Analytic & Numerical Study

ON

Anisotropic Holographic Superfluids

Based on
arXiv:1609.08241

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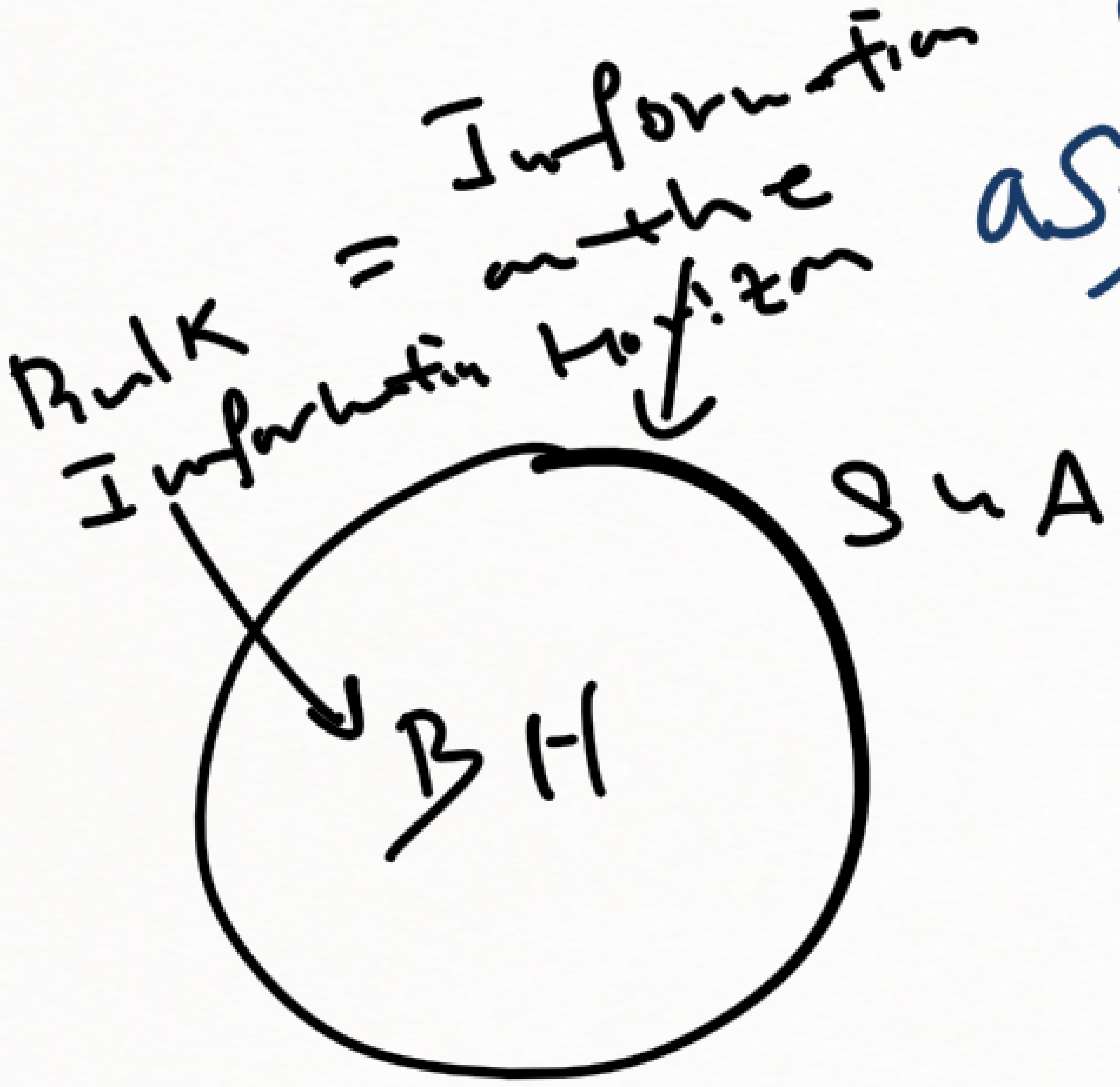
Holography



Gravity theory in asymptotically AdS_{d+1} space $\longrightarrow$ Certain field theory on its Boundary
$\underbrace{\text{its}}_{\text{d-dimensional}}$

