Skyrmion Models of Nuclei

Nicholas S. Manton

DAMTP, University of Cambridge
N.S.Manton@damtp.cam.ac.uk

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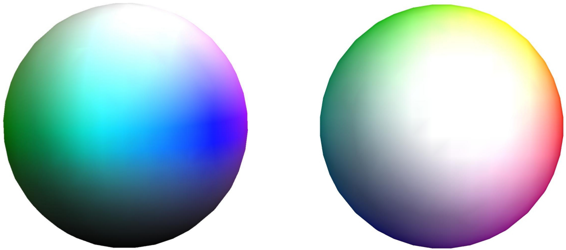
Skyrme Model

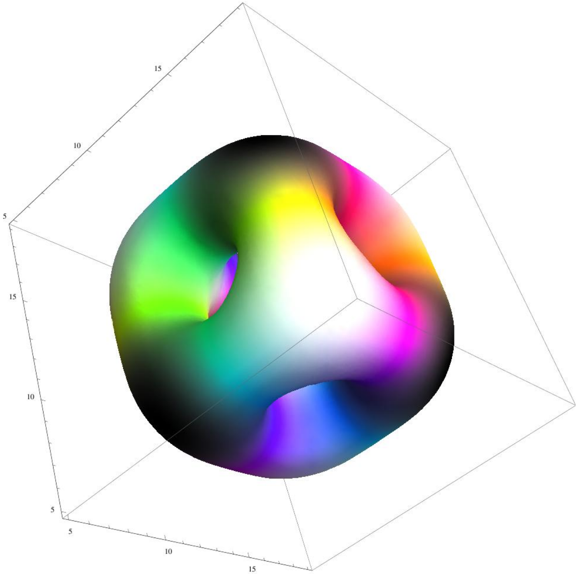
- Skyrme field is an SU(2)-valued field
$$U(\mathbf{x}) = \sigma(\mathbf{x})\,\mathbf{1} + i\boldsymbol{\pi}(\mathbf{x})\cdot\boldsymbol{\tau}\,.$$
($\mathbf{1}$ = unit matrix, $\boldsymbol{\tau}$ = Pauli matrices.) $\sigma(\mathbf{x})$ and $\boldsymbol{\pi}(\mathbf{x})$ are sigma and pion fields satisfying $\sigma^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi} = 1$.
- Vacuum (boundary condition at spatial infinity) is $U = \mathbf{1}$.
- Baryon number (Atomic number) is an integral involving the L(SU(2))-valued "current" $R_i = (\partial_i U)U^{-1}$.
$$B = -\frac{1}{24\pi^2}\int_{\mathbb{R}^3} \varepsilon_{ijk}\mathrm{Tr}(R_iR_jR_k)\,d^3x\,,$$
and is a (positive) integer, constant in time.
- The Skyrme energy is
$$\begin{aligned} E \quad = \quad & \int_{\mathbb{R}^3}\left\{-\frac{1}{2}\mathrm{Tr}(R_iR_i) - \frac{1}{16}\mathrm{Tr}([R_i,R_j][R_i,R_j]) \right. \\ & \left. + m_\pi^2\mathrm{Tr}(\mathbf{1} - U)\right\}d^3x\,. \end{aligned}$$

- Skyrmions are energy minima and low-lying saddle points for each B. They are free to rotate as rigid bodies in space and isospace (pion space).
- Quantizing the orientational degrees of freedom of the $B = 1$ Skyrmion gives nucleon states with spin and isospin $1/2$. Delta resonance states have spin and isospin $3/2$ [Adkins, Nappi and Witten].
- Vibrational degrees of freedom are now recognised as important, and can lead to break-up into Skyrmion subclusters. Some modes are essentially harmonic; others connect energy minima via saddle points (sphalerons) and need nonlinear treatment.
- One must identify saddle points and their unstable modes in order to construct a vibrational manifold. Tunnelling between minima goes through (under) saddle points.

