Particle Dynamics Around a Charged Black Hole

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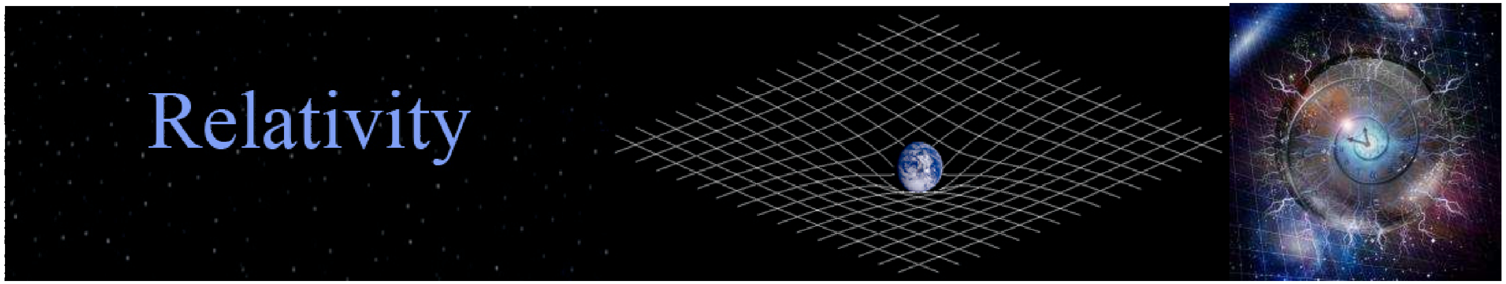
Layout of the Talk

- Basic Concepts
- Dynamics of Neutral and Charged Particles
- Concluding Remarks

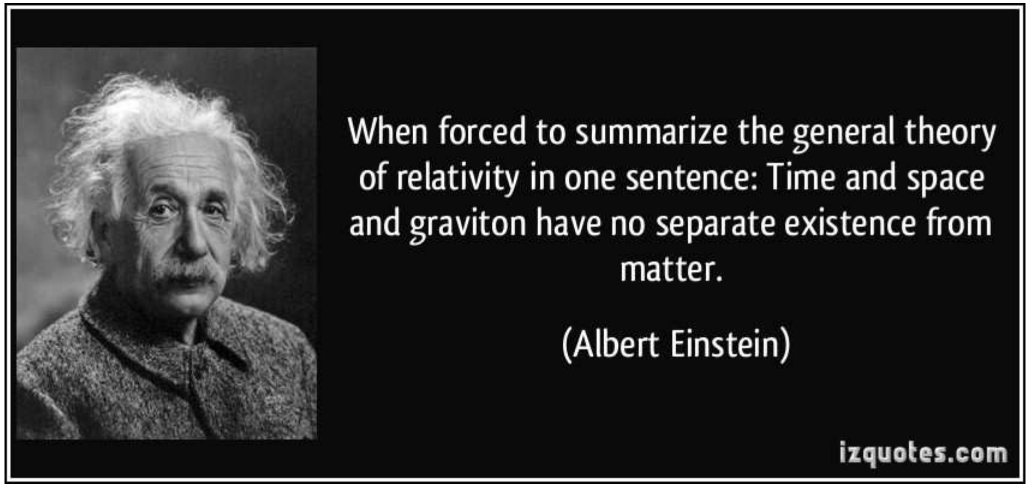
Basic Concepts



An important aspect of nature is the relation between space and time that results from Einstein theory of Relativity.

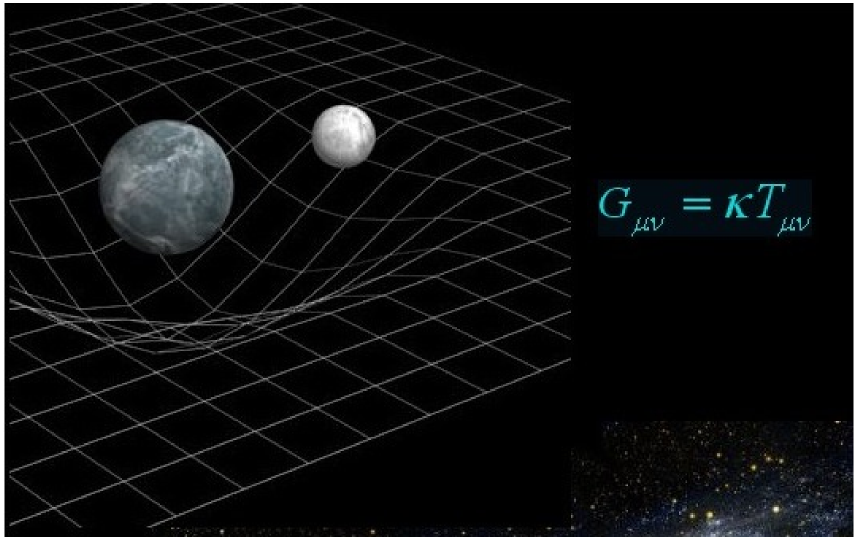
- Theory of Special Relativity (SR) was published in (1905).

Basic Concepts



- Einstein made a number of attempts between 1908-1914 to find a theory of gravity that was consistent with SR. Finally, in 1915 he proposed theory of General Relativity (GR)

Basic Concepts



General relativity demonstrates the most dominant attribute of the universe, "gravity" in terms of geometry of the curved spacetime.

Basic Concepts

It can be expressed in concise form as *"spacetime tells matter how to move; matter tells spacetime how to curve"* *(J. A. Wheeler)*.