Non-normalization excitations $\longrightarrow$ Deformation on the dual theory

Normalizable excitations $\longrightarrow$ States (Non-trivial) on the the dual field theory.

$$\int_{\partial\Phi = \Phi_0} [D\Phi]\, e^{iS_g[\Phi]} = \int [D\psi]\, e^{iS_{CFT}(\psi) + \int d^dx\, \mathcal{O}(\psi)\Phi_0}$$

Fluid – gravity duality (an example)

Conformal field theory in

① low frequency $(\omega, k) \ll l_{mfp}^{-1}$ (mean free path)

② with excitations $T(x^\mu)$, $u_\mu(x^\mu)$, $J^\mu(x^\mu)$ ...

fluid dynamics equations

Boundary energy – momentum tensor

Dual gravity theory

A certain Boosted Black Brane Solution which is slowly varying along Boundary directions.

$\lambda \gg r_0$

wave length

horizon size

with $u_\mu(x^\mu)$, $T(x^\mu)$ and $J^\mu(x^\mu)$ satisfying

$$T^{\mu\nu} = T^4(\eta^{\mu\nu} + 4u^\mu u^\nu) \Leftarrow \partial_\mu T^{\mu\nu} = 0 \, , \quad \partial_\mu J^\mu = 0 .$$

if $d=4$

current conservation.

Thermodynamic phase Transition.

Dual gravity Models

(asym) AdS_{d+1} + Scalar theory (ϕ)

below a certain T_c, a normalizable Mode of the scalar field appears

$\phi = 0 \quad T > T_c$

$\phi \neq 0 \quad T < T_c$

$\Longleftrightarrow$

The corresponding Fluids

Below $T = T_c$,

There is a scalar condensation

explaining

Normal / Super } conductor → phase Transition.

Normal or Super } fluid → phase Transition.

$\langle \psi\psi \rangle$

Cooper pair

Holographic p-wave superconductor / fluids.

(asym) AdS_{d+1} + Yang-Mill fields (SU(2)) → $B^a_\mu \tau^a = b(r)\tau^3 dt + \omega(r)\tau^1 dx^1$

→ at a certain chemical potential $\mu = \mu_c$, a normalizable mode along a certain spatial direction on the boundary appears → $\omega(r)$

This corresponds to a vector order (p-wave superconductor)

$\mu < \mu_c \qquad \omega = 0$

$\mu > \mu_c \qquad \omega \neq 0$

especially $\mu \equiv \dfrac{\tilde{\mu}}{r_h} = 4$ → analytic solution exists.

<Extend to Non-conformal cases>

Holographic Model

Dual field theory

Background geometry

$$ds^2 = r^{2\alpha}\left(-r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 d\vec{x}^2\right) \Longleftrightarrow$$

Hyperscaling Violation
&
Lifshitz

a Normalizable Mode of excitation showing anisotropy along a boundary direction. $\Longleftrightarrow$ a Vector order parameter

e.g.) p-wave (superconductor
superfluid

Question is

Is there thermodynamic phase Transition at arbitrary τ and α, from Isotropic to Anistropic phase in Holographic Superconductor/Fluids?

SU(2) x U(1)

Start with Einstein − Scalar − U(2) theory

$$S = \frac{1}{\kappa_5^2}\int d^5x\sqrt{-g}\Big[R - \frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi + \frac{V_0}{L^2}e^{\gamma\phi} - \frac{\kappa_5^2}{4g_U^2}e^{\lambda_U\phi}F_{MN}F^{MN}$$
$$- \frac{\kappa_5^2}{4g_{YM}^2}e^{\lambda_{YM}\phi}G^a_{MN}G^{aMN}\Big],$$

a real scalar field

scalar potential

U(1) − Field strength

SU(2) − Field strength

SU(2) − gauge field

$G^a_{MN} = \partial_M B^a_N - \partial_N B^a_M - \epsilon^{abc} B^b_M B^c_N$

TRY to obtain a black brane solution with asymptotically

① hyperscaling violation and Lifshitz like geometry.

