Gravitational Scattering Amplitudes and Closed String Field Theory in the Proper-Time Gauge

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Quantum Gravity and String Field Theory

Classical General Relativity Derived from Quantum Gravity

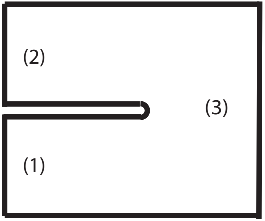
Boulware and Deser, Ann. Phys. 89 (1975):
"A quantum particle description of local (noncosmological) gravitational phenomena necessarily leads to a classical limit which is just a metric theory of gravity. As long as only helicity ± 2 *gravitons are included, the theory is precisely Einsteins general relativity."*

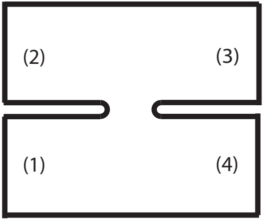
Closed String Field Theory

Closed string theory contains massless spin 2 particles in its spectrum. The low energy limit of the covariant interacting closed string field theory must be the Einstein's general relativity. The closed string field theory may provide a consistent framework to describe a finite quantum theory of the spin 2 particles, the gravitons. We need to examine the graviton scattering amplitudes of the covariant string field theory and compare them with those of the perturbation theory of the gravity in the low energy region.

I. Open string field theory in the proper-time gauge

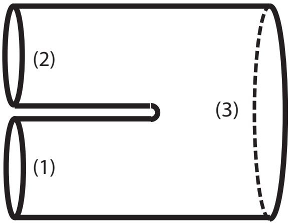
We constructed a covariant string field theory on *Dp*-branes, and calculated three-string scattering amplitude and the four-string scattering amplitude in the low energy limit.
TL, Phys. Lett. B **768**, 248 (2017); arXiv:1609.01473; arXiv:1703.06402 [hep-th]
S. H. Lai, J. C. Lee, Y. Yang and TL, arXiv:1706.08025.

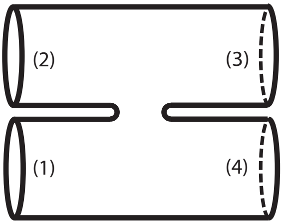




II. Closed string field theory in the proper-time gauge

We will construct a covariant string field theory and calculate three-string scattering amplitude and the four-string scattering amplitude in the low energy limit by extending the previous works on the open string field theory.





Fock Space Representation of the Closed String Field Theory in the Proper-Time Gauge

Closed String Field Theory in the Proper-Time Gauge

$$S = \langle \Phi | \mathcal{K} \Phi \rangle + \frac{g}{3} \Big(\langle \Phi | \Phi \circ \Phi \rangle + \langle \Phi \circ \Phi | \Phi \rangle \Big).$$

The closed string field theory in the proper-time gauge generates the string scattering diagrams, which can be represented by the Polyakov string path integrals:

$$S_P = -\frac{1}{4\pi\alpha'} \int_M d\tau d\sigma \sqrt{-h} h^{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu}, \qquad \mu, \nu = 0, \dots, d-1.$$

String Scattering Amplitudes

$$\mathcal{A}_M = \int D[X] D[h] \exp \left[-i \int_M d\tau d\sigma \mathcal{L} \right].$$

String Scattering Amplitudes of Closed String Field Theory and Polyakov String Path Integral

Strategy of Calculation of String Scattering Amplitudes

1. Construct the covariant closed string field theory
2. Rewrite the scattering amplitudes generated by the closed string field theory by using the Polyakov string path integral
3. Re-express the Polakov string path integrals in terms of the oscillator operators
4. Identify the Fock space (operator) representations of the string field theory vertices
5. Choose appropriate external string states, corresponding to the various particles and evaluate the scattering amplitudes.

Closed String Theory: Review

Free String Theory

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \, \partial X \cdot \partial X .$$

Decomposition of X in terms of left-movers and right-movers

$$X(\tau, \sigma) = X_L(\tau + \sigma) + X_R(\tau - \sigma).$$

Mode expansions

$$\begin{aligned} X_L(\tau, \sigma) &= x_L + \sqrt{\frac{\alpha'}{2}} p_L(\tau + \sigma) + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n e^{-in(\tau+\sigma)}, \\ X_R(\tau, \sigma) &= x_R + \sqrt{\frac{\alpha'}{2}} p_R(\tau - \sigma) + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n e^{-in(\tau-\sigma)}, \end{aligned}$$

where $x = x_L + x_R$.