In GR, the spacetime geometry is related to the matter by the famous Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}$$

Basic Concepts

◇ We cannot directly observe a black hole.

◇ They can be detected by their gravitational effect on other nearby objects also by the radiation given off as material is heated in the accretion disk.

◇ Since black holes can bend light, it would be expected that this phenomenon would be observable.

◇ Black holes in X-ray binaries and active galaxies can be strong deflectors.

Basic Concepts

Geodesics

"A geodesic is defined to be the straightest line in space"

- ▷ in space
- ▷ on sphere
- ▷ on flat space

Types of Geodesics

- Null (lightlike)
- Timelike
- Spacelike

Basic Concepts

▷ Geodesics for the line element $ds^2 = g_{ij}dx^i dx^j$ can be derived from the Euler Lagrange equations [Chandrasekhar, S.: *The Mathematical Theory of Black Holes* (Clarendon Press, 1998)]

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^j}\right) - \frac{\partial \mathcal{L}}{\partial x^j} = 0, \quad \dot{x}^j = \frac{dx^j}{d\lambda}$$

where

$$2\mathcal{L} = g_{ij}\dot{x}^i \dot{x}^i.$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^j} = g_{ij}\dot{x}^i, \quad \frac{\partial \mathcal{L}}{\partial x^j} = \frac{1}{2}g_{ik,j}\dot{x}^i \dot{x}^k.$$

Basic Concepts

After some calculations, we obtain

$$\ddot{x}^l + \Gamma^l_{ik}\dot{x}^i\dot{x}^k = 0.$$

Let us define

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{x}^j} = g_{ij}\dot{x}^i.$$

Then the Euler-Lagrange equation reduces to

$$\frac{d}{d\lambda}p_j = \frac{\partial \mathcal{L}}{\partial x^j}.$$

Basic Concepts

▷ The normalization condition is given as

$$g_{ij}u^iu^j = \epsilon, \quad u^i = \dot{x}^i.$$

- for null geodesics, $\epsilon = 0$
- for timelike geodesics, $\epsilon = -1$
- for spacelike geodesics, $\epsilon = +1$

Applications

- Physical properties of the gravitational fields
- Significance of null and timelike geodesics
- Information regarding physical process in the gravitational fields

Equations of Motion:

Neutral Particle

The metric for a dyonic Reissner Nordstr$\ddot{o}$m (DRN) black hole is given as

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q_e^2 + Q_m^2}{r^2}.$$

Particle Dynamics Around a Charged Black Hole

The corresponding horizons (for $0 < Q_e^2 + Q_m^2 < M^2$) can be obtained by solving $f(r) = 0$.

$$r_\pm = M \pm \sqrt{M^2 - Q_e^2 - Q_m^2}.$$

The extremal condition is defined at $M^2 = Q_e^2 + Q_m^2$, where the horizons degenerate. The case $Q_e^2 + Q_m^2 > M^2$ leads to naked singularity.

Particle Dynamics Around a Charged Black Hole

The integrals of motion are generated by the following Killing vectors

$$\xi^{\mu}_{(t)}\partial_{\mu} = \partial_t, \quad \xi^{\mu}_{(\phi)}\partial_{\mu} = \partial_{\phi},$$

where $\xi^{\mu}_{(t)} = (1, 0, 0, 0)$ and $\xi^{\mu}_{(\phi)} = (0, 0, 0, 1)$ are two Killing vector fields.

Particle Dynamics Around a Charged Black Hole

The conserved quantities are the specific energy E (per unit mass) and azimuthal angular momentum L_z (per unit mass) given as

$$E = \dot{t} f(r), \quad L_z = \dot{\phi} r^2 \sin^2\theta,$$

The total angular momentum of the particle is defined as [V. Frolov, The Galactic Black Hole, eds. by H. Falcke, F.H. Hehl. IoP (2003)]

$$L^2 = r^4 \dot{\theta}^2 + \frac{L_z^2}{\sin^2\theta} = r^2 v_\perp^2 + \frac{L_z^2}{\sin^2\theta}.$$

Here $v_\perp^2 \equiv -r\dot{\theta}$.

Particle Dynamics Around a Charged Black Hole

Using normalization condition $u^\mu u_\mu = -1$, we obtain

$$\dot{r}^2 = E^2 - f(r)(1 + \frac{L^2}{r^2 \sin^2 \theta}),$$

At turning points of moving particles $\dot{r} = 0$, it follows that

$$E^2 = f(r)(1 + \frac{L^2}{r^2 \sin^2 \theta}) \equiv V_{eff}(r),$$

The innermost stable circular orbit (ISCO) for the DRN BH in the equatorial plane ($\theta = \frac{\pi}{2}$) can be obtained by solving

$V_{eff,r} = 0$

$$Mr_0^3 - (L + Q_e^2 + Q_m^2)r_0^2 + 3ML^2 r_0 - 2L^2(Q_e^2 + Q_m^2) = 0.$$

Particle Dynamics Around a Charged Black Hole

The energy and azimuthal angular momentum corresponding to r_0 are

$$L_{z0} = \sqrt{\frac{r_0^2(Q_e^2+Q_m^2-Mr_0)}{3Mr_0-r_0^2-2(Q_e^2+Q_m^2)}},$$

$$E_0^2 = \frac{(r^2-2Mr+Q_e^2+Q_m^2)^2}{r_0^2(2(Q_e^2+Q_m^2)-3Mr_0+r_0^2)}.$$

Particle Dynamics Around a Charged Black Hole

After collision the angular momentum and energy (at the equatorial plane) become

$$L^2 = L_{\mathrm{z}0}^2 + r_0^2 v_\perp^2, \quad E = \sqrt{E_0^2 + v_\perp^2 f(r_0)}.$$