Solution

a simple

Temporal parts of the gauge fields.

$$
\begin{aligned}
ds^2 &= r^{2\alpha}\left(-r^{2z}f(r)\sigma^2(r)dt^2+\frac{dr^2}{r^2f(r)}+r^2h^{-4}(r)dx_1^2+r^2h^2(r)(dx_2^2+dx_3^2)\right)\\
B^a\tau^a &= b(r)\tau^3dt+\omega(r)\tau^1dx_1,\\
A &= a(r)dt\\
\phi &= \phi(r),
\end{aligned}
$$

spatial anisotropy (if $\omega \neq 0$ Isometry $SO(3) \to SO(2)$)

& so $h \neq 1$

$$
\begin{aligned}
f(r) &= 1-mr^{-3\alpha-z-3}+\frac{\kappa_5^2}{g_{YM}^2}\tilde{\mu}^2r^{-6\alpha-4-2z}, \quad \sigma(r)=1, \quad h(r)=1\\
G^3_{rt} &= \partial_r b(r)=\tilde{\mu}e^{-\sqrt{\frac{\alpha+z-1}{6(\alpha+1)}}\phi_0}\sqrt{6(\alpha+1)(3\alpha+z+1)}r^{-3\alpha-z-2}, \quad G^1_{rx_1}=0,\\
F_{rt} &= \partial_r a(r)=\frac{g_U^2}{\kappa_5^2}e^{\frac{2\alpha+3}{\sqrt{6(\alpha+1)(\alpha+z-1)}}\phi_0}\sqrt{2(z-1)(3\alpha+z+3)}r^{3\alpha+z+2},\\
\phi &= \phi_0+\sqrt{6(\alpha+1)(\alpha+z-1)}\ln r,
\end{aligned}
$$

And $\omega = 0$

So this is a spatially isotropic solution.

We want to find a $\omega(r) \neq 0$ solution, which is normalizable mode of excitation, which gives spontaneous sym-breaking on the dual field theory (Fluid / CM-system)

To explore this system, especially its thermodynamic phase transition, we employ both analytic and numerical Methods.

Sturm-Liouville Method

what about $\omega \neq 0$ solution? Let us get a solution is a probe limit meaning that $\boxed{\frac{\kappa_5^2}{g_{YM}^2} \to 0}$

$$X \equiv \frac{1}{\sqrt{-g}}\partial_M(\sqrt{-g}g^{MN}\partial_N\phi) + V_0\gamma e^{\gamma\phi} - \frac{\kappa_5^2\lambda_U}{4g_U^2}e^{\lambda_U\phi}F_{MN}F^{MN} - \cancel{\frac{\kappa_5^2\lambda_{YM}}{4g_{YM}^2}}e^{\lambda_{YM}\phi}G^a_{MN}G^{aMN} = 0,$$

$$Y^N \equiv \frac{1}{\sqrt{-g}}\partial_M(\sqrt{-g}e^{\lambda_U\phi}F^{MN}) = 0,$$

$$\boxed{\mathcal{Y}^{aN} \equiv \frac{1}{\sqrt{-g}}\partial_M(\sqrt{-g}e^{\lambda_{YM}\phi}G^{aMN}) + e^{\lambda_{YM}\phi}\epsilon^{abc}G^{bMN}B^c_M = 0,}$$

$$W_{MN} \equiv R_{MN} + \frac{V_0}{3}g_{MN}e^{\gamma\phi} - \frac{1}{2}\partial_M\phi\partial_N\phi - \frac{\kappa_5^2}{2g_U^2}e^{\lambda_U\phi}\left(F_{PM}F^P_N - \frac{1}{6}g_{MN}F_{PQ}F^{PQ}\right) - \cancel{\frac{\kappa_5^2}{2g_{YM}^2}}e^{\lambda_{YM}\phi}\left(G^a_{PM}G^{aP}{}_N - \frac{1}{6}g_{MN}G^a_{PQ}G^{aPQ}\right) = 0.$$

Then, there are no backreactions of G^a_{MU} to the other fields!

And we solve this equation only in such probe limit! with $\omega(v) \neq 0$.

Sturm–Liouville Method

e.g.) Estimation of the ground state of Hydrogen atom.

An arbitrary wave function: $\psi = \sum_{i=0}^{\infty} a_i \psi_i$, $\psi_0 \rightarrow$ ground state wf.