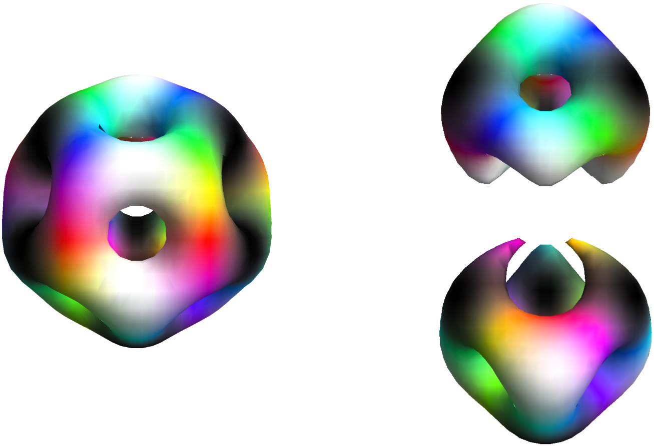
- Skyrmions are found by numerical energy minimisation. Initial configurations are made using multi-layer “rational map” ansatz or crystal chunks. The “half-Skyrmion” crystal condenses into $B = 4$ alpha particle cubes. Also very useful is the FCC “Skyrmion crystal” realised in the lightly-bound model. This has tetrahedrally symmetric crystal chunks.
- Visualising Skyrmions – the Runge colour sphere records the normalised (unit-vector) pion field $\pi/|\pi|$. The colours are superposed on a constant energy density surface.
- Figures by R. Battye and P. Sutcliffe, D. Feist, P.H.C. Lau, C.J. Halcrow, J.I. Rawlinson, D. Harland.



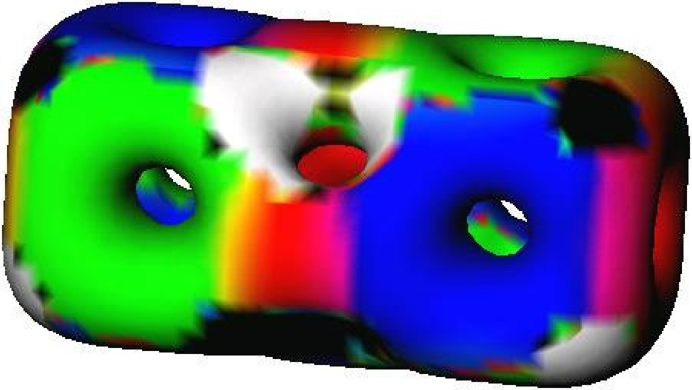
$B = 1$ Skyrmion (two different orientations)



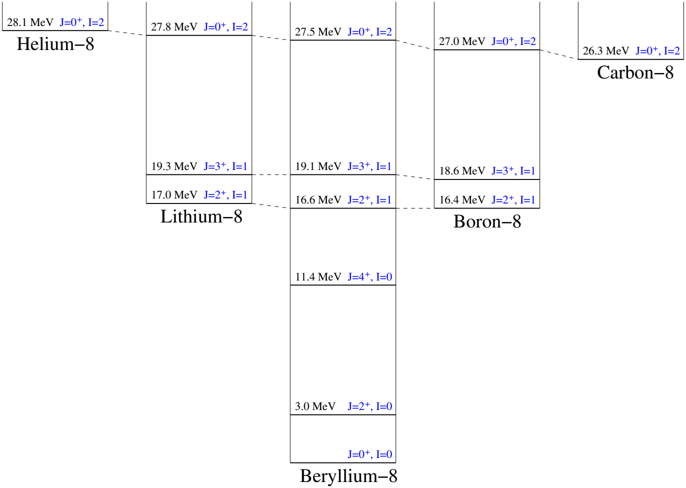
$B = 4$ Skyrmion



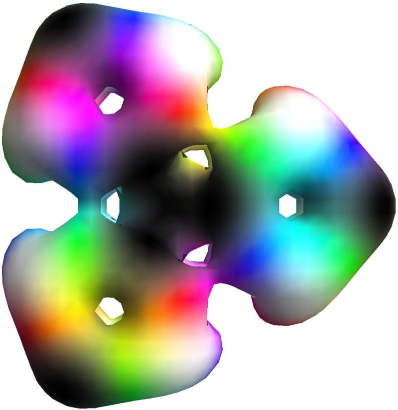
$B = 7$ Skyrmion and its deformation into clusters