The particle will escape to infinity if $E \geq 1$ yielding

$$|v_\perp| \geq \sqrt{(1 - E_0) f(r_0)^{-1}}.$$

Charged Particle

Following [Stuchlik, Z.: Bull. Astron. Inst. Czechosl. **34**(1983)129] we calculate the equations of motion for the electrically charged particles and magnetic monopoles. The electromagnetic field of the DRN BH is given by the vector potential.

$$\mathcal{A}_\mu = \frac{1}{r^2}[rQ_e\delta^t_\mu - r^2 Q_m \cos\theta \delta^\phi_\mu].$$

Particle Dynamics Around a Charged Black Hole

The motion of the test particle with electric charge e and rest mass m is given by the Lorentz equation

$$m\frac{\mathcal{D}u^{\mu}}{\mathcal{D}\tau} = eF^{\mu}_{\nu}u^{\nu},$$

where $u^{\nu} = \frac{dx^{\nu}}{d\tau}$ and

$$F^{\mu}_{\nu} = \mathcal{A}^{\nu}_{;\ \mu} - \mathcal{A}^{\mu}_{;\ \nu}.$$

Particle Dynamics Around a Charged Black Hole

The Lagrangian for a particle of electric charge e is

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} u^{\mu} u^{\nu} - e\mathcal{A}_{\mu} u^{\mu}.$$

The constants of motion take the form

$$E = \dot{t} f(r) + \frac{eQ_e}{r}, \quad L_{\mathrm{z}} = r^2 \sin^2\theta \dot{\phi} + eQ_m \cos\theta.$$

Using normalization condition ($u_{\mu} u^{\mu} = -1$), we obtain

$$\dot{r}^2 = (E - \frac{eQ_e}{r})^2 - f(r)[r^2\dot{\theta}^2 + 1 + r^2 \sin^2\theta (\frac{L_z^2 - eQ_m \cos\theta}{r^2 \sin^2\theta})^2],$$

at tuning points $\dot{r} = 0$ and on equatorial plane $\theta = \frac{\pi}{2}$, we have

$$V_{eff} = E = [f(r)[1 + r^2 (\frac{L_z}{r^2})^2]]^{\frac{1}{2}} + \frac{eQ_e}{r}.$$

Particle Dynamics Around a Charged Black Hole

The motion of particles with a magnetic monopole charge g is given as

$$m\frac{\mathcal{D}u^{\mu}}{\mathcal{D}\tau} = g\mathcal{F}^{\mu}_{\nu}u^{\nu},$$

The tensor $\mathcal{F}^{\mu}_{\nu}$ is the dual to F^{μ}_{ν}, can be easily determined under the transformations $Q_e \rightarrow Q_m$ and $Q_m \rightarrow -Q_e$, also the above equations can be easily found for the motion of magnetic monopole.

Equations of Motion in Dimensionless Form

We introduce the following dimensionless quantities

$$r_g = 2M, \quad \sigma = \frac{\tau}{r_g}, \quad l = \frac{L_z}{r_g}, \quad \rho = \frac{r}{r_g}, \quad \acute{e} = er_g, \quad \acute{g} = gr_g,$$
$$Q_e = q_e r_g, \quad Q_m = q_m r_g,$$

The normalization condition yields

$$
\begin{aligned}
(\frac{d\rho}{d\sigma})^2 &= (E-\frac{\acute{e}q_e}{\rho})^2-(\rho^2-\rho+q_e^2+q_m^2)(\frac{d\theta}{d\sigma})^2\\
&- (\frac{\rho^2-\rho+q_e^2+q_m^2}{\rho^2})[1+\rho^2\sin^2\theta\\
&\times (\frac{l+\rho^2\sin^2\theta(l-\acute{e}q_m\cos\theta}{\rho^2\sin^2\theta})^2].
\end{aligned}
$$

Effective potential and its Consequences

In equatorial plane we have

$$V_{eff} = E = [(1 - \frac{1}{\rho} + \frac{q_e^2 + q_m^2}{\rho^2})[1 + \rho^2(\frac{l}{\rho^2})^2]]^{\frac{1}{2}} + \frac{\acute{e} q_e}{\rho}.$$

After the collision, the particle energy will become

$$E = [E_0^2 + f(\rho_0) v_\perp^2]^{\frac{1}{2}},$$

For ($E \geq 1$), the particle will escape when

$$|v_\perp| \geq \sqrt{(1 - E_0) f(\rho_0)^{-1}}.$$

Solving $V_{,\rho} = 0$ and $V_{,\rho\rho} = 0$, we can obtain l, assuming $q_e = 0.2$ and $q_m = 0.5$.