ψ: normalized. ψ_i: normalized eigen states.

Then, the expectation value of energy

$$\langle E \rangle \Rightarrow \int \psi^*(x) \hat{E} \psi(x)\, dx = \sum_{i,j} \int \psi_i^*(x) E_j \psi_j(x)\, dx$$

$$= \sum_i E_i |a_i|^2 \geq E_0$$

it provides the upper bounds of the ground state energy.

Therefore, if one choose ψ in a smart way (similar to the ground wave function)

$\Rightarrow$ He/she can estimate E_0 well.

Yang-Mill gauge field equations to solve $\boxed{\xi = \frac{1}{r}}$

$$0 = \frac{d}{d\xi}\left(\xi^{2-3\alpha-3z} f(\xi)\frac{d\omega(\xi)}{d\xi}\right) + f^{-1}(\xi)\xi^{-3\alpha-z} b^2(\xi)\omega(\xi),$$
$$0 = \frac{d}{d\xi}\left(\xi^{-3\alpha-z}\frac{db(\xi)}{d\xi}\right) - f^{-1}(\xi)\xi^{-3\alpha-z}\omega^2(\xi)b(\xi).$$

Suppose that $\omega(\xi)$ is small. $\delta(\epsilon)$

$\Rightarrow$ $b(\xi) = \mu(1-\xi^{3\alpha+z+1}) + O(\epsilon^2)$

$\omega(\xi) = \langle J_x^1 \rangle \xi^{3\alpha+2z-1} F(\xi)\epsilon + O(\epsilon^3)$

$\Downarrow$

$F(\xi) = 1 - t\xi^2$

$\left(\begin{array}{l} F(0) = 1 \\ F'(0) = 0 \end{array} \Leftarrow \text{B.C} \right)$

$\mu_1 \to F_1$

$\mu_2 \to F_2$

$\vdots$

$\mu_n \to F_n$

$\mu_1 < \mu_2 < \mu_3 \cdots < \mu_n$

$$0 = \frac{d}{d\xi}\left(K(\xi)\frac{dF(\xi)}{d\xi}\right) - P(\xi)F(\xi) + \mu^2 Q(\xi)F(\xi),$$

$$K(\xi) = \xi^{3\alpha+3z}(1-\xi^{3\alpha+z+3}),$$
$$P(\xi) = -(3\alpha+3z-1)\xi^{3\alpha+3z-1}\frac{d}{d\xi}(1-\xi^{3\alpha+z+3}),$$
$$Q(\xi) = \frac{\xi^{3\alpha+5z-2}(1-\xi^{3\alpha+z+1})^2}{1-\xi^{3\alpha+z+3}}.$$

$\Leftarrow$

$\int_0^1 d\xi\, Q(\xi)F_i(\xi)F_j(\xi) = \delta_{ij}$

$$\boxed{\mu^2 = \frac{\int_0^1 d\xi \left[K(\xi)\left(\frac{dF(\xi)}{d\xi}\right)^2 + P(\xi)F^2(\xi)\right]}{\int_0^1 d\xi Q(\xi)F^2(\xi)}.}$$