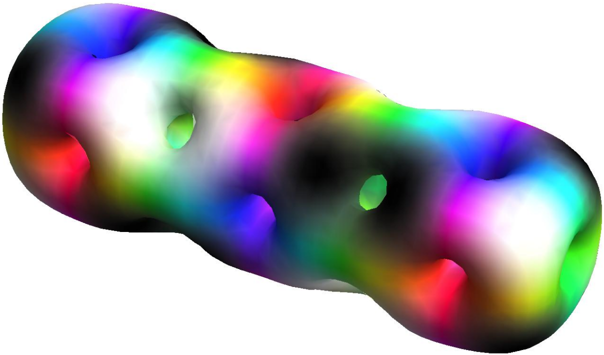
$B = 8$ Skyrmion



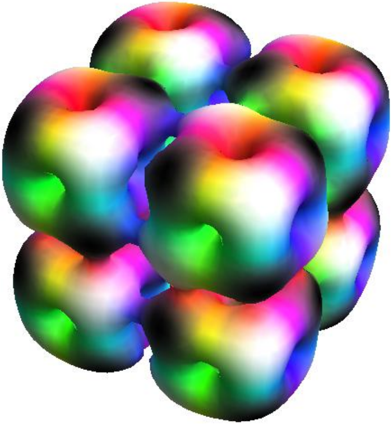
Energy level diagram for nuclei with baryon number 8



$B = 12$ Skyrmion with D_{3h} symmetry

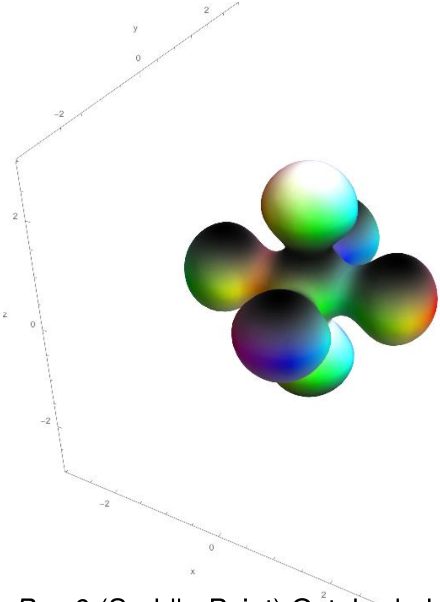


$B = 12$ Skyrmion with D_{4h} symmetry

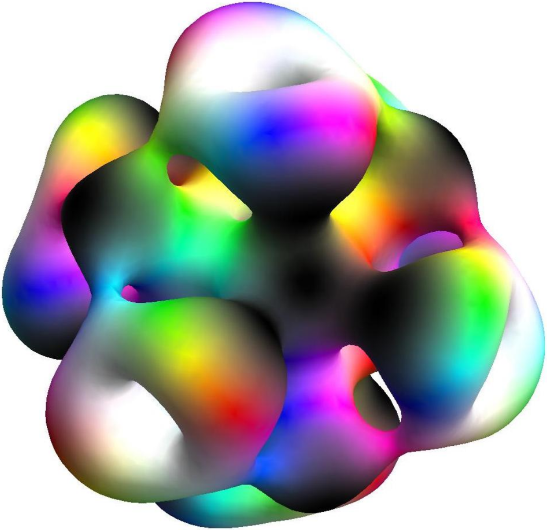


$B = 32$ Skyrmion

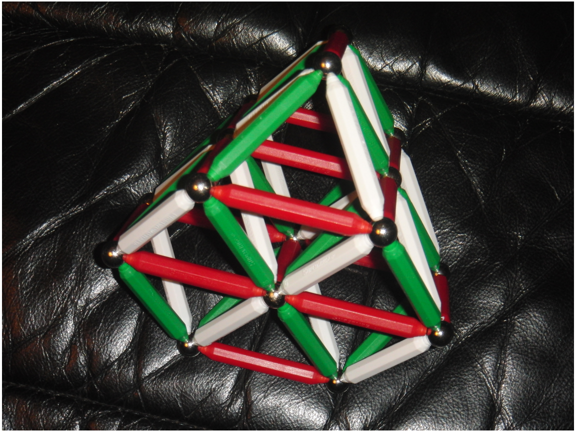
- Some recent work uses the lightly-bound model, developed in Leeds [D. Harland, J.M. Speight et al.]. Skyrmions resemble clusters of $B = 1$ Skyrmions at points of an FCC lattice.
- The FCC clusters relax to solutions of the standard Skyrme model with similar shapes – sometimes with enhanced symmetry (e.g. $B = 4$).
- We have found many solutions up to $B = 85$.
- Basic structural units in the FCC lattice are alternating $B = 4$ tetrahedra and $B = 6$ octahedra, sharing faces.



