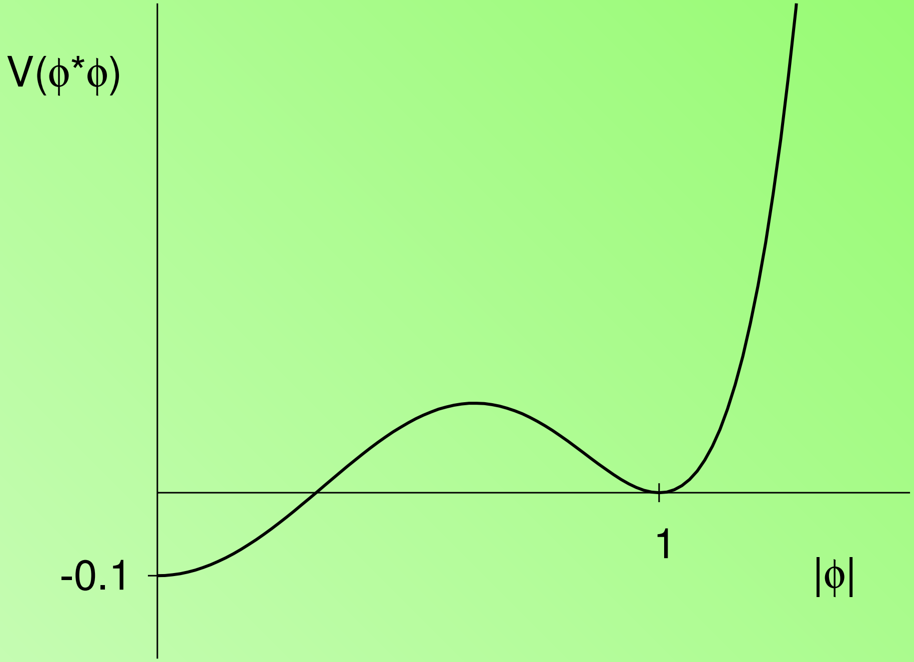
The decay of the false Skyrmion

by

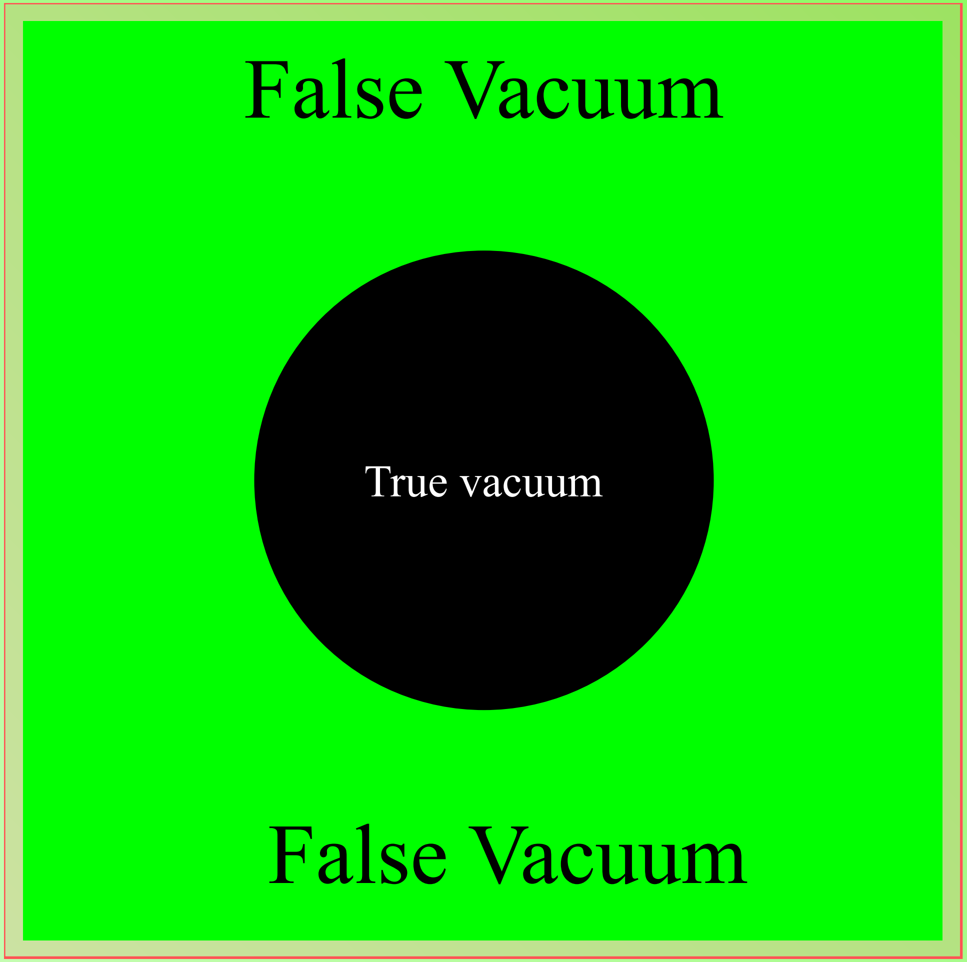
É. Dupuis, M. Haberichter, Richard MacKenzie, M. B.Paranjape, U. A. Yajnik