Canonical commutation relations

$$\begin{aligned}[][x_L, p_L] &= [x_R, p_R] = i\sqrt{\frac{\alpha'}{2}}, \\ [\alpha_m, \alpha_n] &= [\tilde{\alpha}_m, \tilde{\alpha}_n] = m\delta(m+n).\end{aligned}$$

Momentum eigentate with eigenvalue $P_n, n \neq 0$:

$$|P_n\rangle = \sqrt{\frac{1}{\pi n}} \exp\left\{\left(\frac{P_n \alpha_{-n}}{n} + \frac{P_{-n}\tilde{\alpha}_{-n}}{n} - \frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n} - \frac{P_n \cdot P_{-n}}{4n}\right)\right\}|0\rangle.$$

Mapping from cylindrical surface onto the complex plane

$$z = e^{\rho} = e^{\xi + i\eta}, \quad -\pi \leq \eta \leq \pi.$$

Green's function on complex plane ($\xi > \xi'$), $\Delta = |\xi - \xi'|$,

$$\begin{aligned} G_C(z, z') &= \ln|z - z'| \\ &= \max(\xi, \xi') - \frac{1}{2}\sum_{n=1} \frac{e^{-n\Delta}}{n}\left(e^{in(\eta'-\eta)} + e^{-in(\eta'-\eta)}\right). \end{aligned}$$

Closed String Interaction in the Proper Time Gauge

CS mapping from the world sheet of three closed string scattering onto the complex plane. For the three-string vertex in the proper-time gauge,

$$\rho = \ln(z-1) + \ln z.$$

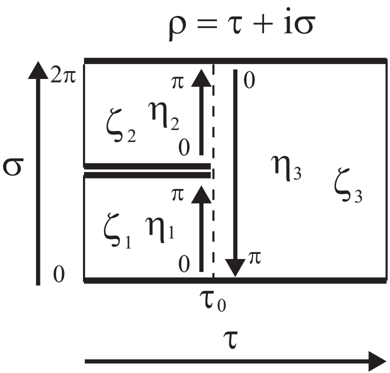
The local coo vrdinates $\zeta_r = \xi_r + i\eta_r,\ r = 1,2,3$ defined on invidual string world sheet patches are related to z as follows:

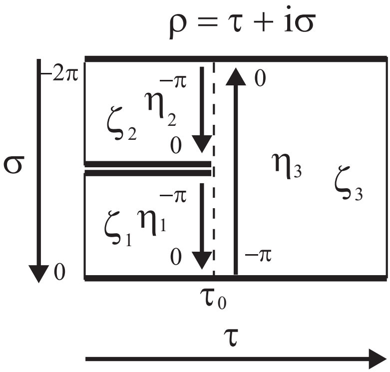
$$\begin{aligned} e^{-\zeta_1} &= e^{\tau_0}\frac{1}{z(z-1)}, \\ e^{-\zeta_2} &= -e^{\tau_0}\frac{1}{z(z-1)}, \\ e^{-\zeta_3} &= -e^{-\frac{\tau_0}{2}}\sqrt{z(z-1)}. \end{aligned}$$

For convenience we choose, by using $SL(2,C)$ invariance

$$Z_1 = 0, \quad Z_2 = 1, \quad Z_3 = \infty.$$

Local Coordinates





Fourier components of the Green's function on complex plane

$$
\begin{aligned}
G_C(\rho_r, \rho'_s) &= \ln |z_r - z'_s| \\
&= -\delta_{rs} \left\{ \sum_{n=1} \frac{e^{-n\Delta}}{2n} \left(e^{in(\eta'_s - \eta_r)} + e^{-in(\eta'_s - \eta_r)} \right) - \max(\xi, \xi') \right\} \\
&\quad + \sum_{n,m} \bar{C}^{rs}_{nm} e^{|n|\xi_r + |m|\xi'_s} e^{in\eta_r} e^{im\eta'_s}.
\end{aligned}
$$

