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Third Quantization of Taub Universe

- Quantum Field Theory of Taub Cosmology

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[Why "Third" Quantization ?]

physical system→ matter + gravity   in which :

matter = dynamical field (to be second quantized)

gravity = background geometry~ arena (to be third quantized)

That is, the gravity is an arena (play ground) on which the (quantum) matter propagates !

Quantum Gravity versus Quantum Cosmology

Quantum Gravity

the degree of freedom to be quantized → metric tensor

the quantization scheme → covariant quantization in which :

metric tensor = background field + quantum fluctuation [perturbation expansion]

Lorentz invariance → preserved

Renormalization → Einstein's general theory of relativity is just

1-loop renormalizable ['tHooft - Veltman]

Quantum Cosmology

the degree of freedom to be quantized → state vector (universe wave function which is a solution to the Wheeler-DeWitt eq.)

the quantization scheme → canonical(Hamiltonian) quantization in which :

universe wave function = creation/annihilation op. *mode functions [mode expansion]

Lorentz invariance → broken [ADM's space-plus-time split formalism]

Issue to be addressed → (time) evolution of the universe state(wave function)

[from “Quantum Mechanics” to “Quantum Field Theory” of cosmology]

Quantum Mechanics (1st Quantization) of the universe wave function(=state)

: due to Wheeler-DeWitt

Classical Hamiltonian constraint H= 0. [time-re parametrization invariance in the Einstein-Hilbert action]

→ [off-shell promotion] the Wheeler-DeWitt (WDW) equation ^H_{op}= 0.

Quantum Field Theory (2nd Quantization) of the universe field operator

: due to Hosoya-Morikawa

a Lagrangian (action)

→[on-shell condition] the Euler-Lagrange’s field equation ~Wheeler-DeWitt (WDW) equation

^H_{op}= 0

-> this is the starting point of :Third Quantization of Taub Universe

- **Quantum Field Theory of Taub Cosmology**

[Bogoliubov transformation] which is responsible for the creation/production of quanta (universes in the Quantum Gravity/Cosmology case) in the 2nd Quantization(i.e., Quantum Field Theory)

In the context of the canonical(Hamiltonian) quantization in ordinary particle field theory, or

[ADM's space-plus-time split formalism] for the case of quantum cosmology,

the choice of mode functions to span the (universe) field operator is NOT unique as each set of mode functions is separately complete, orthonormal (basis in the Hilbert space).

This is because the physical state of the system (subject to the mode expansions) keeps changing during its (time) evolution.

As a result, the out-state (t→=+infty) mode functions can be expanded in terms of the

the in-state (t→=-infty) mode functions and vice versa

→ This (linear) Bogoliubov transformations result in non-trivial mode(or, frequency) mixing,

Which eventually leads to the creation/production of quanta (universes for the case of quantum cosmology) !

[Bogoliubov transformation] in terms of mode functions

the Bogoliubov coefficient C2 is responsible for “frequency-mixing” that associates
the negative-frequency in/out modes with the positive-frequency in/out modes .

[Bogoliubov transformation] in terms of ladder operators

the Bogoliubov coefficient C2 is associated with “mode-mixing” that associates
in/out creation operator with out/in annihilation operator .

à Therefore, this Bogoliubov coefficient C2 is of our major interest and concern as
It is the one that is responsible for the frequency/ mode-mixing !

[What is the Taub Cosmology model ?]

A.H. Taub, Am. Math. 53, 472 (1951)

Bianchi – type IX (Mixmaster) Cosmology

: the most general context of homogeneous but anisotropic cosmology model

→ take a special limit

Taub (universe) Cosmology model

: the simplest homogeneous but anisotropic cosmology model

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Bianchi-type IX (Mixmaster) Versus Taub Cosmologies

The starting set-up to explore classical/Quantum Cosmologies in the context of the Bianchi-type IX/Taub (mini) superspace model(s) is to write the Lagrangian (i.e. action), Hamiltonian and the consequent field equations in terms of the minisuperspace variables $q^{i} = (\alpha, \beta_{+}, \beta_{-})$ and the canonical conjugate momenta $\pi_{i} = (p_{\alpha}, p_{+}, p_{-})$.

(mini) superspace ??
In ADM (Arnowitt–Deser–Misner)'s (3+1) space-plus-time split formalism for GR, the Hilbert space for the metric functions and matter fields on it.

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line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2(t)\, dt^2 + h_{ij}\,\sigma^i \sigma^j$$

$$= -N^2(t)\, dt^2 + e^{2\alpha(t)} \left(e^{2\beta(t)}\right)_{ij} \sigma^i \otimes \sigma^j$$

$$= -\boxed{N^2(t)}\, dt^2 + \boxed{e^{2\alpha(t)}} \left[e^{2\beta_1(t)} \sigma_1^2 + e^{2\beta_2(t)} \sigma_2^2 + e^{2\beta_3(t)} \sigma_3^2 \right]$$

<Lapse function>

$\parallel$ $a(t)$ <scale factor>

In the diagonal Bianchi-type IX metric,

$$\beta_{ij}(t) = \mathrm{diag}(\beta_1, \beta_2, \beta_3)$$

$$\begin{cases} \beta_1 = \beta_+ + \sqrt{3}\beta_-, \\ \beta_2 = \beta_+ - \sqrt{3}\beta_-, \\ \beta_3 = -2\beta_+ \end{cases} \Rightarrow \left[\begin{array}{l} \mathrm{Tr}(\beta) = 0, \\ \det(e^{2\beta}) = 1 \end{array} \right]$$

and $\{\sigma^i\}$ form a basis on 3-sphere S^3 and satisfy the SU(2) "Maurer-Cartan" structure equation

$$\begin{cases} \sigma^1 = \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi \\ \sigma^2 = \sin\psi\, d\theta - \cos\psi \sin\theta\, d\phi \\ \sigma^3 = d\psi + \cos\theta\, d\phi \end{cases}$$

$$\boxed{d\sigma^i = \frac{1}{2}\epsilon^{ijk}(\sigma^j \wedge \sigma^k)}$$

$$\left(\begin{array}{l} d\sigma^1 = \sigma^2 \wedge \sigma^3, \\ d\sigma^2 = \sigma^3 \wedge \sigma^1, \\ d\sigma^3 = \sigma^1 \wedge \sigma^2 \end{array} \right)$$

(θ, ϕ, ψ) Euler angles parametrizing S^3

$$\left\{ \begin{array}{l} 0 \le \theta \le \pi, \\ 0 \le \phi < 2\pi, \\ 0 \le \psi \le 4\pi \end{array} \right\}$$

namely, $\{\sigma^i\}$ generators of Lie group SU(2) obeying this Lie Algebra.