arXiv: http://arxiv.org/abs/arXiv:1805.08038

- False vacuum decay is an interesting physical phenomena.
- Decay of a super-heated or super-cooled liquid to the gas or solid phase are common examples from tangible physical systems.
- The decay corresponds to quantum tunnelling transition.



Cartoon of Vacuum Bubble



- The amplitude for such a decay is suppressed by the usual exponential of the classical action: $e^{-S_E/\hbar}$
- This suppression can be reduced or even removed by the presence of perturbations or disturbances which can seed the transition.
- Such cases are called induced vacuum decay.
- We consider the situation that the false vacuum and the true vacuum correspond to the symmetry broken vacua.
- This can be realized in the Skyrme model.

Skyrme Model

- The Skyrme model is a non-linear sigma model where the quantum field takes values in a topologically non-trivial target manifold $\mathcal{M}$.
- The vacuum corresponds to one point in the target manifold, and finite energy configurations typically achieve the vacuum configuration at spatial infinity.
- This gives the topological compactification of space from $\mathbb{R}^3 \to S^3$

$$\mathbb{R}^2 \to S^2$$

- Then finite energy configurations correspond to correspond to maps from S^3 or S^2 to the target manifold. These correspond to the homotopy groups:

$$\Pi_3\left(\mathcal{M}\right)$$

$$\Pi_2\left(\mathcal{M}\right)$$

- These homotopy groups are quite often non-trivial and the minimum energy configuration in each homotopy class give rise solutions of the equations of motion are called topological solitons.
- These solitons are called Skyrmions.

- It can be physically relevant that the potential energy on the target manifold contains several minima, but one global minimum.
- The local minima are called false vacua while the global minimum is the true vacuum.
- Topological solitons that asymptotically go to one of the false minima are correspondingly called false Skyrmions.
- False Skyrmions are metastable, they can decay quantum mechanically by tunnelling.

- We study this decay in the context of the usual Skyrme model, with rescaling $f_\pi/(4e)$ $2/(ef_\pi)$

$$\mathcal{L} = \frac{1}{2}\,\mathrm{Tr}\left[\partial_\mu \mathcal{U}^\dagger \partial^\mu \mathcal{U}\right] + \frac{1}{16}\,\mathrm{Tr}\left[\mathcal{U}^\dagger \partial_\mu \mathcal{U}, \mathcal{U}^\dagger \partial_\nu \mathcal{U}\right]^2$$

- with the added mass term:

$$\mathcal{L}_{\mathrm{mass}} = -\frac{1}{4}\left(m_1^2\,\mathrm{Tr}\left[\mathbb{1} - \mathcal{U}\right] + m_2^2\,\mathrm{Tr}\left[\mathbb{1} - \mathcal{U}^2\right]\right)$$

- $\mathcal{U}(\mathbf{x})$ takes values in: $\mathcal{M} = S^3$

$$\Pi_3(S^3) = \mathbb{Z}$$

- The chiral symmetry is:

$SU(2) \times SU(2)$ $\quad$ $(\mathcal{V}, \mathcal{W}) : \mathcal{U} \to \mathcal{V}^\dagger \mathcal{U} \mathcal{W}$