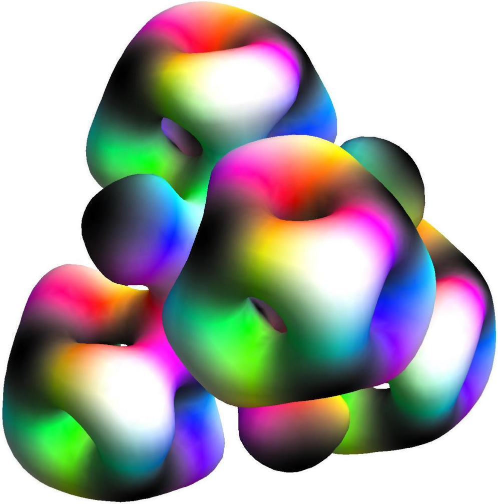
$B = 6$ (Saddle-Point) Octahedral Skyrmion



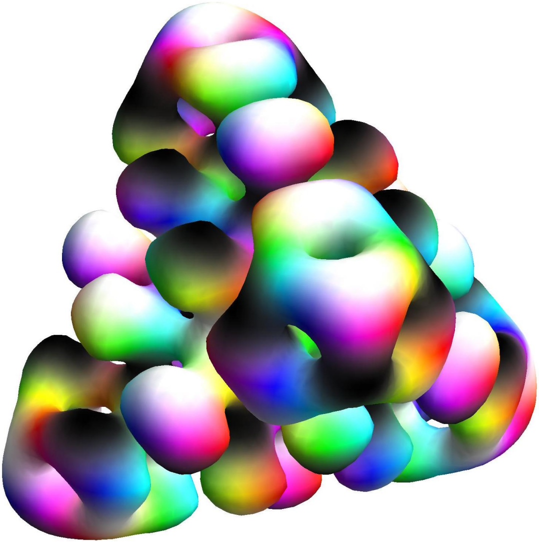
$B = 16$ Skyrmion



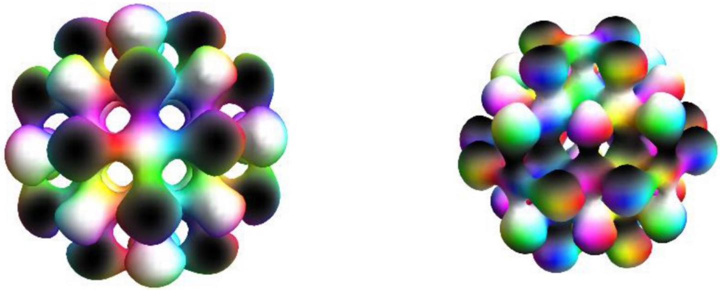
Ball and Stick Model of $B = 16$ Skyrmion