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[Einstein-Hilbert action in the presence of the cosmological constant $(\Lambda > 0)$]

$$S = \frac{1}{16\pi G}\left[\int_M \underbrace{d^4x\sqrt{g}\,(R-2\Lambda)}_{\text{Einstein-Hilbert bulk term}} + 2\int_{\partial M} \underbrace{d^3x\sqrt{h}\,(K-K_0)}_{\text{Gibbons-Hawking surface term}}\right]$$

; trace of the 2nd fundamental form of the spacelike boundary ∂M in the metric $(g_{\mu\nu}, h_{\mu\nu})$ respectively

terms of the mini-super space variables of the Bianchi-type ~~IX~~ Taub model,

$$\left[\begin{array}{l} r^\alpha = (\alpha \equiv \ln a, \beta_+, \beta_-), \\ \pi_\alpha = (p_\alpha, p_+, p_-) \end{array}\right. \quad \text{(where)} \left\{\begin{array}{l} p_\alpha = \frac{\partial S}{\partial \dot{\alpha}} = \frac{1}{N} e^{3\alpha}(-\dot{\alpha}), \\ p_+ = \frac{\partial S}{\partial \dot{\beta}_+} = \frac{1}{N} e^{3\alpha}(\dot{\beta}_+), \\ p_- = \frac{\partial S}{\partial \dot{\beta}_-} = \frac{1}{N} e^{3\alpha}(\dot{\beta}_-) \end{array}\right.$$

Lagrangian (i.e., action), and Hamiltonian of the spatial gravity system are given by :

For convenience, henceforth ; we redefine the action by multiplying it by an overall constant $S \Rightarrow \left(\frac{6\pi}{G}\right)^{-1} S$

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$$S = \int dt\, L_{ADM}$$

$$= \frac{1}{2}\int dt\, N e^{3\alpha}\left\{\frac{1}{N^2}\left(-\dot{\alpha}^2 + \dot{\beta}_+^2 + \dot{\beta}_-^2\right) - \left(e^{-2\alpha}V(\beta) + \frac{2}{3}\Lambda\right)\right\}$$

Mixmaster model potential

$$V(\beta) = -\frac{1}{3}\mathrm{Tr}\left[2e^{-2\beta} - e^{4\beta}\right]$$

$$= \frac{1}{3}e^{-8\beta_+} - \frac{4}{3}e^{-2\beta_+}\cosh 2\sqrt{3}\beta_- + \frac{2}{3}e^{4\beta_+}\left(\cosh 4\sqrt{3}\beta_- - 1\right)$$

..., the ADM Hamiltonian can be read off via the Legendre transformation as

$$S = \int dt\, L_{ADM} = \int dt\left\{\left(p_\alpha\dot{\alpha} + p_+\dot{\beta}_+ + p_-\dot{\beta}_-\right) - H_{ADM}\right\}$$

$$H_{ADM} \equiv N H_0$$

$$H_0 = \frac{1}{2}e^{-3\alpha}\left[-p_\alpha^2 + p_+^2 + p_-^2 + \left(e^{4\alpha}V(\beta) + \frac{2}{3}\Lambda e^{6\alpha}\right)\right]$$

$$\approx 0$$

Hamiltonian constraint $\left\{\text{a primary constraint of the system that results from the time-reparametrization invariance of the Einstein-Hilbert action}\right\}$

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Taub Universe limit of Bianchi-typeIX cosmology

Anisotropy measure of the Bianchi-typeIX model

$$\beta_{ij}(t) = \mathrm{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$$

{ setting $\beta_- = 0$, keeping $\beta_+ \equiv \beta \neq 0$ }

$$\beta_{ij}(t) = (\beta, \beta, -2\beta)$$

< ADM-Lagrangian >

$$S = \int dt\, L_{ADM} = \frac{1}{2}\int dt\, N e^{3\alpha}\left\{\frac{1}{N^2}\left(-\dot{\alpha}^2 + \dot{\beta}^2\right) - \left(e^{-2\alpha}V(\beta) + \frac{4}{3}\Lambda\right)\right\}$$

$$V(\beta) \equiv \frac{1}{3}\left(e^{-8\beta} - 4e^{-2\beta}\right)$$

$$H_0 = \frac{1}{2}e^{-3\alpha}\Big[-p_\alpha^2 + p_\beta^2 + \underbrace{\boxed{e^{4\alpha}\left\{V(\beta) + \frac{4}{3}\Lambda e^{2\alpha}\right\}}}_{\equiv U(\alpha,\beta)}\Big]$$

$$= \frac{1}{2}e^{-3\alpha}\left[\partial_\alpha^2 - \partial_\beta^2 + U(\alpha,\beta)\right] = 0$$

Hamiltonian constraint

< $p_\alpha \to -i\partial_\alpha$, $p_\beta \to -i\partial_\beta$ >

< Wheeler-DeWitt (WD) Equation >

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Quantum Mechanics (1st quantization) of the Taub cosmology

Hamiltonian constraint Equation

$$H(\hat{P}_\alpha, \hat{P}_\beta ; \alpha, \beta)\,\Psi[\alpha,\beta] = 0$$

Wheeler-DeWitt (WD) Equation

$$[\partial_\alpha^2 - \partial_\beta^2 + U(\alpha,\beta)]\,\Psi[\alpha,\beta] = 0$$

where

$$\hat{P}_\alpha = -\frac{\partial}{\partial\alpha}, \quad \hat{P}_\beta = -\frac{\partial}{\partial\beta},$$

$$U(\alpha,\beta) \equiv e^{4\alpha}\left(V(\beta) + \frac{\Lambda}{3} e^{2\alpha}\right),$$

$$V(\beta) = \frac{1}{3}[e^{-8\beta} - 4e^{-2\beta}]$$

Quantum Field Theory (2nd quantization) of the Taub universe (field operator)

We start by noticing that the WD equation above takes the form of interacting Klein-Gordon-type field equation for a scalar field Ψ! ... universe ... In order naturally to second quantize the universe as an ... wave function, we view the WD equations above as an "interacting Klein-Gordon-like" field equation which results from ... an action of the form ...

$$S = \frac{1}{2}\int d\alpha\, d\beta \left[\left(\frac{\partial\Psi}{\partial\alpha}\right)^2 - \left(\frac{\partial\Psi}{\partial\beta}\right)^2 - U(\alpha,\beta)\Psi^2\right]$$

..., we do not consider the higher polynomial terms in the ... of Ψ as they would represent the interaction of universes themselves!!

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§ Quantum Mechanics [1st Quantization] of the universe (state = wave function)

...ntal Hamiltonian constraint > Wheeler–DeWitt

$H_0(p_\alpha, p_\beta; \alpha, \beta) = 0$ {off-shell promotion}

⇓ {off-shell equation} ⟸ WDW eq. >>

Quantum Schrödinger equation

$\hat{H}_{op}[\hat{p}_\alpha \to -i\partial_\alpha, \hat{p}_\beta \to -i\partial_\beta; U(\alpha,\beta)]\Psi[\alpha,\beta] = 0$

§ Quantum Field Theory [2nd Quantization] of the universe (Field operator) ⟸ WDW eq. >>

< Euler–Lagrange field equation

$\hat{H}[\partial_\alpha^2 - \partial_\beta^2 + U(\alpha,\beta)]\Psi[\alpha,\beta] = 0$ Hosoya–Morikawa

⇑ {on-shell condition}

$S = \frac{1}{2}\int d\alpha\, d\beta \left[\left(\frac{\partial\Psi}{\partial\alpha}\right)^2 - \left(\frac{\partial\Psi}{\partial\beta}\right)^2 - U(\alpha,\beta)\Psi^2\right]$

< Theory Action, Lagrangian >

followed by the mode expansion of the field operator $\hat{\Psi}$ ... $c_k^\dagger u_k^*(\alpha,\beta)$ & $[c_k, c_{k'}^\dagger] = \delta_{kk'}$

[Concluding Remarks-after applying to Taub Universe]

[1] Creation of largely-anisotropic universes with (lambda \neq 0) from "nothing"

* Probably, we are living in one of those universes in which the cosmological constant (lambda) was non-zero from the beginning and had large anisotropy and at the same time slowly-varying anisotropy trend since they would have been produced in large numbers at the time of creation !