- We could add a mass term (potential) of the form $$\mathcal{L}_{\text{mass}} = \sum_k C_k \, \text{Tr}\left[\mathcal{U}^k\right]$$
- Then writing

$$\mathcal{U} = e^{i\zeta\hat{\mathbf{n}}\cdot\boldsymbol{\tau}} = \cos\zeta + i\hat{\mathbf{n}}\cdot\boldsymbol{\tau}\sin\zeta$$

- we get $$\mathcal{L}_{\text{mass}} = \sum_k C_k \cos(k\zeta)$$
- which is the Fourier decomposition of an arbitrary potential $V(\zeta)$.
- In general, we could add a potential which is an essentially arbitrary function on the target space.
- The only constraint is that the ensuing pion mass be small.

- With $\mathcal{U} = e^{i\zeta\hat{\mathbf{n}}\cdot\boldsymbol{\tau}} = \cos\zeta + i\hat{\mathbf{n}}\cdot\boldsymbol{\tau}\sin\zeta$
- The potential becomes:

$$V(\zeta) = m_1^2\sin^2\zeta/2 + m_2^2\sin^2\zeta$$

- The global minium is at $\zeta = 0$ with energy 0, while a false minimum appears at $\zeta = \pi$ with energy m_1^2.
- The Skyrmion field is obtained via the ansatz,

$$\mathcal{U} = e^{if(r)\hat{\mathbf{n}}(\hat{\mathbf{x}})\cdot\boldsymbol{\tau}}$$

- and the ensuing equation for $f(r)$ $f(0) = 2\pi\,,\quad f(\infty) = \pi$

$$\left(r^2 + 2B\sin^2 f\right) f'' + 2f'r +$$

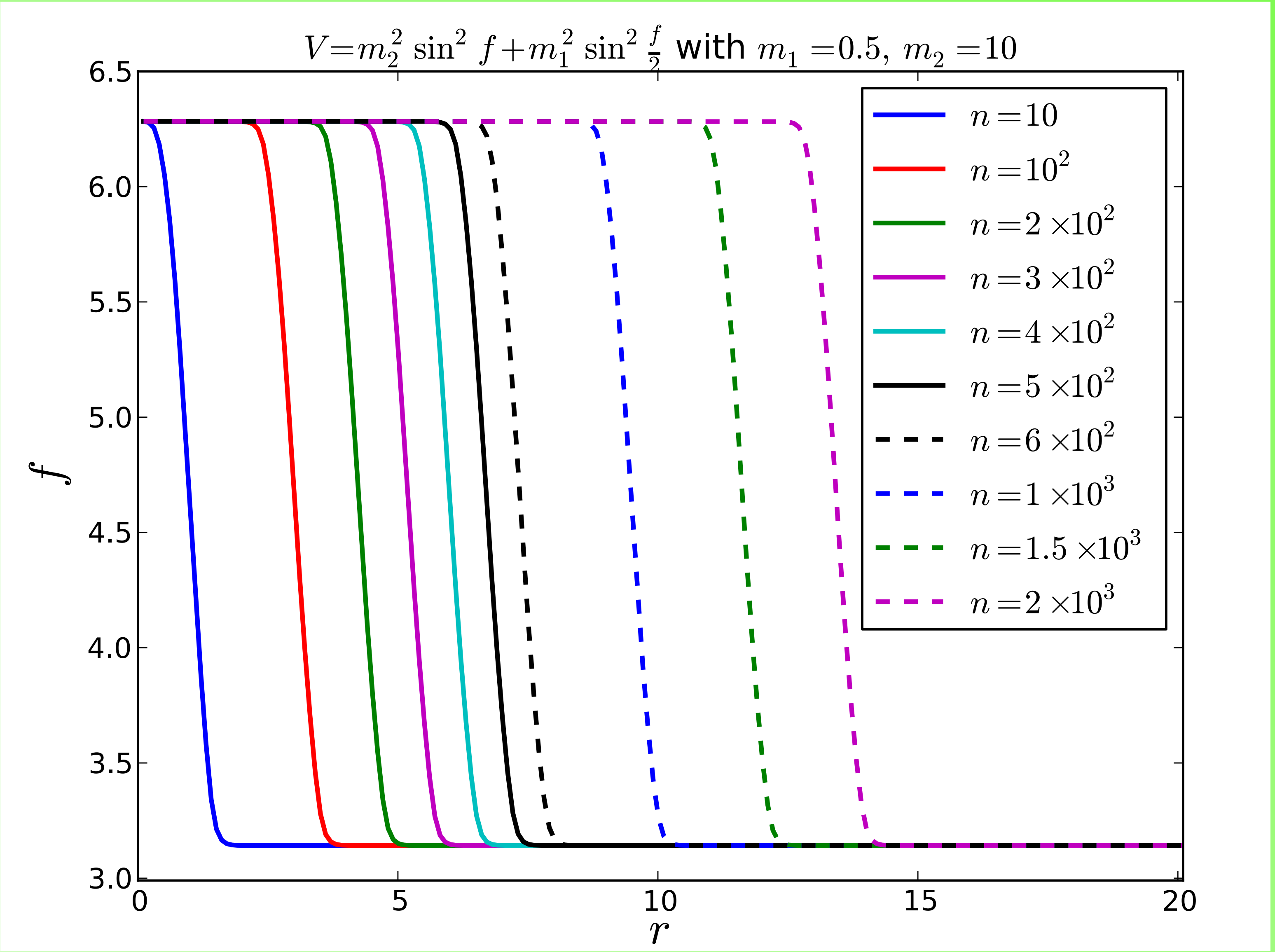
$$+\sin 2f\left(B\left(f'^2 - 1\right) - \frac{\mathcal{I}\sin^2 f}{r^2}\right) - \frac{r^2}{2}\frac{\partial V}{\partial f} = 0$$

