x}^2+ie\,\dot{x}\cdot A_{\rm ext}(x)]} \\ & \times V[k_1,\varepsilon_1]V[k_2,\varepsilon_2]\cdots V[k_N,\varepsilon_N]\,. \end{aligned} \tag{31}$$

Applying the path decomposition and some nontrivial calculations we arrive to the following x-space representation of the dressed scalar propagator in a constant background field:

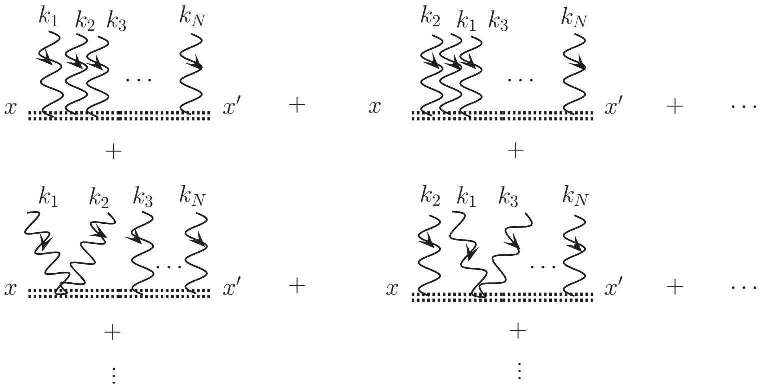
in x-space:

$$
\begin{aligned}
D^{xx'}(F \mid k_1, \varepsilon_1; \cdots ; k_N, \varepsilon_N) \quad = \quad & (-ie)^N \int_0^\infty dT\, \mathrm{e}^{-m^2 T} (4\pi T)^{-\frac{D}{2}} \det{}^{\frac{1}{2}} \left[\frac{\mathcal{Z}}{\sin \mathcal{Z}} \right] \mathrm{e}^{-\frac{1}{4T} x_- \mathcal{Z} \cot \mathcal{Z} x_-} \\
& \times \int_0^T d\tau_1 \cdots \int_0^T d\tau_N \; \mathrm{e}^{\sum_{i=1}^N \left(\varepsilon_i \cdot \frac{x_-}{T} + ik_i \cdot \frac{x_- \tau_i}{T} + ik_i \cdot x' \right)} \\
\times \exp\Bigg[\sum_{i,j=1}^N \Big(k_i \; \underset{\smile}{\Delta}_{ij} \; k_j - 2i\varepsilon_i \; {}^\bullet\underset{\smile}{\Delta}_{ij} \; k_j - \varepsilon_i \; {}^\bullet\underset{\smile}{\Delta}^\bullet{}_{ij} \; \varepsilon_j \Big) & + \frac{2e}{T} x_- \sum_{i=1}^N \Big(F \; {}^\circ\underset{\smile}{\Delta}_i \; k_i - iF \; {}^\circ\underset{\smile}{\Delta}^\bullet{}_i \; \varepsilon_i \Big) \Bigg] \Big|_{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_N} .
\end{aligned}
\tag{32}
$$

left (right) ‘open circle’ on $\underset{\smile}{\Delta}(\tau, \tau')$ denotes an integral $\int_0^T d\tau$ ($\int_0^T d\tau'$).

For the special case of a purely magnetic field, this x - space master formula was obtained already in 1994 by McKeon and Sherry (Mod. Phys. Lett. A9 (1994) 2167).

which describes the following Feynman diagrams:



and in p-space

$$
\begin{aligned}
D^{pp'}(F|k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N) \quad &= \quad (-ie)^N(2\pi)^D\delta\Big(p+p'+\sum_{i=1}^N k_i\Big)\int_0^\infty dT\,\mathrm{e}^{-m^2T}\,\frac{1}{\det^{\frac{1}{2}}\big[\cos\mathcal{Z}\big]} \\
&\times\int_0^T d\tau_1\cdots\int_0^T d\tau_N\;\mathrm{e}^{\sum_{i,j=1}^N\left(k_i\,\underset{\smile}{\Delta}_{ij}\,k_j-2i\varepsilon_i\,{}^{\bullet}\underset{\smile}{\Delta}_{ij}\,k_j-\varepsilon_i\,{}^{\bullet}\underset{\smile}{\Delta}{}^{\bullet}{}_{ij}\,\varepsilon_j\right)}\;\mathrm{e}^{-Tb(\frac{\tan\mathcal{Z}}{\mathcal{Z}})b}\Big|_{\varepsilon_1\varepsilon_2\cdots\varepsilon_N}\,.
\end{aligned}
\tag{33}
$$

Here we have defined

$$
b\equiv p+\frac{1}{T}\sum_{i=1}^N\Big[\Big(\tau_i-2ieF\,{}^{\circ}\underset{\smile}{\Delta}_i\Big)k_i-i\Big(1-2ieF\,{}^{\circ}\underset{\smile}{\Delta}{}^{\bullet}{}_i\Big)\varepsilon_i\Big]\,.
\tag{34}
$$

Compton scattering in a constant field

Expanding out the exponentials in (33) and projecting to the terms linear in both polarization vectors, we find (omitting now the global factor for energy-momentum conservation):