Effective potential and its Consequences

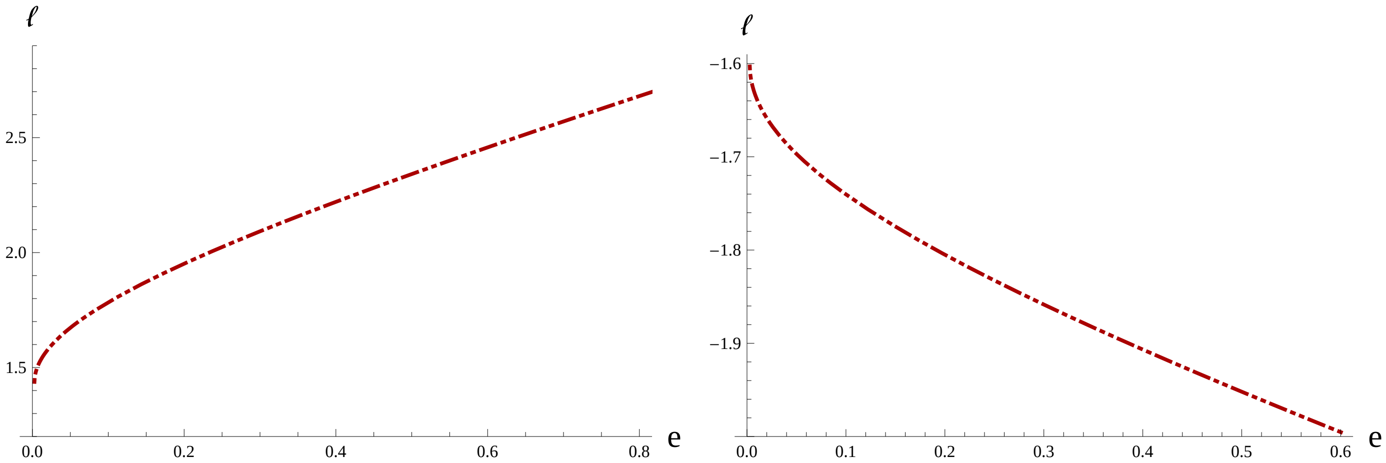


Figure 1: Behavior of angular momentum (l_+ (left), l_- (right)) versus e.

Effective potential and its Consequences

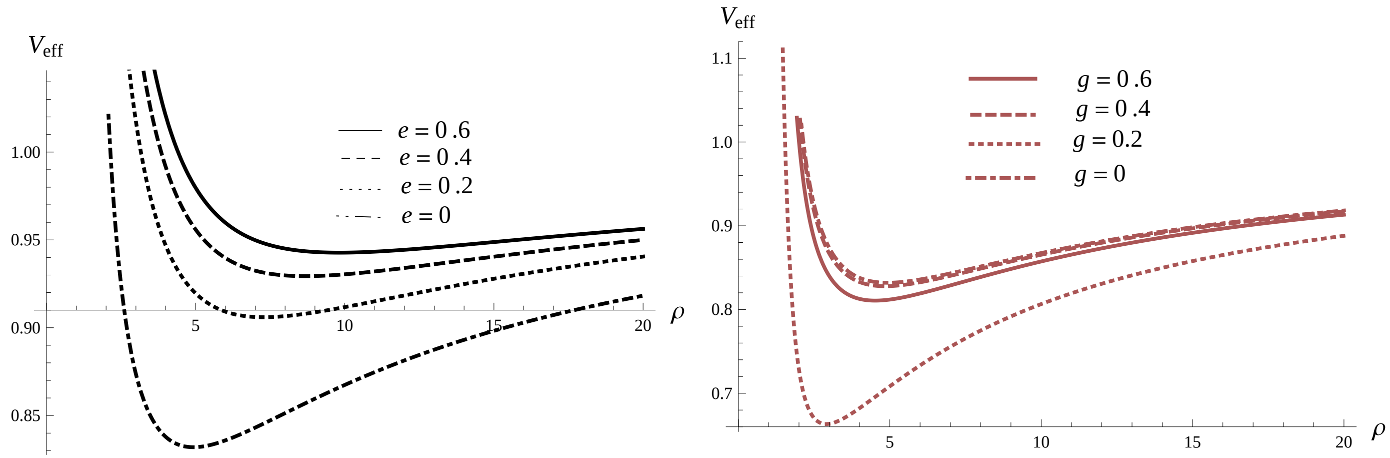


Figure 2: Effective potential as a function of ρ for $q_e = 0.2$ and $q_m = 0.5$ and $l = 2.5$, for the electrically (left) and magnetically (right) charged particles.