- Here $\hat{\mathbf{n}}(\hat{\mathbf{x}})$ is a mapping from $S^2 \to S^2$ a so-called rational map and the winding number of the map corresponds to the baryon number of the Skyrmion, and $\mathcal{I}$ is an integral that only depends on the rational map and approximately $\mathcal{I} \approx 1.28B^2$.

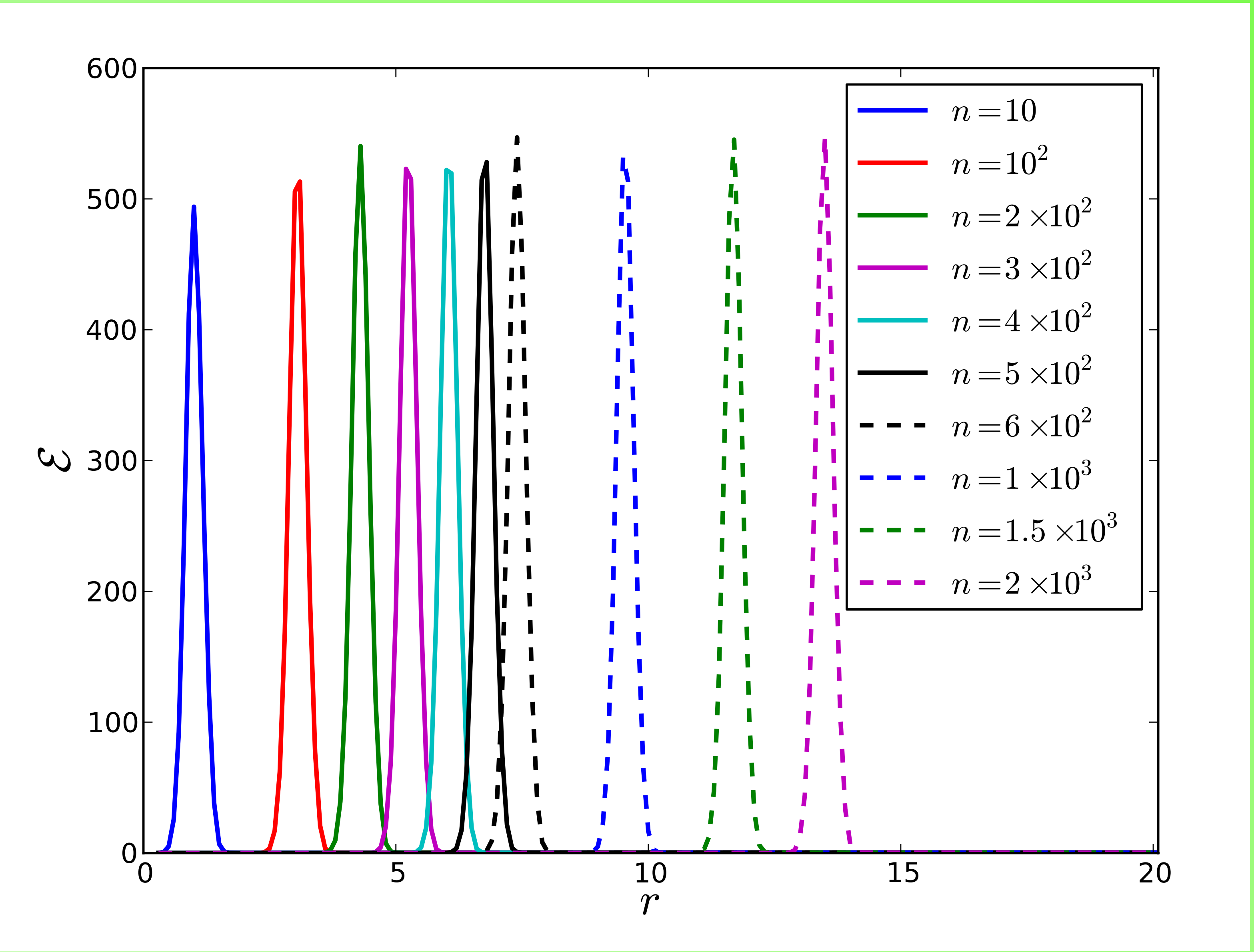
- Previous work has given the expectation of thin wall solitons. In this case, we expect the energy to behave like