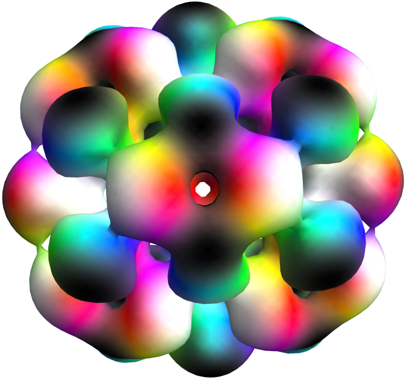
$B = 20$ Skyrmion



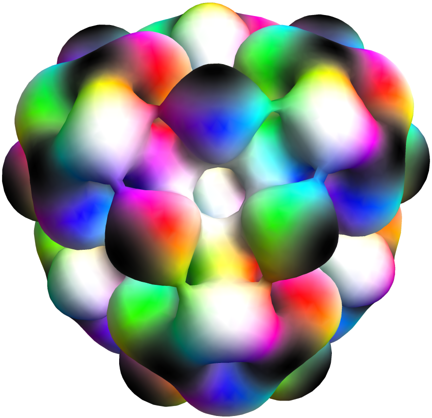
$B = 35$ Skyrmion



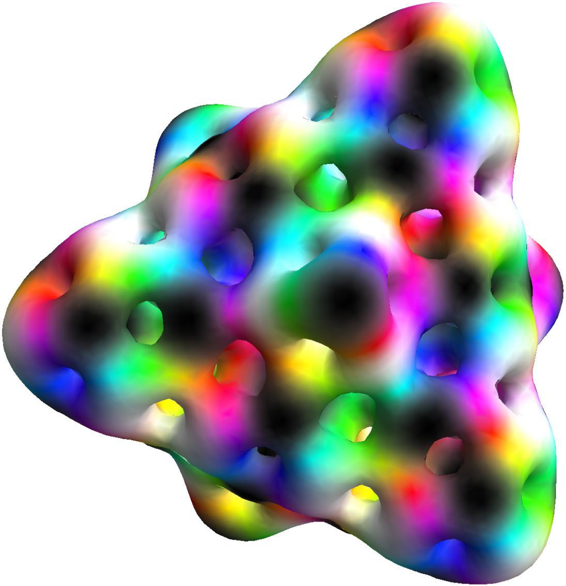
$B = 38$ (Truncated Octahedral) Skyrmion



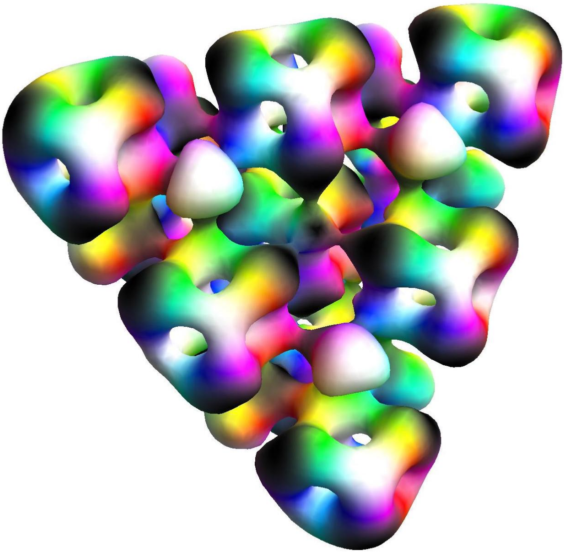
$B = 40$ Skyrmion



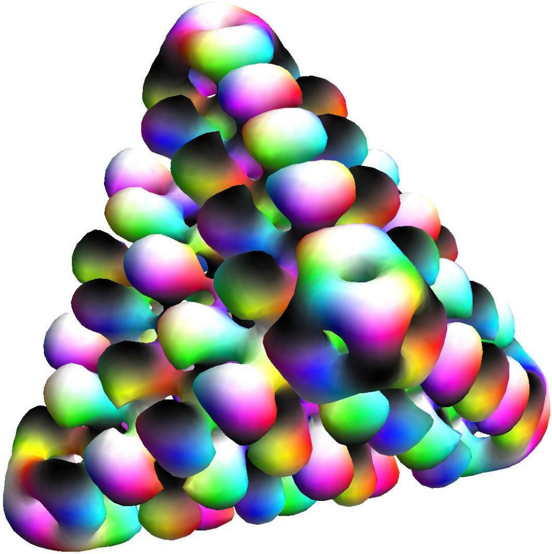
$B = 40$ Skyrmion (another view)



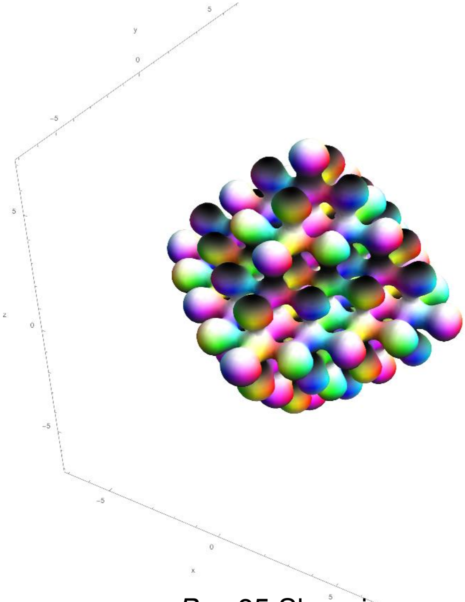
$B = 56$ Skyrmion [Lau]