Effective potential and its Consequences

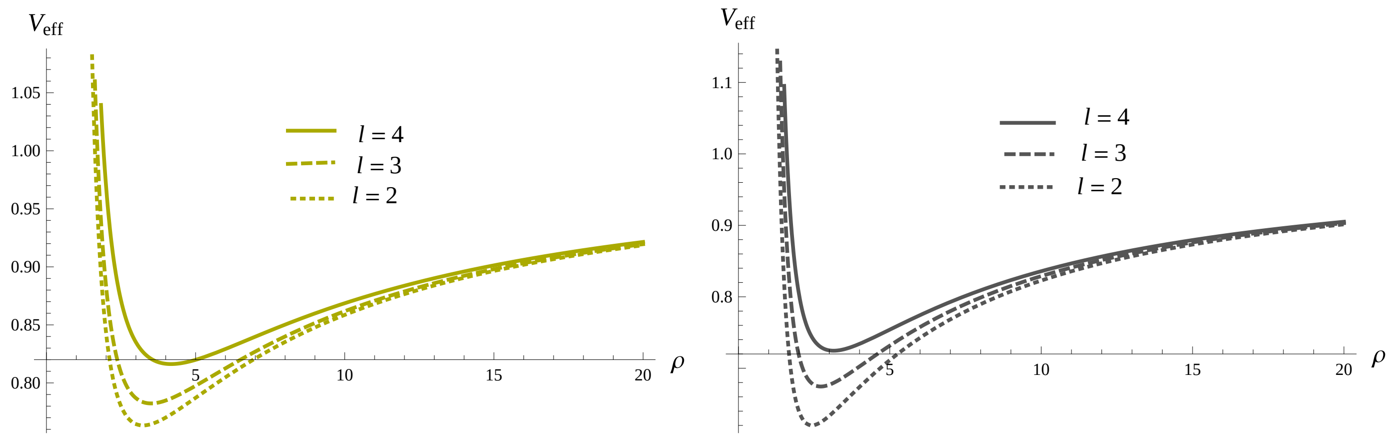
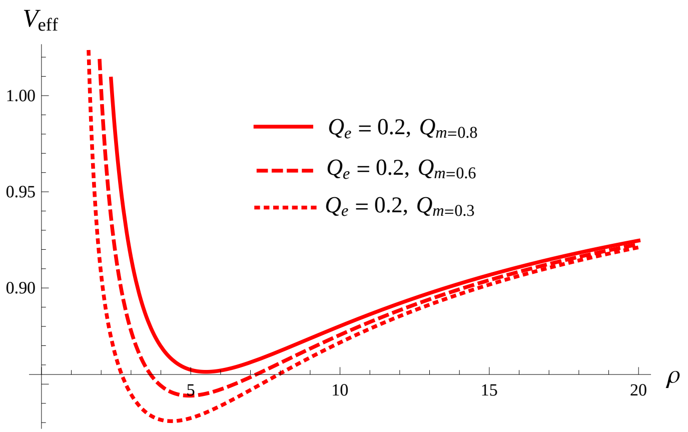


Figure 3: Effective potential as a function of ρ for $q_e = 0.2$ and $q_m = 0.5$, $e = 0.3$ (left) and $g = 0.5$ (right).

Effective potential and its Consequences



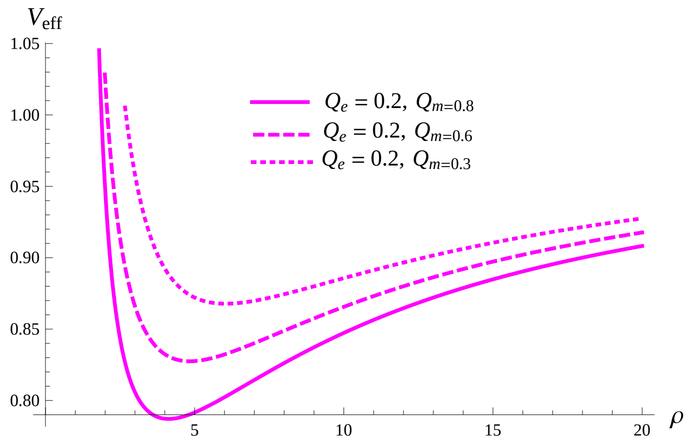


Figure 4: Effective potential as a function of ρ for $l = 2.5$, $e = 0.3$ (left) and $g = 0.5$ (right).

Effective potential and its Consequences

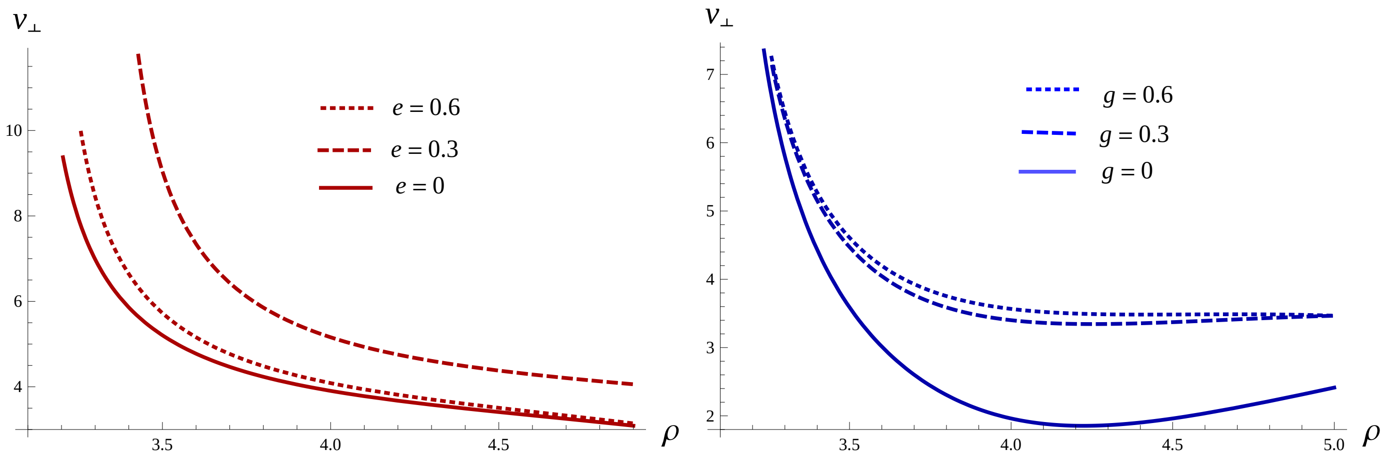


Figure 5: Escape velocity $v_\perp$ as a function of ρ for $q_e = 0.2$, $q_m = 0.5$ and $l = 2.5$.