Integral Formulas for $\bar{C}^{rs}_{nm}$

$$
\begin{aligned}
\bar{C}^{rs}_{00} &= \ln|Z_r - Z_s|, \quad r \neq s, \\
\bar{C}^{rr}_{00} &= -\sum_{i \neq r} \frac{\alpha_i}{\alpha_r} \ln|Z_r - Z_i| + \frac{1}{\alpha_r}\tau_0^{(r)} \\
\bar{C}^{rs}_{n0} &= \bar{C}^{sr}_{0n} = \frac{1}{2n}\oint_{Z_r} \frac{dz}{2\pi i}\frac{1}{z - Z_s} e^{-n\zeta_r(z)}, \quad n \geq 1, \\
\bar{C}^{rs}_{nm} &= \frac{1}{2nm}\oint_{Z_r}\frac{dz}{2\pi i}\oint_{Z_s}\frac{dz'}{2\pi i}\frac{1}{(z-z')^2} e^{-n\zeta_r(z) - m\zeta'_s(z')}, \quad n, m \geq 1.
\end{aligned}
$$

Reality conditions of the Green's function

$$
\bar{C}^{rs}_{nm} = \bar{C}^{*rs}_{-n-m}, \quad \bar{C}^{rs}_{-nm} = 0, \quad n, m \geq 1.
$$

Scattering amplitude

$$
\begin{aligned}
\mathcal{W} &= \int DX \exp\left(i \sum_{r=1}^{M} \int P_r(\sigma)\cdot X(\tau_r,\sigma) d\sigma - \int d\tau d\sigma \mathcal{L} \right) \\
&= [\det \Delta]^{-d/2} \exp\left\{ \frac{1}{4} \left\{ \sum_r \xi_r P_0^2 - \sum_r \sum_{n=1} \frac{1}{n} P_n^{(r)} \cdot P_{-n}^{(r)} \right.\right. \\
&\quad \left.\left. + \sum_{n,m} \bar{C}^{rs}_{nm} e^{|n|\xi_r + |m|\xi'_s} P_{-n}^{(r)} \cdot P_{-m}^{(s)} \right\}\right. \\
&= \langle \mathbf{P} | \exp\left(\sum_r \xi_r L_0^{(r)} \right) | V[N] \rangle .
\end{aligned}
$$

Neumann Functions of Three-Closed-String Vertex

Neumann functions of three-closed string vertex and those of three-open string vertex:

$$
\begin{aligned}
\bar{C}^{rs}_{00} &= \bar{N}^{rs}_{00} = \ln|Z_r - Z_s|, \quad r \neq s, \\
\bar{C}^{rr}_{00} &= \bar{N}^{rr}_{00} = -\sum_{i\neq r} \frac{\alpha_i}{\alpha_r} \ln|Z_r - Z_i| + \frac{1}{\alpha_r}\tau_0, \\
\bar{C}^{rs}_{n0} &= \bar{C}^{rs}_{-n0} = \frac{1}{2}\bar{N}^{rs}_{n0} = \frac{1}{2n}\oint_{Z_r} \frac{dz}{2\pi i} \frac{1}{z - Z_s} e^{-n\zeta_r(z)}, \quad n \geq 1, \\
\bar{C}^{rs}_{nm} &= \bar{C}^{rs}_{-n-m} = \frac{1}{2}\bar{N}^{rs}_{mn} \\
&= \frac{1}{2nm}\oint_{Z_r} \frac{dz}{2\pi i} \oint_{Z_s} \frac{dz'}{2\pi i} \frac{1}{(z-z')^2} e^{-n\zeta_r(z) - m\zeta'_s(z')}, \quad n, m \geq 1, \\
\bar{C}^{rs}_{n-m} &= \bar{C}^{rs}_{-nm} = 0.
\end{aligned}
$$