$B = 56$ Skyrmion [Halcrow]



$B = 84$ Skyrmion



$B = 85$ Skyrmion

Skyrmions as SU(4) Weight Clusters

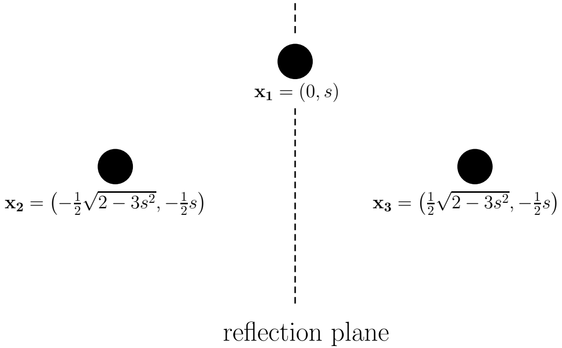
- A useful set of compact, convex, tetrahedrally symmetric FCC clusters are the weight diagrams of SU(4) (C.J. Halcrow, NSM and J.I. Rawlinson).
- The SU(4) root lattice is FCC, and each weight diagram is in one of its FCC cosets (labelled by quadrality).
- Put $B = 1$ Skyrmions at the weight locations (in four distinct orientations) to create a lightly-bound Skyrmion. Relaxation gives a standard Skyrmion.
- B is the cluster number, the number of weights *without multiplicity*.

Rigid Body Quantization

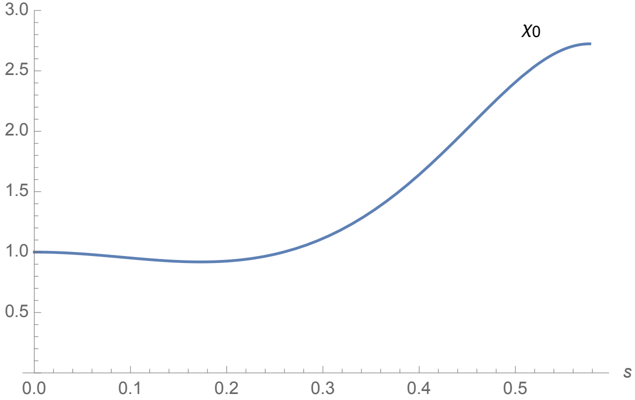
- The basic quantization idea is to treat these clusters as rigid bodies free to rotate in space and isospace (and also move through space). The cluster symmetries restrict the allowed spin/isospin/parity combinations.
- A simple consequence of tetrahedral symmetry is that states with isospin 0 ($N = Z$) and baryon number a multiple of 4 have spin/parity 0^+, 3^-, 4^+, $6^\pm$, 7^-, 8^+, ... Spin 1 and spin 2 excitations require vibrational motion.
- This spectrum is typical of the magic nuclei, e.g. Oxygen-16 and Calcium-40, and tetrahedral clusters representing these are in our list. Shell model also allows for (tetrahedral) deformation from spherical core shape in magic nuclei.
- Rigid body quantization has some further success, e.g. for $B = 38$, but certainly doesn't describe all low-energy states.

Quantum States of ^{12}C

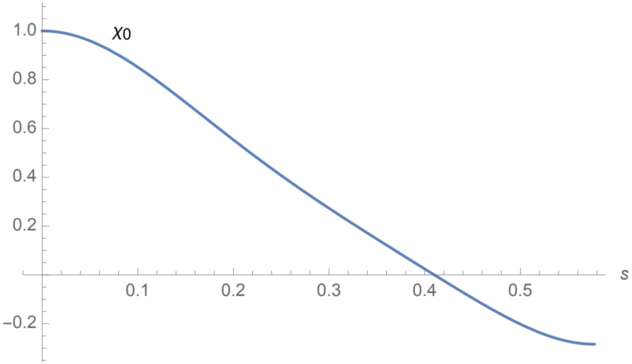
- A simple approach by P.H.C. Lau and NSM combines the distinct states arising from the rigid body quantization of the D_{3h}-symmetric $B = 12$ Skyrmion (triangle) and the D_{4h}-symmetric Skyrmion (chain).
- This gives a ground state band ($0^+, 2^+, 3^-, 4^\pm, 5^-, \ldots$) and a Hoyle state band ($0^+, 2^+, 4^+, \ldots$). The Hoyle state band has smaller E v. $J(J+1)$ slope.
- However, this approach suppresses the dynamical relationship between triangle and chain, and the intermediate "bent arm", and fails to capture vibrational states, starting with 1^-.
- J.I. Rawlinson has considered the bending mode connecting triangle to chain, and has quantized this coupled to rotations. Quantization takes place on a graph in the shape space of triangles.