Effective potential and its Consequences

Effective Force on the Particle

The effective force on the particle (in equatorial plane) using the effective potential for the charged particle is calculated as

$$F = -\frac{M}{r^4}[3L^2 + r^2] + \frac{L^2}{r^3} + (\frac{Q_e^2 + Q_m^2}{r^5})[2L^2 + r^2] + \frac{2eQ_e}{r^5}(\frac{eQ_e}{r} - 1).$$

Effective potential and its Consequences

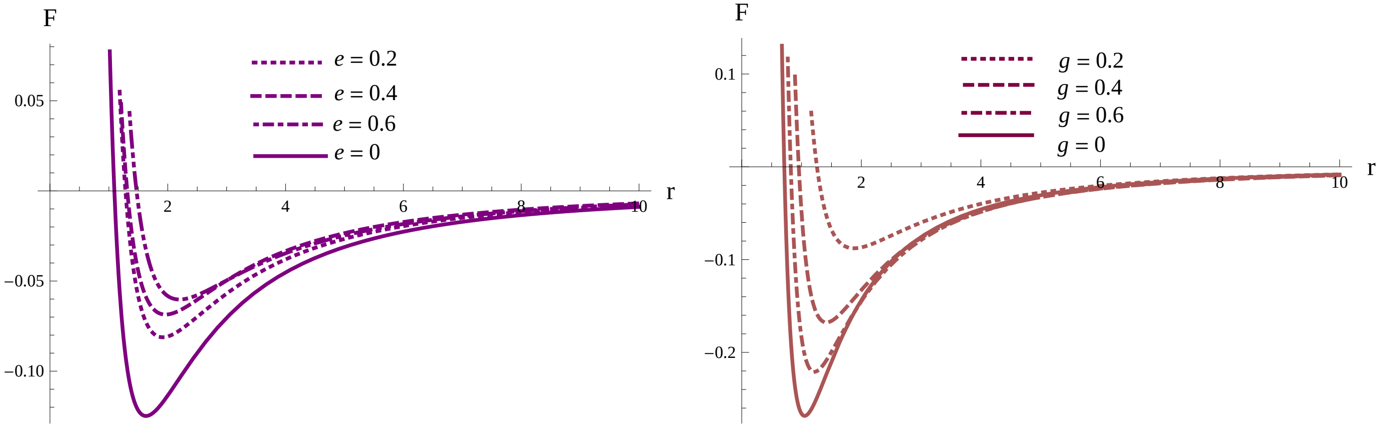


Figure 6: Behavior of the effective force on the particle with respect to r. Here, $L = 2,\ M = 1$ and $Q_e = 0.2, Q_m = 0.4$.

Lyapunov Exponent

The Lyapunov exponent is given as [Cardoso, V. et al.: Phys. Rev. D **79**(2009)064016]

$$
\begin{aligned}
\lambda &= \sqrt{\frac{-V''_{eff}(r_0)}{2\dot{t}(r_0)}} \\
&= [\frac{1}{L^2 r^2}[-2e^2 Q_e^2 r^4 - eQ_e(Q_e^2 - 2r)r^4 + 4L^2 r^2 (Q_e^2 + Q_m^2 - Mr) \\
&+ 3L^2 r^2 (Q_e^2 + Q_m^2 + r(-2M + r)) + (3Q_e^2 + 3Q_m^2 - 2Mr) \\
&\times (L^2 + r^2)(Q_e^2 + Q_m^2 + r(-2M + r))]]^{1/2}|_{r=r_0}.
\end{aligned}
$$

Effective potential and its Consequences

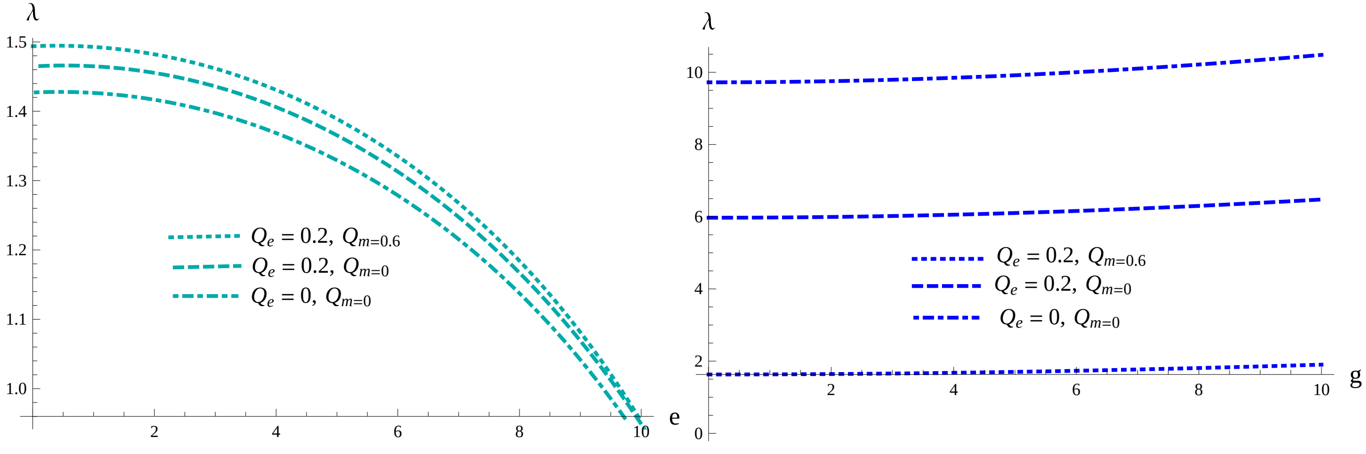


Figure 7: Behavior of Lyapunov exponent with respect to e (left) and g (right).