Since our trial solution is $F(\xi) = 1 - t\xi^2$, Find t

$= \sum_i a_i F_i$

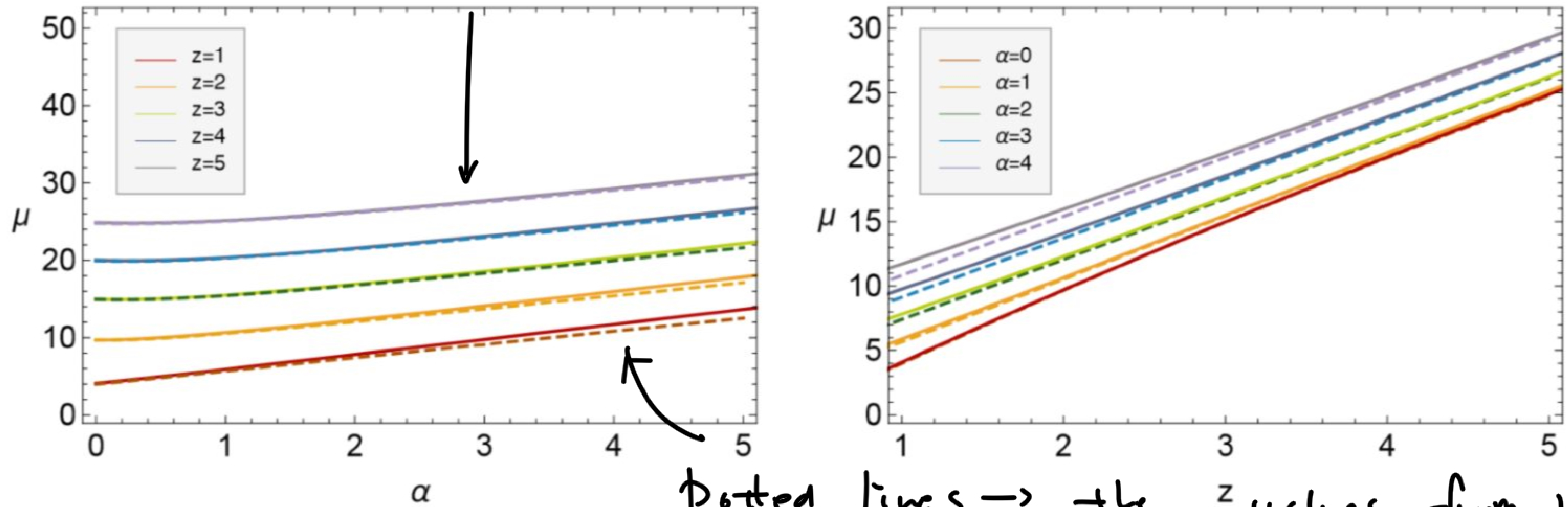
which minimize μ^2.

$\omega \to$ normalizable mode

$$\begin{cases} F(0) = 1 \\ F'(0) = 0 \end{cases}$$

The lowest upper bound of the value of μ for ~~the~~ a normalizable solution to exist.

The solid lines → values from Strum-Liouville method.



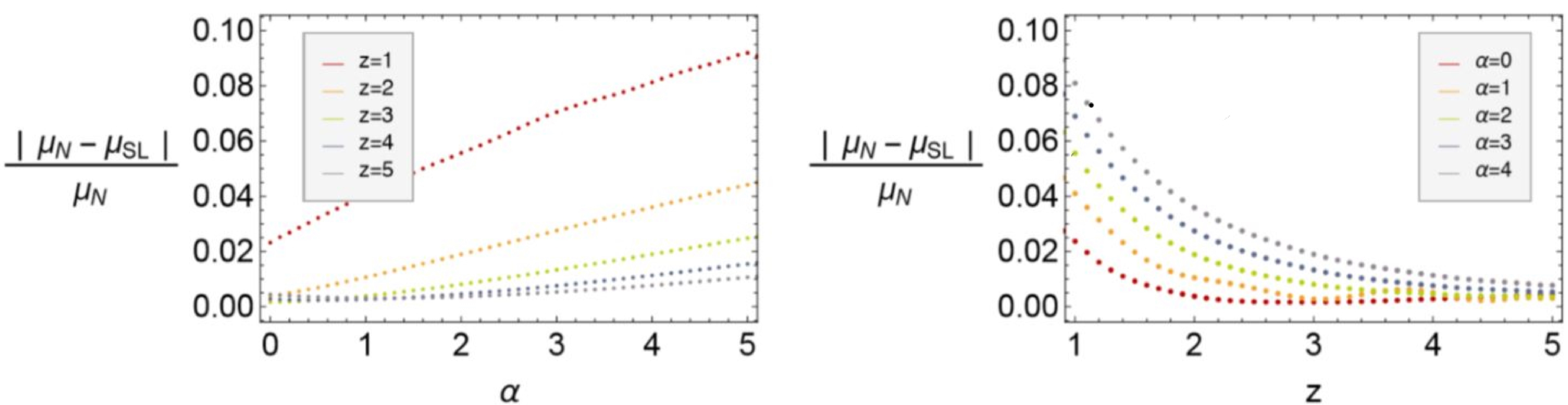
Dotted lines → the values from numerics.

(a) For fixed z

(b) For fixed α

The values of chemical potential (μ), when the normalizable mode of solution $\omega(\xi)$ appears.