* In addition, these (Taub) universes with large anisotropy but with slowly-varying anisotropy trend Would have gone through an exponential expansion (i.e., inflation) period after the creation due to the non-zero cosmological constant (lambda \neq 0).

- Therefore, any initial large anisotropy would have been isotropized rapidly .

* Certainly, this is consistent with the homogeneous, isotropic LSS of the present universe !

[2] Creation of nearly-isotropic universes with (lambda = 0) from "nothing"

* Probably, we are living in one of those universes in which the cosmological constant (lambda) was zero from the beginning and had small anisotropy and at the same time slowly-varying anisotropy trend since they would have been produced in large numbers at the time of creation !

*Certainly, once again, this is consistent with the homogeneous, isotropic LSS of the present universe !

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§ Chapter I

Creation of

largely-anisotropic universes with $(\Lambda \neq 0)$

$\left(\begin{array}{l} \beta \rightarrow \infty, \\ V(\phi) \rightarrow 0 \end{array} \right)$

from "Nothing"

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<Proof> We are now ready to compute the Average # of universe created in a state designated by the mode index p:

$$N_p = \langle 0, in | \hat{N}_p^{out} | 0, in \rangle$$

$$= \langle 0, in | C_p^{\dagger out} C_p^{out} | 0, in \rangle$$

$$= \langle 0, in | \sum_q \{ -c_2(p,q)\, C_q^{in} + c_1(p,q)\, C_q^{\dagger in} \} \{ c_1^*(p,r)\, C_r^{in} - c_2^*(p,r)\, C_r^{\dagger in} \} | 0, in \rangle$$

$$= \sum_{q,r} \langle 0, in | \{ -c_2(p,q)\, c_1^*(p,r)\, C_q^{in} \underbrace{C_r^{in}}_{0} + \boxed{c_2(p,q)\, c_2^*(p,r)}\, C_q^{in} C_r^{\dagger in} + c_1(p,q)\, c_1^*(p,r)\, \underbrace{C_q^{\dagger in} C_r^{in}}_{0} - c_1(p,q)\, c_2^*(p,r)\, \underbrace{C_q^{\dagger in} C_r^{\dagger in}}_{0} \} | 0, in \rangle$$

$$= \sum_r \langle 0, in | |c_2(p,r)|^2 \delta_{q,r} \delta_{q,r} \Big(\underbrace{[C_p^{in}, C_p^{\dagger in}]}_{=1} + \underbrace{C_p^{\dagger in} C_p^{in}}_{0} \Big) | 0, in \rangle$$

$$= |c_2(p,p)|^2 = \frac{1}{e^{\frac{2}{3}\pi T/c} - 1} \left\{ \begin{array}{l} \text{restore } \hbar \\ c = \sqrt{\frac{2\pi G}{3}}\, (=1) \end{array} \right\} = \frac{1}{e^{\frac{2}{3}\pi T/c} - 1}$$

Planckian distribution

with temperature

$$T \sim \sigma = \left(\frac{2\pi G}{3}\right)^{1/2}$$

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Explicit computation of Bogoliubov coefficients

[Universe field equation ≈ (WD equation)]

$$\left[\partial_\alpha^2 - \partial_\beta^2 + U(\alpha,\beta)\right]\Psi[\alpha,\beta] = 0$$

$$\hat{P}_\alpha = -i\partial_\alpha, \quad \hat{P}_\beta = -i\partial_\beta, \quad U(\alpha,\beta) = e^{4\alpha}\left[V(\beta) + \frac{2}{3}\Lambda e^{2\alpha}\right],$$

$$V(\beta) = \frac{1}{3}\left(e^{-8\beta} - 4e^{-2\beta}\right)$$

our particular concern is in the region of minisuperspace where $[\beta \to \infty \;\&\; V(\beta) = 0]$

and there is assuming the separation of variables:

$$\Psi[\alpha,\beta] = A(\alpha)B(\beta)$$

$$\underbrace{-e^{4\alpha}\left[\frac{1}{A}\frac{\partial^2 A}{\partial\alpha^2} + \frac{2}{3}\Lambda e^{2\alpha}\right]}_{\text{function of } \alpha\text{-alone}} = \underbrace{\frac{1}{B}\frac{\partial^2 B}{\partial\beta^2}}_{\text{function of } \beta\text{-alone}} = \underbrace{-p^2}_{\text{constant}}$$

β-sector

$$\frac{\partial^2 B}{\partial\beta^2} + p^2 B = 0$$

$$B_p(\beta) = (\text{const.})\, e^{ip\beta}$$

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(2) α-sector

$$\frac{\partial^2 A}{\partial \alpha^2} + \left(\frac{4}{3}\Lambda e^{6\alpha} + p^2\right) A = 0 \qquad \nu = i\frac{1}{3}|p|$$

$$\tilde{a} = e^{3\alpha}, \quad x = \frac{2}{3\sqrt{3}}\sqrt{\Lambda}\,\tilde{a}$$

$$x^2 \frac{d^2 A}{dx^2} + x\frac{dA}{dx} + (x^2 - \nu^2) A = 0$$

$$A(\alpha) = J_\nu[x] = Z_\nu\left[\frac{2}{3}\sqrt{\frac{\Lambda}{3}}\,\tilde{a}\right]$$

$$u[\alpha, \beta] = A(\alpha) B(\beta) = N Z_\nu\left[\frac{2}{3}\sqrt{\frac{\Lambda}{3}}\, e^{3\alpha}\right] e^{ip\beta} \qquad \left(\nu = i\frac{1}{3}|p|\right)$$

normalized $\left\{ \begin{matrix} \text{in} \\ \text{out} \end{matrix} \right\}$-state mode functions

$$u^{in}[\alpha,\beta] = N^{in} e^{ip\beta} J_\nu\left[\frac{2}{3}\sqrt{\frac{\Lambda}{3}}\, e^{3\alpha}\right] \quad (\text{for } \alpha \to -\infty), \qquad N^{in} = \boxed{\left(\frac{\pi}{6}\right)^{1/2}\left[\sinh\frac{|p|}{3}\pi\right]^{-1/2}}$$

$$u^{out}[\alpha,\beta] = N^{out} e^{ip\beta} H^{(2)}_\nu\left[\frac{2}{3}\sqrt{\frac{\Lambda}{3}}\, e^{3\alpha}\right] \quad (\text{for } \alpha \to +\infty), \qquad N^{out} = \boxed{\frac{1}{2}\left(\frac{\pi}{3}\right)^{1/2} e^{-\frac{|p|}{6}\pi}}$$

$$\left\{ Z_\nu \,\middle|\, x^2 Z_\nu''(x) + x Z_\nu'(x) + (x^2 - \nu^2) Z_\nu(x) = 0 \right\}$$