Collision Energy of Two Particles

We consider the collision of two neutral particles having the same rest mass m_0 moving in the same plane with 4-velocities u_1^μ and u_2^ν. The conserved angular momentum and energy of two particles are L_1, L_2, E_1 and E_2. The center of mass energy (CME, E_{cm}) of these particles is defined as

$$\frac{E_{cm}}{\sqrt{2}m_0} = \sqrt{1 - g_{\mu\nu}u_1^\mu u_2^\nu},$$

where

$$u_i^\mu = (\frac{E_i}{f(r)}, -K_i, 0, \frac{L_i}{r^2}), \quad K_i = \sqrt{E_i^2 - f(r)(1 + \frac{L_i^2}{r^2})}, \quad i = 1, 2.$$

Collision Energy of Two Particles

Inserting these values in above equation, it follows that

$$\frac{E_{cm}}{\sqrt{2}m_0} = \sqrt{1+\frac{E_1E_2}{f(r)}-\frac{K_1K_2}{f(r)}-\frac{L_1L_2}{r^2}}$$
$$= [1+\frac{1}{2}(\frac{E_1^2+E_2^2}{E_1E_2})+\frac{f(r)}{E_1E_2}(L_1^2+L_2^2)$$
$$+ \frac{1}{2r^2}(\frac{E_2^2L_1^2+E_1^2L_2^2}{E_1E_2})+L_1L_2(\frac{L_1L_2f(r)-4E_1E_2r^2}{4E_1E_2r^4})]^{\frac{1}{2}}.$$

Substituting the value of $f(r)$ and taking $E_1=E_2=1$ for simplicity, we obtain CME for the DRN BH. Near the horizon (r_+), the above equation takes the form

$$\frac{E_{cm}}{\sqrt{2}m_0} = [\frac{4r_+^2+(L_1-L_2)^2}{2r_+^2}]^{\frac{1}{2}}.$$

Collision Energy of Two Particles

Now we consider the collision of charged particles. In this case, the 4-velocity of the particle is

$$u_i^\mu = (\frac{\acute{E}_i}{f(r)}, -K_i, 0, \frac{L_i}{r^2}), \quad \acute{E}_i = (E_i - \frac{eQ_e}{r}),$$

$$K_i = [\acute{E}_i^2 - f(r)(1 + r^2(\frac{L_i^2}{r^2})]^{1/2}.$$

Collision Energy of Two Particles

The corresponding CME can be written as

$$
\begin{aligned}
\frac{E_{cm}}{\sqrt{2}m_0} &= [1+\frac{\acute{E}_1\acute{E}_2}{f(r)}-\frac{K_1K_2}{f(r)}-\frac{L_1L_2}{r^2}]^{1/2} \\
&= [1+\frac{1}{2}(\frac{\acute{E}_1^2+\acute{E}_2^2}{\acute{E}_1\acute{E}_2})+\frac{f(r)}{\acute{E}_1\acute{E}_2}(L_1^2+L_2^2) \\
&+ \frac{1}{2r^2}(\frac{\acute{E}_2^2L_1^2+\acute{E}_1^2L_2^2}{\acute{E}_1\acute{E}_2})-\frac{f(r)}{4\acute{E}_1\acute{E}_2} \\
&+ L_1L_2(\frac{L_1L_2f(r)-4\acute{E}_1\acute{E}_2r^2}{4\acute{E}_1\acute{E}_2r^4})]^{\frac{1}{2}}.
\end{aligned}
$$

The CME for magnetic monopole charge can be obtained by the above equation with $\acute{E}_i=(E_i-\frac{gQ_m}{r})$

Concluding Remarks

The results are summarized as follows:

- The effective potential attains smaller values for the neutral particle $(e = 0 = g)$ than the charged particle.
- We have observed that the particles have more chances to escape to infinity in the presence of electromagnetic field.
- The effective force on the particle initially shows repulsive behavior and then becomes attractive.
- The behavior of Lyapunov exponent indicates the instabilities in orbits.
- The CME of two colliding particles remains finite if the

Concluding Remarks

angular momentum is finite (in the absence or presence of electromagnetic field).

Thank You

The corresponding dynamical equations can be written as

$$
\begin{aligned}
\ddot{\theta} &= -\frac{2}{r}\dot{r}\dot{\theta} + \sin\theta\cos\theta[\frac{L_z^2 - eQ_m^2\cos\theta}{r^2\sin^2\theta}]^2 - \frac{eQ_m^2}{r^2} \\
&\times \sin\theta[\frac{L_z^2 - eQ_m^2\cos\theta}{r^2\sin^2\theta}], \\
\ddot{r} &= (r - 2M + \frac{(Q_e^2 + Q_m^2)}{r})[\dot{\theta}^2 + (\frac{L_z^2 - eQ_m^2\cos\theta}{r^2\sin^2\theta})] \\
&- [(E - \frac{eQ_e^2}{r})^2 - \dot{r}^2]\frac{2Mr - 2(Q_e^2 + Q_m^2)}{2r^2(r - 2M + Q_e^2 + Q_m^2)} - \frac{eQ_e^2}{r^2}(E - \frac{eQ_e^2}{r}).
\end{aligned}
$$