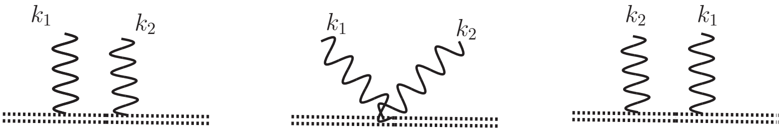
$$D^{pp'}(F \mid k_1, \varepsilon_1; k_2, \varepsilon_2) = e^2 \int_0^\infty dT \, \frac{\mathrm{e}^{-m^2 T}}{\det^{\frac{1}{2}} \left[\cos \mathcal{Z}\right]} \times \int_0^T d\tau_1 \int_0^T d\tau_2 \, \mathrm{e}^{-Tb_0 (\frac{\tan \mathcal{Z}}{\mathcal{Z}}) b_0 + \sum_{i,j=1}^2 k_i \underset{\smile}{\Delta}_{ij} k_j} \varepsilon_1 M_{12} \varepsilon_2 \,,$$

with

$$b_0 \equiv p + \frac{1}{T} \sum_{i=1}^{2} \left(\tau_i - 2ieF \, {}^{\circ}\underset{\smile}{\Delta}_i \right) k_i \,, \qquad (35)$$

and

$$M_{12} \equiv 2\, {}^{\bullet}\underset{\smile}{\Delta}{}^{\bullet}{}_{12} - \frac{2}{T} \left(1 + 2ie\, {}^{\circ}\underset{\smile}{\Delta}{}^{\bullet}{}_1 F \right) \frac{\tan \mathcal{Z}}{\mathcal{Z}} \left(1 - 2ieF \, {}^{\circ}\underset{\smile}{\Delta}{}^{\bullet}{}_2 \right) + 4 \left[\left(1 + 2ie\, {}^{\circ}\underset{\smile}{\Delta}{}^{\bullet}{}_1 F \right) \frac{\tan \mathcal{Z}}{\mathcal{Z}} b_0 - \sum_{i=1}^{2} {}^{\bullet}\underset{\smile}{\Delta}_{1i} \, k_i \right] \left[b_0 \frac{\tan \mathcal{Z}}{\mathcal{Z}} \left(1 - 2ieF \, {}^{\circ}\underset{\smile}{\Delta}{}^{\bullet}{}_2 \right) - \sum_{i=1}^{2} k_i \underset{\smile}{\Delta}{}^{\bullet}{}_{i2} \right] . \qquad (36)$$


k1
k2
k1
k2
k2
k1

Squaring, and performing the sum over the photon polarizations via

$$\sum_{\mathrm{pol}} \varepsilon_i^{*\mu} \varepsilon_i^{\nu} \longrightarrow g^{\mu\nu} , \tag{37}$$

we get the following for the Compton cross section:

$$\begin{aligned} \sum_{\mathrm{pol}} \mathcal{T}^* \mathcal{T} \quad = \quad & \frac{e^4}{|D(p,F)|^2 \, |D(p',F)|^2} \\ & \times \int_0^\infty dT' \, \frac{\mathrm{e}^{-m^2 T'}}{\det^{\frac{1}{2}} \left[\cos \mathcal{Z}' \right]} \int_0^{T'} d\tau_1' \int_0^{T'} d\tau_2' \; \mathrm{e}^{-T' b_0^* (\frac{\tan \mathcal{Z}'}{\mathcal{Z}'}) b_0^* + \sum_{i,j=1}^{2} k_i \, \underset{\smile}{\Delta}'_{ij} \, k_j} \\ & \times \int_0^\infty dT \, \frac{\mathrm{e}^{-m^2 T}}{\det^{\frac{1}{2}} \left[\cos \mathcal{Z} \right]} \int_0^{T} d\tau_1 \int_0^{T} d\tau_2 \; \mathrm{e}^{-T b_0 (\frac{\tan \mathcal{Z}}{\mathcal{Z}}) b_0 + \sum_{i,j=1}^{2} k_i \, \underset{\smile}{\Delta}_{ij} \, k_j} \, \mathrm{tr}(M_{12}'^{\dagger} M_{12}) \, . \end{aligned} \tag{38}$$

After writing the integrand explicitly this expression is suitable for numerical integration.

Previous studies

Daugherty and **Harding** (The Astrophysics Journal, **309**, 362, 1986) have derived the relativistic cross section for Compton scattering by electron in strong magnetic fields. The derivation assumes that the electrons are initially in their lowest Landau levels, but considered transitions to arbitrary Landau levels in the final state.

An immediate conclusion to be drown from their results is that the cross section for scattering to higher Landau states becomes comparable to the ground state cross section for energy of several cyclotron frequencies in teragauss fields. In fact the Comptonization of incident high energy photons may prove an important mechanism for maintaining a significant population of electrons in excited Landau levels. Scatterings which leave the electron in an excited state effectively split the energy of the incident photon into two or more photons of lower energy. The presence of the magnetic field therefore softens the scattered photons spectrum.

Conclusion

- We have rederived the momentum-space Bern-Kosower type master formula for the tree-level scalar propagator dressed by an arbitrary number of photons, starting directly from the worldline path integral representation of this amplitude. We have also generalized this master formula to the x - space propagator.

- Using the worldline path integral formalism, we have derived a Bern- Kosower type master formula for the scalar propagator in QED, in a constant field and dressed by an arbitrary number of photons.

- Our master formula is valid off-shell, and combines the various orderings of the N photons along the scalar line. It can thus be used as a convenient starting point for the construction of higher- loop scalar QED processes in a constant field.

- Compton scattering in magnetic fields is a process of potential relevance for astrophysics, but, to the best of our knowledge, so far has been studied only in the strong-field limit.

- The spinor QED generalization is basically done, and about to be published.

- Next, its generalization to gravity.