$J_\nu(x)$ Bessel function of the 1st kind

$$N_\nu(x) = \frac{\cos\nu\pi\, J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi}$$ Bessel function of the 2nd kind = Neumann function

$$H^{(1)}_\nu(x) = J_\nu + iN_\nu = \frac{i}{\sin\nu\pi}\left[e^{-i\nu\pi} J_\nu(x) - J_{-\nu}(x)\right],$$ Hankel function of the 1st kind

$$H^{(2)}_\nu(x) = J_\nu - iN_\nu = \frac{-i}{\sin\nu\pi}\left[e^{i\nu\pi} J_\nu(x) - J_{-\nu}(x)\right]$$ of the 2nd kind

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[Bogolinbov coefficients]

$\boxed{(p,q)} = -(u_p^{out}, u_q^{*in}) = -\int d\beta \, i\left(u_p^{*out}\frac{\partial}{\partial\alpha}u_q^{*in} - u_q^{*in}\frac{\partial}{\partial\alpha}u_p^{*out}\right)$

sing

$u_p^{in}[\alpha,\beta] = N_{in} J_\nu[x] e^{ip\beta}$

$u_p^{*in}[\alpha,\beta] = N_{in} J_\nu^*[x] e^{-ip\beta} = N_{in} J_{-\nu}[x] e^{-ip\beta}$

$\frac{\partial}{\partial\alpha}u_p^{in} = \frac{\partial x}{\partial\alpha}\frac{\partial}{\partial x}u_p^{in} = N_{in}(\partial x) J_\nu'[x] e^{ip\beta}$

$\frac{\partial}{\partial\alpha}u_p^{*in} = \frac{\partial x}{\partial\alpha}\frac{\partial}{\partial x}u_p^{*in} = N_{in}(\partial x) J_{-\nu}'[x] e^{-ip\beta}$

here) $x = \frac{2}{3}\sqrt{3}\, e^{\alpha}$, $\nu = -\frac{i}{3}|p|$

$\boxed{N_{in} \equiv \left(\frac{\pi}{3}\right)^{1/2}\left[\sinh\frac{|p|}{3}\pi\right]^{-1/2}}$

$u_p^{out}[\alpha,\beta] = N_{out} H_\nu^{(2)}[x] e^{ip\beta}$

$u_p^{*out}[\alpha,\beta] = N_{out} H_\nu^{*(2)}[x] e^{-ip\beta}$

$\frac{\partial}{\partial\alpha}u_p^{out} = \frac{\partial x}{\partial\alpha}\frac{\partial}{\partial x}u_p^{out} = N_{out}(\partial x) H_\nu^{(2)\prime}[x] e^{ip\beta}$

$\frac{\partial}{\partial\alpha}u_p^{*out} = \frac{\partial x}{\partial\alpha}\frac{\partial}{\partial x}u_p^{*out} = N_{out}(\partial x) H_\nu^{*(2)\prime}[x] e^{-ip\beta}$

here) $N_{out} = \frac{1}{2}\left(\frac{\pi}{3}\right)^{1/2} e^{-\frac{|p|}{6}\pi}$

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Determination of normalization constant for $\left\{ \begin{smallmatrix} in \\ out \end{smallmatrix} \right\}$-state mode functions

in-state mode function

$$u_p^{in}[\alpha,\beta] = N_{in} J_\nu[x] e^{ip\beta}$$

$$u_p^{*in}[\alpha,\beta] = N_{in}^* J_\nu^*[x] e^{-ip\beta} = N_{in} J_{-\nu}[x] e^{-ip\beta}$$

$$\left(x = \frac{?}{?} e^{\alpha}, \quad \nu = -i\frac{1}{?}|p| \right)$$

$$\frac{\partial}{\partial\alpha} u_p^{in} = \frac{\partial x}{\partial\alpha}\frac{\partial}{\partial x} u_p^{in} = N(\partial x) J_\nu' e^{ip\beta}$$

$$\frac{\partial}{\partial\alpha} u_p^{*in} = \frac{\partial x}{\partial\alpha}\frac{\partial}{\partial x} u_p^{*in} = N(\partial x) J_{-\nu}' e^{-ip\beta}$$

orthonormality condition

$$\boxed{\int d\beta\, i\left(u_p^{*}\frac{\partial}{\partial\alpha} u_{\tilde p} - u_{\tilde p}\frac{\partial}{\partial\alpha} u_p^{*} \right) = \delta_{p\tilde p}}$$

$$N_{in}^2 (\partial x)\left(J_{\nu_1} J_{\nu_2}' - J_{\nu_2} J_{\nu_1}' \right) \underbrace{\int d\beta\, e^{i(\tilde p - p)\beta}}_{\delta_{p\tilde p}} = \delta_{p,\tilde p}$$

$$\left[\begin{array}{l} \nu_1 = i\frac{1}{?}|p| \\ \nu_2 = -i\frac{1}{?}|p| = -\nu_1 \end{array} \right]$$

$$N_{in}^2 (\partial x)\left(J_{\nu_1} J_{-\nu_1}' - J_{-\nu_1} J_{\nu_1}' \right) = 1$$

<Wronskian

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$$N_{in}^{2} = \lambda \frac{\pi}{6} \left(\sinh \mu \pi\right)^{-1}$$

$$N_{in} = (\lambda)^{1/2} \left(\frac{\pi}{6}\right)^{1/2} \left[\sin\left(\lambda \frac{1}{3} |\mu| \pi\right)\right]^{-1/2}$$

$$= (\lambda)^{1/2} \left(\frac{\pi}{6}\right)^{1/2} \left[\lambda \sinh \frac{|\mu|}{3}\pi\right]^{-1/2} = \left(\frac{\pi}{6}\right)^{1/2} \left[\sinh \frac{|\mu|}{3}\pi\right]^{-1/2}$$

$$u_p^{in}[\alpha, \beta] = \left(\frac{\pi}{6}\right)^{1/2} \left[\sinh \frac{1}{3}|\mu| \pi\right]^{-1/2} e^{i\mu\beta} J_\nu \left[\frac{2}{3}\sqrt{\frac{\lambda}{3}}\, e^{3\alpha}\right]$$

→ out-state mode function

$$u_p^{out}[\alpha, \beta] = N_{out} H_\nu^{(2)}[x]\, e^{i\mu\beta},$$

$$u_p^{*out}[\alpha, \beta] = N_{out} H_\nu^{*(2)}[x]\, e^{-i\mu\beta}$$

$$\left(x = \frac{2}{3}\sqrt{\frac{\lambda}{3}}\, e^{3\alpha}, \quad \nu = -i \frac{1}{3}|\mu|\right)$$

$$\frac{\partial}{\partial \alpha} u_p^{out} = \frac{\partial x}{\partial \alpha} \frac{\partial}{\partial x} u_p = N_{out} (3x) H_\nu^{(2)\prime}[x]\, e^{i\mu\beta},$$

$$\frac{\partial}{\partial \alpha} u_p^{*out} = \frac{\partial x}{\partial \alpha} \frac{\partial}{\partial x} u_p^{*} = N_{out} (3x) H_\nu^{(2)*\prime}[x]\, e^{-i\mu\beta}$$