Factorization of Three-Closed-String Scattering Amplitude

$$
\begin{aligned}
\mathcal{A}[1,2,3] \quad = \quad & g\langle\{\mathbf{k}^{(r)}\}| \\
& \exp\left\{\sum_{r,s}\Big(\sum_{n,m\ge 1}\frac{1}{2}\bar{N}^{rs}_{nm}\,\frac{\alpha^{(r)\dagger}_n}{2}\cdot\frac{\alpha^{(r)\dagger}_m}{2}+\sum_{n\ge 1}\bar{N}^{rs}_{n0}\frac{\alpha^{(r)\dagger}_n}{2}\cdot\frac{p^{(s)}}{2}\Big)\right\} \\
& \exp\left\{\tau_0\sum_r\frac{1}{\alpha_r}\left(\frac{1}{2}\left(\frac{p^{(r)}}{2}\right)^2-1\right)\right\} \\
& \exp\left\{\sum_{r,s}\Big(\sum_{n,m\ge 1}\frac{1}{2}\bar{N}^{rs}_{nm}\,\frac{\tilde{\alpha}^{(r)\dagger}_n}{2}\cdot\frac{\tilde{\alpha}^{(r)\dagger}_m}{2}+\sum_{n\ge 1}\bar{N}^{rs}_{n0}\frac{\tilde{\alpha}^{(r)\dagger}_n}{2}\cdot\frac{p^{(s)}}{2}\Big)\right\} \\
& \exp\left\{\tau_0\sum_r\frac{1}{\alpha_r}\left(\frac{1}{2}\left(\frac{p^{(r)}}{2}\right)^2-1\right)\right\}|0\rangle.
\end{aligned}
$$

Factorization of Three-Closed-String Scattering Amplitude

Factorization of three-closed-string scattering amplitude

$$\mathcal{A}_{\text{closed}}[1,2,3] = \mathcal{A}_{\text{open}}[1,2,3]\,\mathcal{A}_{\text{open}}[1,2,3].$$

Scattering amplitude of three closed strings can be completely factorized into those of three open strings except for the zero modes.

Question: Can we factorize general closed string scattering amplitudes into those of open string theory?

Realization of the Kawai-Lewellen-Tye (KLT) relations in the framework of the second quantized string theory .

Three-Graviton Scattering Amplitude

Decomposition of the spin-2 field into graviton, anti-symmetric tensor, and scalar field

$$h_{\mu\nu} = \left\{\frac{1}{2}\left(h_{\mu\nu} + h_{\nu\mu}\right) - \eta_{\mu\nu}\frac{1}{d}h^{\sigma}{}_{\sigma}\right\} + \left\{\frac{1}{2}\left(h_{\mu\nu} - h_{\nu\mu}\right)\right\} + \eta_{\mu\nu}\left\{\frac{1}{d}h^{\sigma}{}_{\sigma}\right\}.$$

We choose the covariant gauge condition

$$\partial^{\mu} h_{\mu\nu} = 0,$$

which becomes de Donder gauge condition for the graviton

$$\partial^{\mu} h_{\mu\nu} - \frac{1}{d-2}\partial_{\nu} h^{\sigma}{}_{\sigma} = 0.$$

For three-graviton scattering, we choose the external string state as

$$|\Psi_{3G}\rangle = \prod_{r=1}^{3}\left\{h_{\mu\nu}(p^r)\alpha_{-1}^{(r)\mu}\tilde{\alpha}_{-1}^{(r)\nu}\right\}|0\rangle.$$

Three-graviton scattering amplitude

$$\begin{aligned}
\mathcal{A}_{[3-graviton]} &= \int \prod_{r=1}^{3} dp^{(r)} \delta\left(\sum_{r=1}^{3} p^{(r)}\right) \frac{2g}{3} \langle \Psi_{3G} | E^{\text{Closed}}_{[3]}[1,2,3] | 0 \rangle \\
&= \left(\frac{2g}{3}\right) e^{-2\tau_0 \sum_{r=1}^{3} \frac{1}{\alpha_r}} \int \prod_{i=1}^{3} dp^{(i)} \delta\left(\sum_{i=1}^{3} p^{(i)}\right) \\
&\quad \langle 0 | \left\{ \prod_{i=1}^{3} h_{\mu\nu}(p^{(i)}) a_1^{(i)\mu} \cdot \tilde{a}_1^{(i)\nu} \right\} \frac{1}{2^5} \left(\sum_{r,s=1}^{3} \bar{N}^{rs}_{11} a_1^{(r)\dagger} \cdot a_1^{(s)\dagger} \right) \\
&\quad \left(\sum_{t=1}^{3} \bar{N}^{t}_{1} a_1^{(t)\dagger} \cdot \mathbf{p} \right) \frac{1}{2^5} \left(\sum_{l,m=1}^{3} \bar{N}^{lm}_{11} \tilde{a}_1^{(l)\dagger} \cdot \tilde{a}_1^{(m)\dagger} \right) \\
&\quad \left(\sum_{n=1}^{3} \bar{N}^{n}_{1} \tilde{a}_1^{(n)\dagger} \cdot \mathbf{p} \right) |0\rangle .
\end{aligned}$$