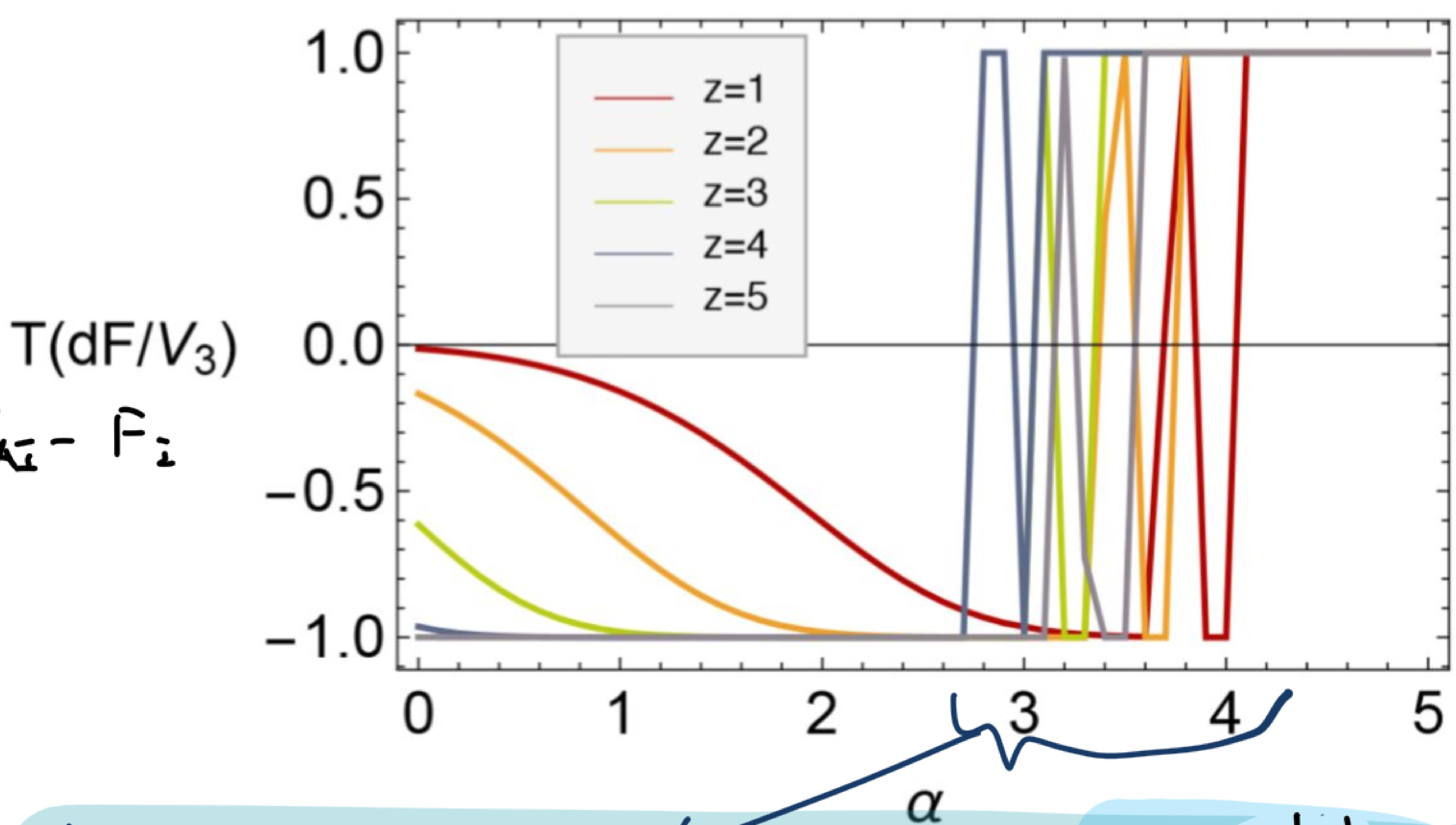
A comparison between SL-method and numerics.



(a) For fixed z (b) For fixed α

The differance becomes larger as α increases or z decreases.

Free energy (numerics)



$= F_{Ai} - F_i$

This region Free energy is not well-behaved
we did not determine our conclusion.