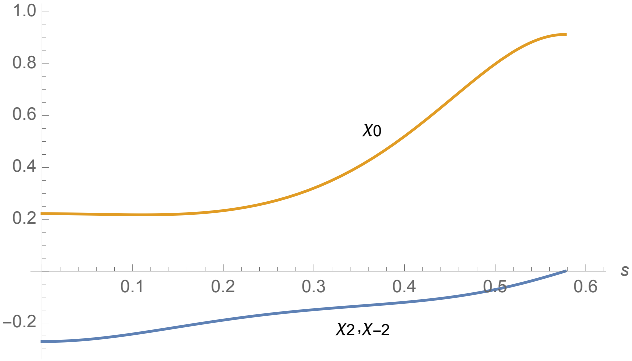
Isosceles triangle of three alpha-particles. *s* is related to bending angle.



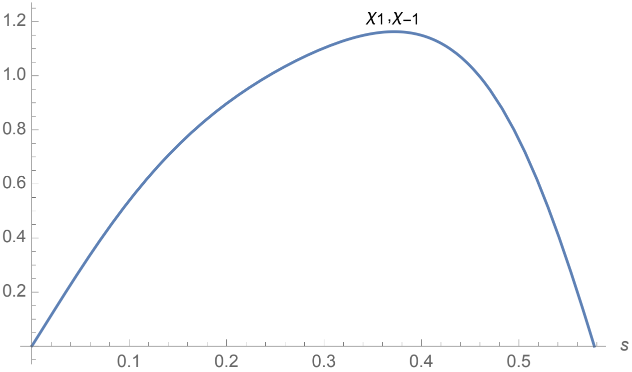
Ground state wavefunction, concentrated at triangle



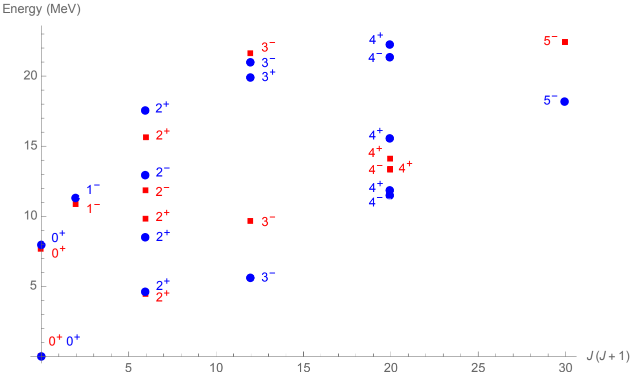
Hoyle state wavefunction, concentrated at chain