Using these quantities, the above equations become

$$\begin{aligned}
\frac{d^2\theta}{d\sigma^2} &= -\frac{2}{\rho}(\frac{dr}{d\sigma}\frac{d\theta}{d\sigma}) - \sin\theta\cos\theta[\frac{l - \acute{e}q_m^2\cos\theta}{\rho^2\sin^2\theta}]^2 \\
&- \frac{\acute{e}q_m^2}{r^2}\sin\theta[\frac{l_z - \acute{e}q_m^2\cos\theta}{\rho^2\sin^2\theta}], \\
\frac{d^2\rho}{d\sigma^2} &= (\rho - 1 + \frac{(q_e^2 + q_m^2)}{\rho})[(\frac{d\theta}{d\sigma})^2 + (\frac{l^2 - \acute{e}q_m^2\cos\theta}{\rho^2\sin^2\theta})] \\
&- [(E - \frac{\acute{e}q_e^2}{\rho})^2 - (\frac{d\rho}{d\sigma})^2]\frac{\rho - 2(q_e^2 + q_m^2)}{2\rho^2(\rho^2 - \rho + q_e^2 + q_m^2)} \\
&- \frac{\acute{e}q_e^2}{\rho^2}(E - \frac{\acute{e}q_e^2}{\rho}).
\end{aligned}$$

The homogenous Maxwell Field equations can also be written in an equivalent form $\partial_\mu \tilde{F}_{\mu\nu} = 0$ with dual electromagnetic field tensor $F_{\mu\nu} = -\frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$. In terms of the elements replacement of $F^{\mu\nu} \rightarrow \tilde{F}^{\mu\nu}$ correspond to the dual rotation of **E** $\rightarrow$ **B** and **B** $\rightarrow$ **-E** [Einstein's Physics: Atoms, Quanta, and Relativity - Derived, Explained, and Appraised By Ta-Pei Cheng (OUP Oxford, 2013)].

Maxwell's Equations: [Black Hole Gravitohydromagnetics By B. Punsly (Springer Science and Business Media, 2013); Finite Element Methods for Maxwell's Equations By Peter Monk (Clarendon Press, 2003)]

Gauss's law for electricity: $\nabla.E = \frac{\rho}{\varepsilon_o} = 4\pi k\rho$.

Gauss's law for electricity: $\nabla.B = 0$.

Faraday's law of induction: $\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$.

Ampere's law: $\nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}j$.

Geodesic equation (remaining solution): [Chandrasekhar, S.: *The Mathematical Theory of Black Holes* (Clarendon Press, 1998)]

$$\frac{d}{d\lambda}(g_{ij}\dot{x}^i) - \frac{1}{2}g_{ik,j}\dot{x}^i\dot{x}^k = 0,$$

$$g_{ij}\ddot{x}^i + (g_{ij,k} - \frac{1}{2}g_{ik,j})\dot{x}^i\dot{x}^k = 0,$$

or alternatively,

$$g_{ij}\ddot{x}^i + \frac{1}{2}(g_{ij,k} + g_{kj,i} - g_{ik,j})\dot{x}^i\dot{x}^k = 0.$$

Contracting with g^{jl}, we obtain

$$g_{ij}\ddot{x}^i\Gamma^l_{ik}\dot{x}^i\dot{x}^k = 0.$$

Both stable and unstable obits exist according to the minima and maxima of V_{eff}. This correspond to [Kološ, M. et al.: Class. Quantum Grav. **32**(2015)165009] where both (stable/unstable) motion of particles is possible near the Schwarzschild BH immersed in the uniform magnetic field.

The effective potential attains smaller values for the neutral particle ($e = 0 = g$) than the charged particle which yields that the neutral particle travels in the stable orbit (when the radius is small). This behavior is similar to [Stuchlik, Z. and Hledik, S.: Acta Physica Slovaca **52**(2002)363] where the properties of RN BH (also the naked singularity) with non vanishing cosmological constant has been studied.

The study of circular orbits yield that the stable region exists for all asymptotically anti-de Sitter BHs as well as for some asymptotically de Sitter BHs.

The case motion of neutral particle for DRN BH also corresponds to RN BH [Pugliese, D.: Phys. Rev. D **83**(2011)024021]. In DRN BH there exists at leat one minimum point that leads to stable orbit, similarly in RN BH there is a single zone of stability.

The effective potential for the electrically charged particle attains higher values for large values of charge which indicates instability in the orbits while in the case of magnetic monopole charge the effective potential corresponds to small vales. We also note that the large values angular momentum leads to stable motion.

We have also studied the effect of magnetic field and electric as well as magnetic charges on the escape velocity of particles. We have observed that the particles have more chances to escape to infinity in the presence of electromagnetic field.

The escape velocity also depends on the electric, magnetic charges and the angular momentum of the particle. The situation for rotating magnetized BH [Stuchlik, Z. and Martin Kološ, M.: Eur. Phys. J. C **76**(2016)32] is slightly different as their motion is not only influenced by the uniform magnetic field but also the spin of the BH.

The CME of two colliding particles does not diverge in the absence or presence of magnetic field. The effective force on the particle initially shows repulsive behavior and then becomes attractive. This contrast to the magnetized Kerr BH [Tursunov, A. et al.: Phys. Rev. D **93**(2016)084012] where four types of motion is possible corresponding to the spin of BH and angular momentum of the particle.

The behavior of Lyapunov exponent indicates that the instabilities in orbits increase in the presence electric and magnetic charges (with external electromagnetic field) is greater than the Schwarzschild as well as RN BHs. We can conclude that both electric and magnetic charges as well as external electromagnetic field deviate the particles from their original path.

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