For the given values of z, $(1 \le z \le 5)$, the anisotrophic phase is stable when

$$\boxed{0 \lesssim \alpha \lesssim 3}$$

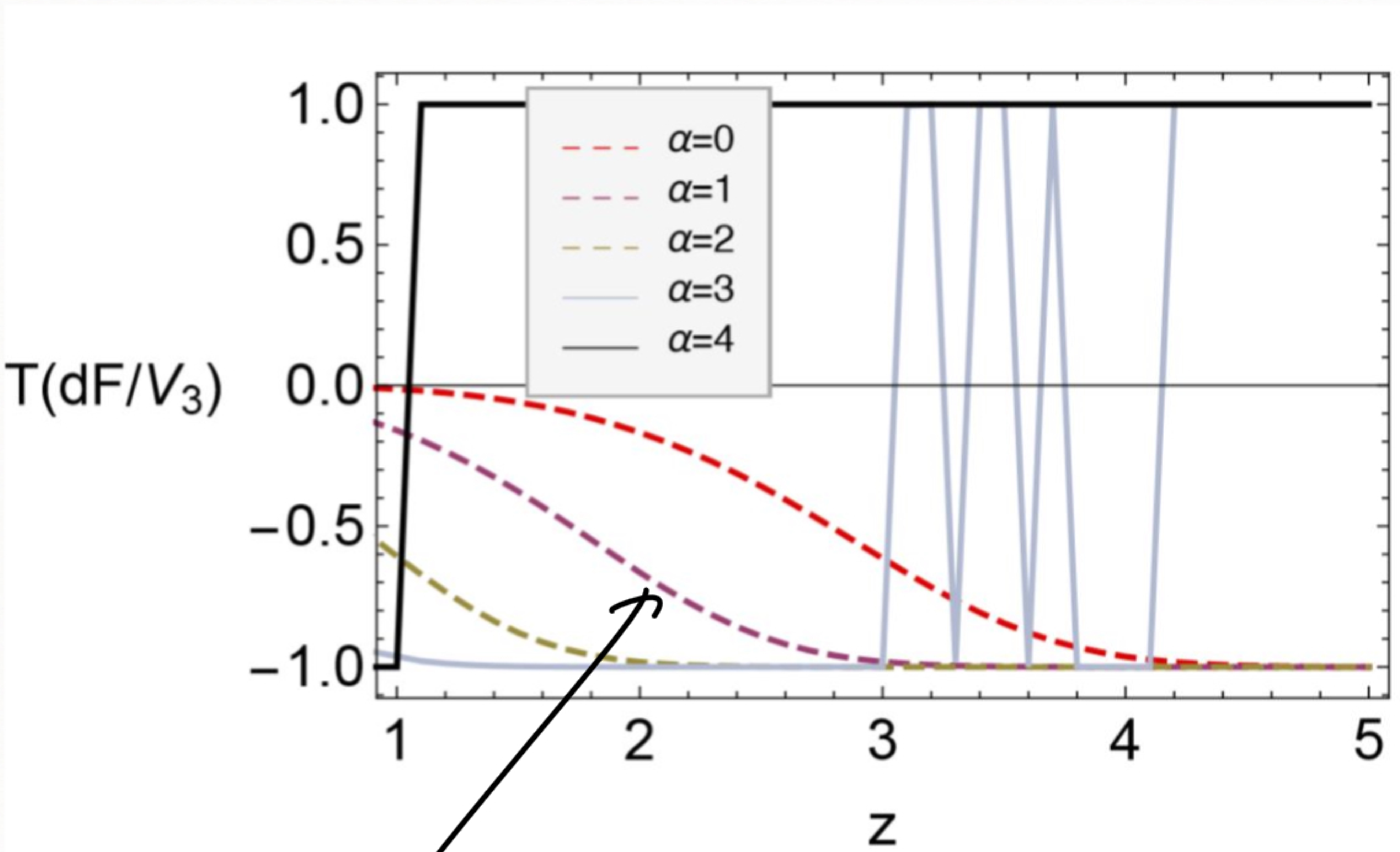
Phase Transition

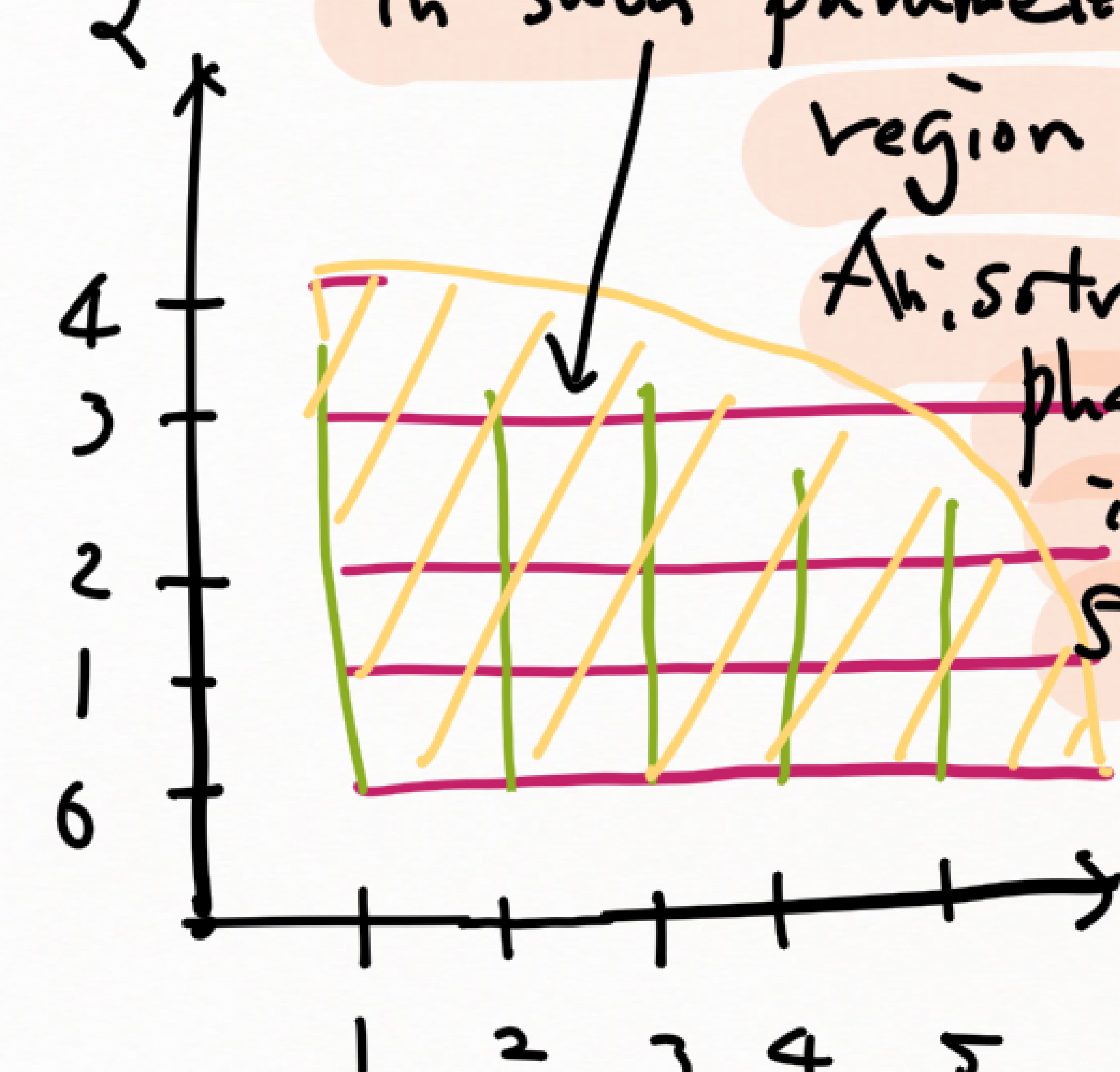
Isotrophic phase

$\downarrow$

Anisotrophic phase

It is certain that in such parameter region Anisotrophic phase is stable.





For the given "α's", the anisotrophic phase is stable in such region.

Summary

① We explore holographic p-wave superconductor / Fluid by utilizing Sturm–Liouville and numerical Methods.

② In a certain region of α and z, the anisotropic phase is stable above a certain chemical potential μ_c.