Again, from normalization condition;

$$\int d\beta\, x \left(u_p^{*out} \frac{\partial}{\partial x} u_q^{out} - u_q^{out} \frac{\partial}{\partial x} u_p^{*out}\right) = \frac{i}{3\pi}$$

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$$i N_{out}^{2} (2\pi)\, \boxed{\left(H_{\nu_1}^{(1)*} H_{\nu_2}^{(1)} - H_{\nu_2}^{(2)} H_{\nu_1}^{(2)*} \right)} \underbrace{\int dp\, e^{i(k-p)\eta}}_{\delta_{p,q}} = \delta_{p,q}$$

$$\left[\begin{array}{l} \nu_1 = i\frac{1}{3}|\beta|, \\ \nu_2 = -i\frac{1}{3}|\beta| = -\nu_1 \end{array} \right]$$

Using

$$\boxed{\left[H_{\nu_1}^{(1)*} H_{-\nu_1}^{(1)*} - H_{-\nu_1}^{(2)} H_{\nu_1}^{(2)*} \right]}$$

$$= 2i \frac{e^{-i\nu_1\pi}}{\sin \nu_1 \pi} \left(J_{\nu_1} J_{-\nu_1}' - J_{-\nu_1} J_{\nu_1}' \right)$$

$$= 2i \left(\frac{e^{-i\nu_1\pi}}{\sin \nu_1 \pi} \right) \times \left[\frac{-2 \sin \nu_1 \pi}{\pi x} \right] = \boxed{-4i \frac{e^{-i\nu_1\pi}}{\pi x}}$$

thus,

$$i N_{out}^{2} (2\pi) \left[-4i \frac{e^{-i\nu_1\pi}}{\pi x} \right] = 1$$

$$N_{out}^{2} = \frac{\pi}{12} e^{i\nu_1\pi} \qquad N_{out} = \frac{1}{2}\left(\frac{\pi}{3}\right)^{1/2} e^{\frac{1}{2} i \nu_1 \pi}$$
$$= \frac{1}{2}\left(\frac{\pi}{3}\right)^{1/2} e^{-\frac{|\beta|}{6}\pi}$$

$$f_{p}^{out}[\alpha, \beta] = \frac{1}{2}\left(\frac{\pi}{3}\right)^{1/2} e^{-\frac{|\beta|}{6}\pi} e^{ip\eta} H_{\nu_1}^{(2)} \left[\frac{2\sqrt{p}}{3} e^{3\alpha} \right]$$

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Note that we are :

$$N_\nu(x) = \frac{\cos\nu\pi J_\nu - J_{-\nu}}{\sin\nu\pi}$$

$$\begin{cases} N_{-\nu}(x) = \dfrac{J_\nu - \cos\nu\pi J_{-\nu}}{\sin\nu\pi} \\ N_\nu'(x) = \dfrac{\cos\nu\pi J_\nu' - J_{-\nu}'}{\sin\nu\pi} \end{cases} \quad \left[\begin{array}{l} H_\nu^{(2)}(x) = J_\nu(x) - i N_\nu(x) \\ H_\nu^{(2)*}(x) = J_\nu + i N_\nu \end{array} \right.$$

"wronskian formula"

$$\left[J_{-\nu}(x) J_\nu'(x) - J_{-\nu}'(x) J_\nu(x) \right] = \frac{-2\nu}{x\,\nu!(-\nu)!} = \frac{2\sin\nu\pi}{\pi x}$$

Therefore :

$$\boxed{H_\nu^{(2)} H_\nu^{(2)*\prime} - H_\nu^{(2)\prime} H_\nu^{(2)*}}$$

$$= (J_\nu + iN_\nu)(J_\nu' - iN_\nu') - (J_\nu - iN_\nu)(J_\nu' + iN_\nu')$$

$$= J_\nu J_\nu' + i(N_\nu J_\nu' - J_\nu N_\nu') + N_\nu N_\nu'$$

$$- J_\nu J_\nu' + i(N_\nu J_\nu' - J_\nu N_\nu') - N_\nu N_\nu'$$

$$(J_\nu J_\nu' - J_\nu J_\nu') + (N_\nu N_\nu' - N_\nu N_\nu')$$

$$+ i(N_\nu J_\nu' - J_\nu N_\nu') - i(J_\nu N_\nu' - N_\nu J_\nu')$$

$$2(J_\nu J_\nu' - J_\nu J_\nu') + 2i\frac{\cos\nu\pi}{\sin\nu\pi}(J_\nu J_\nu' - J_\nu J_\nu')$$

$$2\left(\frac{i e^{i\nu\pi}}{\sin\nu\pi}\right)(J_\nu J_\nu' - J_\nu J_\nu') \quad \boxed{\frac{2\sin\nu\pi}{\pi x}}$$

<wronskian formula>

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And their "orthonormality" conditions :

$$\boxed{(p,k)} = -\int d\rho \, i \left(u_p^{*\,\mathrm{out}} \frac{\partial}{\partial\alpha} u_k^{*\,\mathrm{in}} - u_k^{*\,\mathrm{in}} \frac{\partial}{\partial\alpha} u_p^{*\,\mathrm{out}} \right)$$

$$= -i N_{\mathrm{in}} N_{\mathrm{out}} (3\alpha) \left(H_{\nu_1}^{(1)} \boxed{J_{\nu_2}^{*\prime}} - \boxed{J_{\nu_2}^{*}} H_{\nu_1}^{(1)*\prime} \right) \underbrace{\int d\rho \, e^{-i(p+k)\rho}}_{\delta_{p-k} = \delta(|p|,|k|)}$$

$$J_{\nu_2}^{\prime} \qquad J_{\nu_2}$$

$$= -i N_{\mathrm{in}} N_{\mathrm{out}} (3\alpha) \left(H_{\nu_1}^{(1)*} J_{\nu_1}^{\prime} - J_{\nu_1} H_{\nu_1}^{(1)*\prime} \right) \delta_{p-k} \Rightarrow \boxed{-2i\frac{1}{\pi\alpha}}$$

$$\nu_1 = i\frac{1}{3}|p|$$

$$-\nu_2 = i\frac{1}{3}|k| = \nu_1$$

$$\langle \delta_{|p|,|k|} \rangle$$

$$= -i N_{\mathrm{in}} N_{\mathrm{out}} (3\alpha) \left[-2i \frac{1}{\pi\alpha} \right] \delta_{p-k}$$

$$= -\frac{\sqrt{\pi}}{\sqrt{3}} \left[\sinh \frac{|p|}{3}\pi \right]^{-1/2} \frac{1}{2}\left(\frac{\pi}{3}\right)^{1/2} e^{-\frac{1}{6}|p|\pi} \boxed{\left(-6i\frac{1}{\pi}\right)} \delta_{p-k}$$

$$= -\frac{1}{\sqrt{2}} \left[\sinh \frac{|p|}{3}\pi \right]^{-1/2} e^{-\frac{1}{6}|p|\pi} \delta_{p-k}$$

$$= -\left[\frac{1}{2 e^{\frac{1}{3}|p|\pi}} \frac{2}{e^{\frac{1}{3}|p|\pi} - e^{-\frac{1}{3}|p|\pi}} \right]^{1/2} \delta_{p-k}$$