Three-Graviton Scattering Amplitude

We note that $\mathcal{A}_{[3-graviton]}$ can be written also as

$$\begin{aligned}\mathcal{A}_{[3-graviton]} &= \left(\frac{2g}{3}\right)\frac{1}{2^8}\int\prod_{i=1}^{3}dp^{(i)}\delta\left(\sum_{i=1}^{3}p^{(i)}\right) \\ &\quad \langle 0|\left\{\prod_{i=1}^{3}h_{\mu\nu}(p^{(i)})a_1^{(i)\mu}\cdot\tilde{a}_1^{(i)\nu}\right\}E^{\text{Open}}_{[3-Gauge]}\,\tilde{E}^{\text{Open}}_{[3-Gauge]}|0\rangle.\end{aligned}$$

Making use of the Neumann functions of the open string

$$\begin{aligned}\bar{N}^{11}_{11} &= \frac{1}{2^4},\quad \bar{N}^{22}_{11}=\frac{1}{2^4},\quad \bar{N}^{33}_{11}=2^2, \\ \bar{N}^{12}_{11} &= \bar{N}^{21}_{11}=\frac{1}{2^4},\quad \bar{N}^{23}_{11}=\bar{N}^{32}_{11}=\frac{1}{2},\quad \bar{N}^{31}_{11}=\bar{N}^{13}_{11}=\frac{1}{2}, \\ \bar{N}^{1}_{1} &= \bar{N}^{2}_{1}=\frac{1}{4},\quad \bar{N}^{3}_{1}=-1,\end{aligned}$$

We are able evaluate the three-graviton interaction term as follows

$$
\begin{aligned}
\mathcal{A}_{[3-graviton]} \quad = \quad & \left(\frac{2g}{3}\right) 2^6 \left(\frac{1}{2^5}\right)^2 \int \prod_{i=1}^{3} dp^{(i)} \delta\left(\sum_{i=1}^{3} p^{(i)}\right) \\
& h_{\mu_1\nu_1}(p^{(1)}) h_{\mu_2\nu_2}(p^{(2)}) h_{\mu_3\nu_3}(p^{(3)}) \\
& \Big\{ -\frac{1}{2^4}\eta^{\mu_1\mu_2}\mathbf{p}^{\mu_3} + \frac{1}{2^3}\eta^{\mu_1\mu_3}\mathbf{p}^{\mu_2} + \frac{1}{2^3}\eta^{\mu_2\mu_3}\mathbf{p}^{\mu_1} \\
& -\frac{1}{2^4}\eta^{\mu_2\mu_1}\mathbf{p}^{\mu_3} + \frac{1}{2^3}\eta^{\mu_3\mu_1}\mathbf{p}^{\mu_2} + \frac{1}{2^3}\eta^{\mu_3\mu_2}\mathbf{p}^{\mu_1} \Big\} \\
& \Big\{ -\frac{1}{2^4}\eta^{\nu_1\nu_2}\mathbf{p}^{\nu_3} + \frac{1}{2^3}\eta^{\nu_1\nu_3}\mathbf{p}^{\nu_2} + \frac{1}{2^3}\eta^{\nu_2\nu_3}\mathbf{p}^{\nu_1} \\
& -\frac{1}{2^4}\eta^{\nu_2\nu_1}\mathbf{p}^{\nu_3} + \frac{1}{2^3}\eta^{\nu_3\nu_1}\mathbf{p}^{\nu_2} + \frac{1}{2^3}\eta^{\nu_3\nu_2}\mathbf{p}^{\nu_1} \Big\}.
\end{aligned}
$$

Three-Graviton Scattering Amplitude

$\mathcal{A}_{[3-graviton]}$ is precisely the three-graviton interaction term which may be obtained from the Einstein's gravity action.

$$\begin{aligned}\mathcal{A}_{[3-graviton]} &= \kappa \int \prod_{i=1}^{3} dp^{(i)} \delta \left(\sum_{i=1}^{3} p^{(i)} \right) \\ & h_{\mu_1\nu_1}(p^{(1)}) h_{\mu_2\nu_2}(p^{(2)}) h_{\mu_3\nu_3}(p^{(3)}) \\ & \left\{ \eta^{\mu_1\mu_2} p^{(1)\mu_3} + \eta^{\mu_2\mu_3} p^{(2)\mu_1} + \eta^{\mu_3\mu_1} p^{(3)\mu_2} \right\} \\ & \left\{ \eta^{\nu_1\nu_2} p^{(1)\nu_3} + \eta^{\nu_2\nu_3} p^{(2)\nu_1} + \eta^{\nu_3\nu_1} p^{(3)\nu_2} \right\}\end{aligned}$$

where $\kappa = \frac{g}{2^7 \cdot 3} = \sqrt{32\pi G_{10}}$.