Lowest 2^+ state, in ground state band



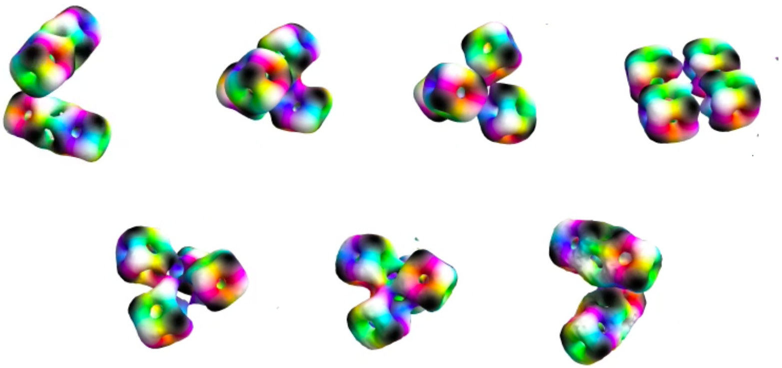
Lowest vibrational 1^- state



Carbon-12 Energy Levels: Experiment, Theory

Vibrational Quantum States of $^{16}\mathrm{O}$

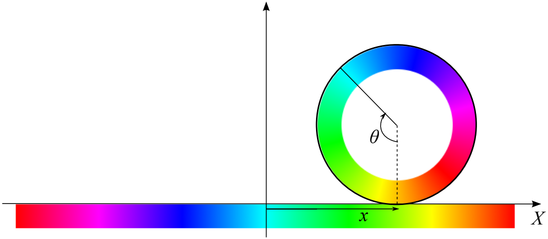
- Vibrational modes of tetrahedron of alpha particles classified as *A* (breather), *E* (towards square) and *F* (towards equilateral triangle). Skyrme model predicts *E* mode has lowest frequency. Coupling to rotations gives rovibrational states.
- C.J. Halcrow, C. King and NSM have constructed an *E* vibrational surface of D_2-symmetric configurations. Surface itself has O_h symmetry and allows for four alpha particles in a tetrahedron, square or bent rhomb. Quantum states on surface are interpreted as multiple *E*-phonons. Model explains splitting of 2^+ and 2^- single-phonon *E* states, which are degenerate in other models.
- Model explains many states of Oxygen-16 (maybe all) up to 20 MeV, provide *A* and *F* phonons included (C.J. Halcrow thesis, C. King thesis). Electromagnetic transition strengths (C.J. Halcrow, in progress) are needed to confirm assignment of states.



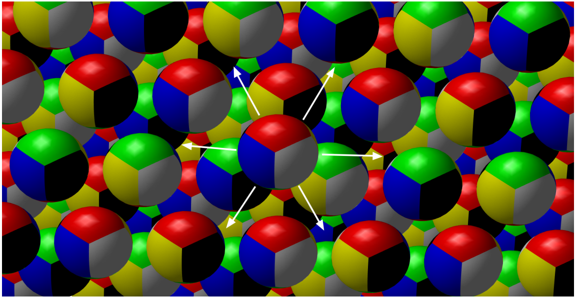
B=16 *E*-mode deformation from tetrahedron to square

Rolling Motion and Spin-Orbit Coupling

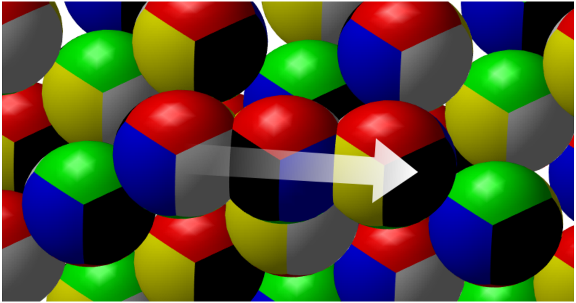
- Spin-orbit coupling is an important strong-interaction effect in nuclear physics. Can be understood in the Skyrme model because the potential energy of a Skyrmion depends on its orientation.
- A $B = 1$ Skyrmion moving over a nuclear surface prefers to roll rather than slip or anti-roll. Why is this?
- D. Harland and NSM see this effect classically using the lightly-bound model, and have also calculated the quantum mechanical correlation of spin and momentum. The effect is non-perturbative and needs a quite strong interaction potential.
- This analysis extends the earlier 2D model of rolling cogs by C.J. Halcrow and NSM to 3D.



Coloured disc on a fixed coloured rail.



A Skyrmion above a half-filled lattice of Skyrmions (Leeds colouring scheme).



The path of a rolling Skyrmion. The sense of spin (about the red-green axis) is unambiguous.

Conclusions

- Based on insight from rational maps, the FCC crystal in the lightly-bound Skyrme model, weight diagrams of SU(4), and the clustering of alpha-particle cubes, we have (numerically) found Skyrmions in the standard Skyrme model for many B values. Skyrmions often have tetrahedral or cubic symmetry.
- The Skyrme model has produced good spectra for Carbon-12 and Oxygen-16, allowing for rotations and shape changes. A vibrational manifold (or its approximation by a graph) is essential for understanding the spectrum.
- Spin-orbit coupling can be modelled for a $B = 1$ Skyrmion rolling over a hexagonally-symmetric surface of the FCC Skyrmion crystal. What about rolling motion over the surface of a tetrahedrally symmetric cluster?
- Further work needed on Coulomb and other E.M. effects, and on weak decays.