$$= -\left[\frac{1}{e^{\frac{2}{3}|p|\pi} - 1} \right]^{1/2} \delta_{p-k}$$

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$$\boxed{\left(H^{(2)*}_{\nu_1} J'_{\nu_1} - J_{\nu_1} H^{(2)*\prime}_{\nu_1}\right)}$$

$$\left[H^{(2)*}_{\nu_1} = J_{\nu_1} + i N_{\nu_1}\right]$$

$$= (\cancel{J_{\nu_1}} + i N_{\nu_1}) J'_{\nu_1} - J_{\nu_1}(\cancel{J'_{\nu_1}} + i N'_{\nu_1})$$

$$= i\left(N_{\nu_1} J'_{\nu_1} - J_{\nu_1} N'_{\nu_1}\right)$$

$$= i\left[\left(\frac{\cancel{\cos\nu_1\pi J_{\nu_1}} - J_{-\nu_1}}{\sin\nu_1\pi}\right) J'_{\nu_1} - J_{\nu_1}\left(\frac{\cancel{\cos\nu_1\pi J'_{\nu_1}} - J'_{-\nu_1}}{\sin\nu_1\pi}\right)\right]$$

$$= i\frac{1}{\sin\nu_1\pi}\left(J_{\nu_1} J'_{-\nu_1} - J_{-\nu_1} J'_{\nu_1}\right)$$

$$= i\frac{1}{\sin\nu_1\pi}\left(-2\frac{\sin\nu_1\pi}{\pi x}\right) \leftarrow \boxed{-2i\frac{1}{\pi x}}$$

here) we used the Wronskian Formula :

$$\left[J_\nu(x) J'_{-\nu}(x) - J'_\nu(x) J_{-\nu}(x)\right] = \frac{-2\nu}{x\,\nu!\,(-\nu)!} = \frac{-2\sin\nu\pi}{\pi x}$$

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$$\boxed{G(p,q)} = \left(u_p^{out}, u_q^{in}\right) = \int d\beta\, i \left(u_p^{*out} \frac{\partial}{\partial\alpha} u_q^{in} - u_q^{in} \frac{\partial}{\partial\alpha} u_p^{*out} \right)$$

$$= i N_{in} N_{out} (3x) \left(H_{\nu_1}^{(2)*} J_{\nu_2}' - J_{\nu_2} H_{\nu_1}^{(2)*\prime} \right) \underbrace{\int d\beta\, e^{i(q-p)\beta}}_{\delta_{p,q}}$$

$$= i N_{in} N_{out} (3x) \boxed{\left(H_{\nu_1}^{(2)*} J_{-\nu_1}' - J_{-\nu_1} H_{\nu_1}^{(2)*\prime} \right)} \delta_{p,q} \Rightarrow \boxed{-2i \frac{1}{\pi x} e^{-i\nu_1 \pi}}$$

$\nu_1 = i\frac{1}{3}|p|$

$\nu_2 = -i\frac{1}{3}|q| = -\nu_1$

$\langle \delta_{p,q} \rangle$

$$= i N_{in} N_{out} (3x) \left[-2i \frac{1}{\pi x} e^{-i\nu_1 \pi} \right] \delta_{p,q} \quad \boxed{-6i \frac{1}{\pi} e^{-i\nu_1\pi}} \delta_{p,q}$$

$$= i \left(\frac{\pi}{6}\right)^{1/2} \left[\sinh \frac{|p|}{3}\pi \right]^{-1/2} \frac{1}{2} \left(\frac{\pi}{3}\right)^{1/2} e^{-\frac{1}{6}|p|\pi} \left(-6i \frac{1}{\pi} e^{\frac{1}{3}|p|\pi} \right)$$

$\langle \nu_1 = i\frac{1}{3}|p| \rangle$

$$= \frac{1}{\sqrt{2}} \left[\sinh \frac{|p|}{3}\pi \right]^{-1/2} e^{\frac{1}{6}|p|\pi} \delta_{p,q}$$

$$= \left[\frac{e^{\frac{1}{3}|p|\pi}}{2} \frac{2}{e^{\frac{1}{3}|p|\pi} - e^{-\frac{1}{3}|p|\pi}} \right]^{1/2} \delta_{p,q}$$

$$= \left[\frac{1}{1 - e^{-\frac{2}{3}|p|\pi}} \right]^{1/2} \delta_{p,q}$$

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$$\boxed{\left(H_{\nu_1}^{(2)*}J'_{-\nu_1} - J_{-\nu_1}H_{\nu_1}^{(2)*\prime}\right)} \qquad \left[H_{\nu_1}^{(2)*} = J_{\nu_1} + iN_{\nu_1}\right]$$

$$= \left(J_{\nu_1} + iN_{\nu_1}\right)J'_{-\nu_1} - J_{-\nu_1}\left(J'_{\nu_1} + iN'_{\nu_1}\right)$$

$$= \left(J_{\nu_1}J'_{-\nu_1} - J_{-\nu_1}J'_{\nu_1}\right) + i\left(N_{\nu_1}J'_{-\nu_1} - J_{-\nu_1}N'_{\nu_1}\right)$$

$$= \left(J_{\nu_1}J'_{-\nu_1} - J_{-\nu_1}J'_{\nu_1}\right) + i\frac{\cos\nu_1\pi}{\sin\nu_1\pi}\left(J_{\nu_1}J'_{-\nu_1} - J_{-\nu_1}J'_{\nu_1}\right)$$

$$= i\frac{e^{-i\nu_1\pi}}{\sin\nu_1\pi}\left(J_{\nu_1}J'_{-\nu_1} - J_{-\nu_1}J'_{\nu_1}\right) \Leftarrow \text{Wronskian Formula} \quad \frac{-2\sin\nu_1\pi}{\pi x}$$

$$= i\frac{e^{-i\nu_1\pi}}{\sin\nu_1\pi}\left(-2\frac{\sin\nu_1\pi}{\pi x}\right) = \boxed{-2i\frac{1}{\pi x}e^{-i\nu_1\pi}}$$

...we can check that Bogoliubov coefficients c_1 (B.2) and c_2 (B.2) satisfy the probability conservation property

$$|c_1|^2 - |c_2|^2 = \left[\frac{1}{1-e^{-\frac{2}{3}\mu\pi}}\right] - \left[\frac{1}{e^{\frac{2}{3}\mu\pi}-1}\right]$$

$$= \frac{e^{\frac{2}{3}\mu\pi} - 1}{e^{\frac{2}{3}\mu\pi} - 1} = 1\ !$$

...essentially results from the orthonormality of mode functions

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《Physical Interpretation》

[Universe field equation ⇐ (Wheeler–DeWitt equation)]

$$\left[\partial_\alpha^2 - \partial_\beta^2 + U(\alpha,\beta)\right]\Psi[\alpha,\beta] = 0$$

$$U(\alpha,\beta) = e^{4\alpha}\left[V(\beta) + \frac{4}{3}\Lambda e^{2\alpha}\right].$$

$\hat{P}_\alpha = -i\partial_\alpha$, $\hat{P}_\beta = -i\partial_\beta$, $V(\beta) = \frac{1}{3}\left(e^{-8\beta} - 4e^{-2\beta}\right)$

⇒ separation of variables is expected!