Scattering Amplitude of Four Strings

Using the Cremmer-Gervais identity, we may write the scattering amplitude of four closed strings as follows

$$\mathcal{A}[1,2,3,4] = g^2 \int \prod_r dZ_r^2 \frac{|Z_a - Z_b|^2 |Z_b - Z_c|^2 |Z_c - Z_a|^2}{d^2 Z_a d^2 Z_b d^2 Z_c}$$

$$\prod_{r<s} |Z_r - Z_s|^{2\frac{p^{(r)}}{2}\cdot\frac{p^{(s)}}{2}} \prod_{r<s} |Z_r - Z_s|^{2\left\{\frac{\alpha_s}{\alpha_r}\left(1-\frac{1}{2}\left(\frac{p^{(r)}}{2}\right)^2\right)+\frac{\alpha_r}{\alpha_s}\left(1-\frac{1}{2}\left(\frac{p^{(s)}}{2}\right)^2\right)\right\}}$$

$$\langle \{\mathbf{k}^{(r)}\}| \exp\left\{\frac{1}{4}\sum_{r,s}\sum_{n,m\geq 1}\left(\bar{C}^{rs}_{nm}\,\tilde{\alpha}^{(r)\dagger}_n \cdot \tilde{\alpha}^{(s)\dagger}_m + \bar{C}^{rs*}_{nm}\,\alpha^{(r)\dagger}_n \cdot \alpha^{(r)\dagger}_m\right)\right.$$

$$+\sum_{r,s}\left(\sum_{n\geq 1}\left(\bar{C}^{rs}_{n0}\tilde{\alpha}^{(r)\dagger}_n \cdot \frac{p^{(s)}}{2} + \bar{C}^{rs*}_{n0}\,\alpha^{(r)\dagger}_n \cdot \frac{p^{(s)}}{2}\right)\right)$$

$$\left. +2\tau^{(r)}\sum_r \frac{1}{\alpha_r}\left(\frac{1}{2}\left(\frac{p^{(r)}}{2}\right)^2 - 1\right)\right\}|0\rangle.$$

Shapiro-Virasoro Amplitude

Scattering of four closed string tachyons

$$\begin{aligned}\mathcal{A}_{\text{Tachyon}}[1,2,3,4] &= g^2 \int d^2 Z\, |Z|^{2\,\frac{p^{(1)}}{2}\cdot\frac{p^{(2)}}{2}} |1-Z|^{2\,\frac{p^{(2)}}{2}\cdot\frac{p^{(3)}}{2}} \\ &= g^2 \int d^2 Z\, |Z|^{2\left(-\frac{s}{8}-2\right)} |1-Z|^{2\left(-\frac{u}{8}-2\right)} \\ &= 2\pi g^2 \frac{\Gamma\left(-1-\frac{s}{8}\right)\Gamma\left(-1-\frac{t}{8}\right)\Gamma\left(-1-\frac{u}{8}\right)}{\Gamma\left(2+\frac{s}{8}\right)\Gamma\left(2+\frac{t}{8}\right)\Gamma\left(2+\frac{u}{8}\right)}.\end{aligned}$$

Mandelstam variables:

$$s = -(p_1+p_2)^2, \quad t = -(p_1+p_3)^2, \quad u = -(p_1+p_4)^2.$$

The Koba-Nielsen variables:

$$Z_1 = 0, \quad Z_2 = Z, \quad Z_3 = 1, \quad Z_4 = \infty.$$

Conclusions and Discussions

1. Construction of covariant closed string field theory in the proper-time gauge.
2. Neuman functions and Fock space representations of closed string vertices.
3. Scattering amplitude of three closed strings.
4. Complete factorization of three-closed-string amplitudes.
5. Three-graviton scattering and its relation to the three-gauge-particle scattering.
6. Generalized Kawai-Lewellen-Tye (KLT) relations.