$\boxed{u[\alpha,\beta]} = A(\alpha)B(\beta)$: positive-frequency mode functions of $\Psi[\alpha,\beta]$

⇒ Next, remind that our particular concern is in the region of mini-superspace where [$\beta \to \infty$, $V(\beta) \to 0$]

there,

$$\underbrace{\frac{1}{A}\partial_\alpha^2 A + \frac{4}{3}\Lambda e^{6\alpha}}_{\text{function of } \alpha\text{-alone}} = \underbrace{\frac{1}{B}\partial_\beta^2 B}_{\text{function of } \beta\text{-alone}} = \boxed{-p^2}\ \langle\text{constant}\rangle$$

⇒ $$\frac{\partial^2 B}{\partial\beta^2} + p^2 B = 0 \rightarrow B_p(\beta) \propto e^{\pm ip\beta}$$

$$\hat{P}_\beta^2 B(\beta) = \left(-i\frac{\partial}{\partial\beta}\right)^2 B(\beta) = -\frac{\partial^2 B}{\partial\beta^2} = p^2 B(\beta)$$

The physical meaning of the separation constant $\boxed{p}$ is that it is an eigenvalue of the momentum operator conjugate to the mini-superspace variable β (degree of anisotropy)

thus p probes "anisotropy changing rate"!

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To summarize: For a given $a = e^{\alpha}$,

small values of p > Taub universes with slowly-varying anisotropy

large values of p > Taub universes with rapidly-varying anisotropy

Consequently, from the (Planckian) distribution of the universes created in the p-th mode;

$$\boxed{N_p = \frac{1}{e^{3|p|\pi/c} - 1}}$$

we can conclude that:

(Taub) universes with large anisotropy $[p \to \infty]$ and at the same time,

rapidly-varying $|p| \sim$ large
slowly-varying $|p| \sim$ small
anisotropy are created [in small numbers / large numbers]

(ex) cosmological constant Λ is present)

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To conclude :

robably we are living in one of those
iverses in which the cosmological constant
s non-zero from the beginning and
ad large anisotropy and at the same time
owly-varying anisotropy trend since they
ould have been produced in large numbers
the time of creation.

addition, these (Tanb) universes with
rge but slowly-varying anisotropy
ould have gone through an
ponential expansion (Inflation) period
on after the creation due to the
on-zero cosmological constant ($\Lambda > 0$).
erefore, any initial large anisotropy
ould have been isotropized rapidly.
rtainly, this is consistent with the
mogeneous, isotropic LSS of the present universe!

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§ Chapter II

creation of

nearly-isotropic

$\left(\begin{array}{l}\beta \to 0, \\ V(\beta) \to -1\end{array}\right)$ universes but without ($\Lambda=0$)

from "Nothing"

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Proof >> The Average # of universes created in a state designated by the mode index p:

$$N_p = \langle 0, in | N_p^{out} | 0, in \rangle$$

$$= \langle 0, in | c_p^{\dagger out} c_p^{out} | 0, in \rangle$$

$$= \langle 0, in | \sum_q \{ -c_2(\eta, q) c_q^{in} + c_1(\eta, q) c_q^{in\dagger} \} \sum_r \{ c_1^*(\eta, r) c_r^{in} - c_2^*(\eta, r) c_r^{\dagger in} \} | 0, in \rangle$$

$$= \sum_{q,r} \langle 0, in | \{ -c_2(\eta, q) c_1^*(\eta, r) c_q^{in} c_r^{in} + \boxed{c_2(\eta, q) c_2^*(\eta, r)} c_q^{in} c_r^{\dagger in}$$

$$+ c_1(\eta, q) c_1^*(\eta, r) c_q^{\dagger in} c_r^{in} - c_1(\eta, q) c_2^*(\eta, r) c_q^{\dagger in} c_r^{\dagger in} \} | 0, in \rangle$$

$$= \sum_{q,r} \langle 0, in | \, |c_2(\eta, p)|^2 \delta_{q,r} \delta_{r,-q} \Big(\underbrace{[c_p^{in}, c_p^{\dagger in}]}_{=1} + c_p^{\dagger in} c_p^{in} \Big) | 0, in \rangle$$

$$= |c_2(\eta, p)|^2$$

$$= \frac{1}{e^{\ldots} - 1} \qquad \boxed{\frac{1}{e^{\hbar\omega/c} - 1}}$$

$$\left\{ \sqrt{\frac{\ldots}{3}} \; (=1) \right\}$$

Planckian Distribution with temperature $T \sim c = \left(\frac{8\pi e}{3}\right)^{1/2}$

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Explicit computation of Bogolubov coefficients

[Universe field equation]

$$\left[\partial_\alpha^2 - \partial_\beta^2 + U(\alpha,\beta)\right]\Psi[\alpha,\beta] = 0$$

$$U(\alpha,\beta) = e^{4\alpha}\left[V(\beta) + \tfrac{4}{3}\Lambda e^{2\alpha}\right]$$

$$V(\beta) = \tfrac{1}{3}\left(e^{-8\beta} - 4e^{-2\beta}\right)$$

For the case at hand, our particular interest is in the region of mini superspace where

$$\left[\beta \to 0 \ \& \ V(\beta) = -1\right]$$

therefore;

$$\left\{\partial_\alpha^2 - \partial_\beta^2 - e^{4\alpha}\left[1 - \tfrac{4}{3}\Lambda e^{2\alpha}\right]\right\}\Psi[\alpha,\beta] = 0$$

like before, assuming the separation of variables

$$u[\alpha,\beta] = A(\alpha)B(\beta)$$

$$\underbrace{\frac{1}{A}\frac{\partial^2 A}{\partial\alpha^2} - e^{4\alpha}\left[1 - \tfrac{4}{3}\Lambda e^{2\alpha}\right]}_{\{\text{function of } \alpha \text{ alone}\}} = \underbrace{\frac{1}{B}\frac{\partial^2 B}{\partial\beta^2}}_{\{\text{function of } \beta \text{ alone}\}} = \underbrace{-p^2}_{\{\text{a constant}\}}$$

a) β-sector

$$\frac{\partial^2 B}{\partial\beta^2} + p^2 B = 0$$

$$B(\beta) = (\text{const})\, e^{ip\beta}$$

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(2) α-section

$$\frac{\partial^2 A}{\partial \alpha^2} + \left\{ e^{4\alpha}\left[-1+\frac{4}{3}\Lambda e^{2\alpha}\right] + p^2 \right\} A = 0$$

or in terms of $a = e^{\alpha}$,

$$a^2\frac{\partial^2 A}{\partial a^2} + a\frac{\partial A}{\partial a} + \left\{ a^4\left[-1+\underbrace{\frac{4}{3}\Lambda a^2}_{=0}\right] + p^2 \right\} A = 0$$

$(\Lambda = 0)$

In the absence of the cosmological constant, an analytic, exact closed-form solution to this ODE is available. Indeed, the inclusion of the cosmological constant is not essential, but optional. Thus deleting Λ, one ends up with

$$\boxed{\frac{\partial^2 A}{\partial a^2} + \frac{1}{a}\frac{\partial A}{\partial a} + \left[-a^2 + \frac{p^2}{a^2}\right] A = 0}$$

Obviously, this is NOT precisely Bessel equation but it is a slightly-modified version of the Bessel equations of which the solution is available in an analytic, closed-form and it is given by

$$\frac{d^2 u}{dz^2} + \frac{(1-2\alpha)}{z}\frac{du}{dz} + \left[(\beta\gamma z^{\gamma-1})^2 + \frac{\alpha^2-\nu^2\gamma^2}{z^2}\right] u = 0$$

$$\Rightarrow u(z) = z^{\alpha} Z_{\nu}(\beta z^{\gamma})$$

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the ODE of our interest, namely, the α-section of the universe field equation corresponds to the case where $\left[\alpha = 0 \quad \beta = \frac{\lambda}{2} \quad \gamma = 2 \quad \nu = -i\frac{p}{2} \right]$

and therefore, the α-section of the mode function turns out to be

$$A(\alpha) = Z_{-i\frac{p}{2}}\left(\frac{\lambda}{2}a^2\right)$$

$$= Z_{-i\frac{p}{2}}\left(\frac{\lambda}{2}e^{2\alpha}\right)$$

$$u(\alpha,\beta) = A(\alpha)\, B_p(\beta)$$

$$= N\, Z_\nu\left[\frac{\lambda}{2}e^{2\alpha}\right] e^{ip\beta}$$

normalized $\left\{ \begin{matrix} \text{in} \\ \text{out} \end{matrix} \right\}$-state mode functions

$$u_{in}(\alpha,\beta) = \boxed{\left(\frac{\pi}{4}\right)^{1/2}\left[\sinh\frac{\pi|p|}{2}\right]^{-1/2}} e^{ip\beta} J_\nu\left[\frac{\lambda}{2}e^{2\alpha}\right] \quad \text{for } \alpha \to -\infty$$

$N_{in} \Leftarrow$

$$u_{out}(\alpha,\beta) = \boxed{\frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2} e^{-\frac{\pi|p|}{4}}} e^{ip\beta} H_\nu^{(2)}\left[\frac{\lambda}{2}e^{2\alpha}\right] \quad \text{for } \alpha \to +\infty$$

$N_{out} \Leftarrow$

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Bogolinbov coefficients

$$\boxed{\beta_2(p,q)} = -\left(u_p^{out}, u_q^{*in}\right) = -\int d^3x\, i\left(u_p^{out}\frac{\partial}{\partial x}u_q^{*in} - u_q^{*in}\frac{\partial}{\partial x}u_p^{out}\right)$$

$$= -i N_{in} N_{out}(\pi x)\left(H_{\nu}^{*}\boxed{J_{\nu}'} - \boxed{J_{-\nu}'}H_{\nu}^{*'}\right)\underbrace{\int d^3x\, e^{-i(p+q)x}}_{\delta_{p,-q}}$$

$$J_\nu^* \qquad J_{-\nu}$$

$$= -i N_{in} N_{out}(\pi x)\left(H_\nu^{*}J_\nu' - J_\nu H_\nu^{*'}\right)\delta_{p,-q} \rightarrow \boxed{-2i\frac{1}{\pi x}}$$

$$\nu_1 = -\nu = i\tfrac{1}{2}|p|$$
$$\nu_2 = -i\tfrac{1}{2}|q| = i\tfrac{1}{2}|p| = -\nu_1$$

$$= -i N_{in} N_{out}(\pi x)\left(-2i\frac{1}{\pi x}\right)\delta_{p,-q} \qquad \boxed{(-1)(i\cdot 2)}\ \delta_{p,-q}$$

$$= -\left(\frac{\pi}{4}\right)^{1/2}\left[\sinh\tfrac{1}{2}|p|\pi\right]^{-1/2}\times\frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2} e^{-\frac{1}{4}|p|\pi}$$

$$= -\frac{1}{2}\cancel{\pi}\left[\sinh\tfrac{1}{2}|p|\pi\right]^{-1/2} e^{-\frac{1}{4}|p|\pi}\,\delta_{p,-q}$$

$$= -\left[\frac{1}{2e^{\frac{1}{2}|p|\pi}}\times\frac{1}{e^{\frac{1}{2}|p|\pi}-e^{-\frac{1}{2}|p|\pi}}\right]^{1/2}\delta_{p,-q}$$

$$= -\left[\frac{1}{e^{|p|\pi}-1}\right]^{1/2}\delta_{p,-q}$$

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$$\boxed{C_1(p,q)} = (u_p^{out}, u_q^{in}) = \int dx\, i\left(u_p^{*out} \frac{\partial}{\partial x} u_q^{in} - u_q^{in} \frac{\partial}{\partial x} u_p^{*out} \right)$$

$$= i N_{in} N_{out} (2\pi) \left(H_\nu^{*} J_\nu' - J_\nu H_\nu^{*\prime} \right) \underbrace{\int dp\, e^{-i(q-p)\rho}}_{\delta_{p,q}}$$

$$= i N_{in} N_{out} (2\pi) \left(H_\nu^{*} J_{-\nu}' - J_{-\nu} H_\nu^{*\prime} \right) \delta_{p,q} \Longrightarrow \boxed{-2i \frac{1}{\pi x} e^{-i\nu_1 \pi}}$$

$$\begin{cases} \nu_1 = -\nu = i\frac{1}{2}|p| \\ \nu_2 = -i\frac{1}{2}|q| = -\nu_1 \end{cases}$$

$$= i\left(\frac{\pi}{4}\right)^{1/2} \left[\sinh \frac{|p|\pi}{2}\right]^{-1/2} \frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2} e^{-\frac{|p|\pi}{4}} \boxed{\left(-4i \frac{1}{\pi} e^{\frac{1}{2}|p|\pi}\right)} \delta_{p,q}$$

$$= \frac{1}{\sqrt{2}} \left[\sinh \frac{|p|\pi}{2}\right]^{-1/2} e^{\frac{|p|\pi}{4}} \delta_{p,q}$$

$$\left[\frac{1}{2 e^{-\frac{|p|\pi}{2}}\left(e^{\frac{|p|\pi}{2}} - e^{-\frac{|p|\pi}{2}}\right)} \right]^{1/2} \delta_{p,q}$$

$$= \boxed{\left[\frac{1}{1 - e^{-|p|\pi}} \right]^{1/2} \delta_{p,q}}$$

Ah, we can check that Bogoliubov coefficients $C_1(p,q)$, $C_2(p,q)$ satisfy the probability conservation property.

$$|C_1|^2 - |C_2|^2 = \left[\frac{1}{1 - e^{-|p|\pi}}\right] - \left[\frac{1}{-1 + e^{|p|\pi}}\right] = \frac{e^{|p|\pi} - 1}{e^{|p|\pi} - 1} = 1 \, !$$

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To conclude: [$\rho \to 0$]

Taub) universes with small anisotropy and
at the same time

apidly-varying $|\rho|$-large
owly-varying $|\rho|$-small

anisotropy are created [small numbers / large numbers]

(when cosmological constant Λ is absent)

obably we are living in one of those
iverses in which the cosmological constant
as zero from the beginning and
d small anisotropy and at the same time
lowly-varying anisotropy trend since
ey would have been produced in
rge numbers at the time
of creation !!