$$E = \alpha R^2 + \frac{\beta}{R} - \frac{1}{9\pi}\epsilon R^3$$

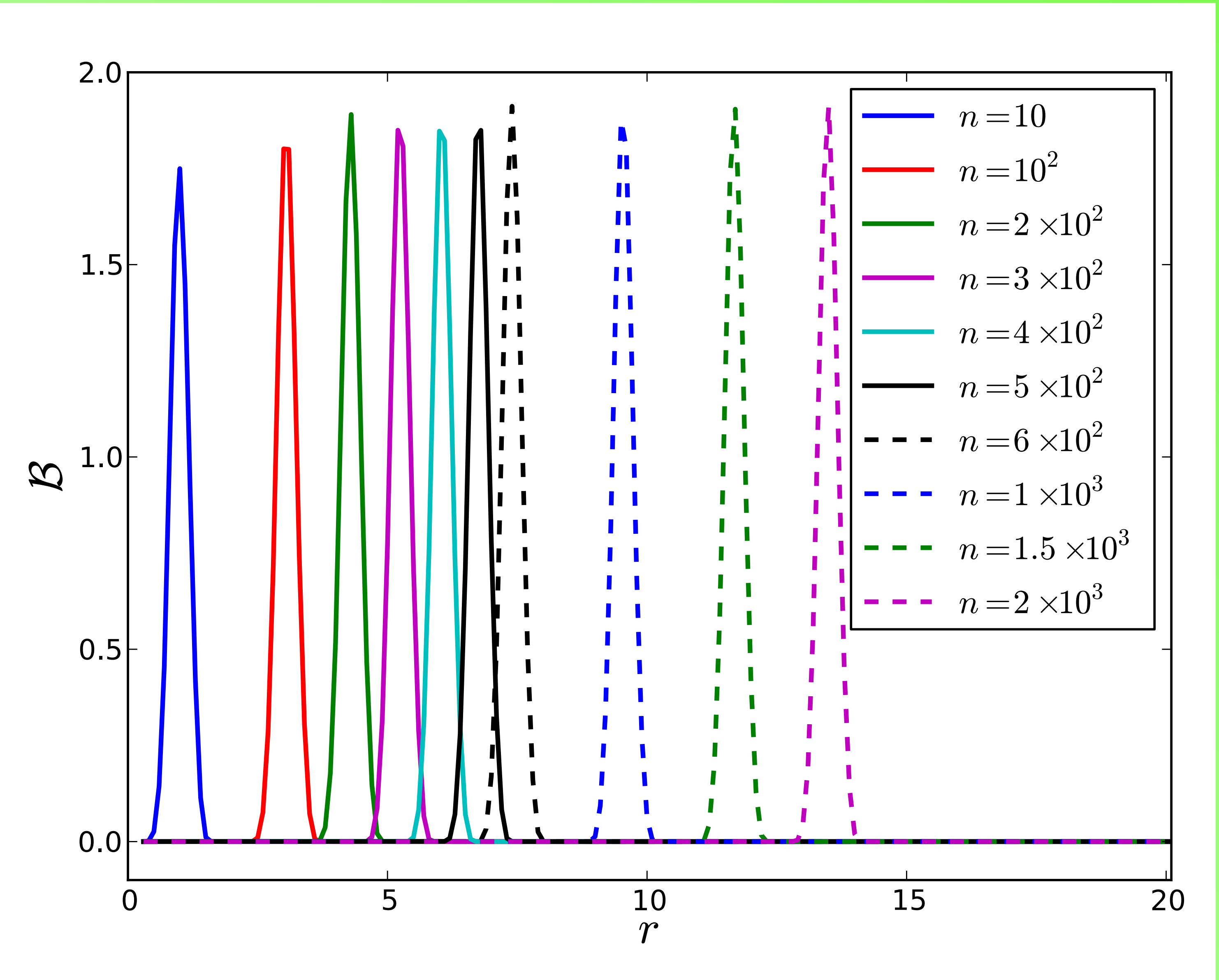
- We find numerically, thin wall solitons for large baryon number:



- The energy density is localized on the wall:



Baryon number density



Effective dynamics of the radius

- The energy of the thin wall Skyrmion separates into three contributions:

$$
\begin{aligned}
E(R) &= \int_0^{R-\Delta} dr\mathcal{E} + \int_{R-\Delta}^{R+\Delta} dr\mathcal{E} + \int_{R+\Delta}^{\infty} dr\mathcal{E} \\
&= E_{\rm int} + E_{\rm wall} + E_{\rm ext}
\end{aligned}
$$

- One can approximate them easily as

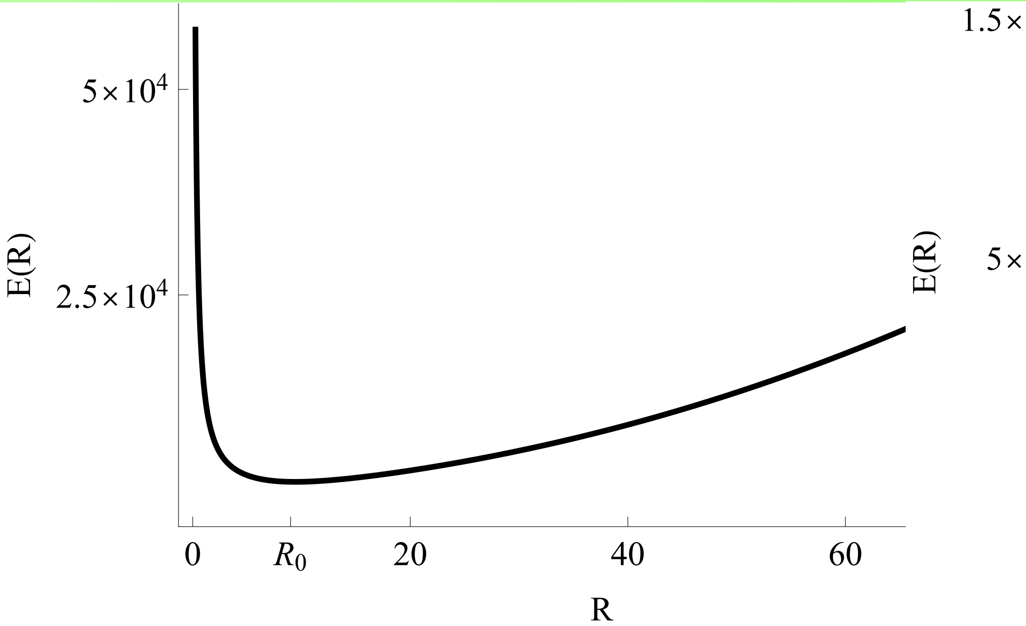
$$
E_{\rm int} = -\frac{1}{9\pi}(R-\Delta)^3\epsilon = -\frac{1}{9\pi}R^3\epsilon\left(1+\mathcal{O}\left(\frac{\Delta}{R}\right)\right)
$$

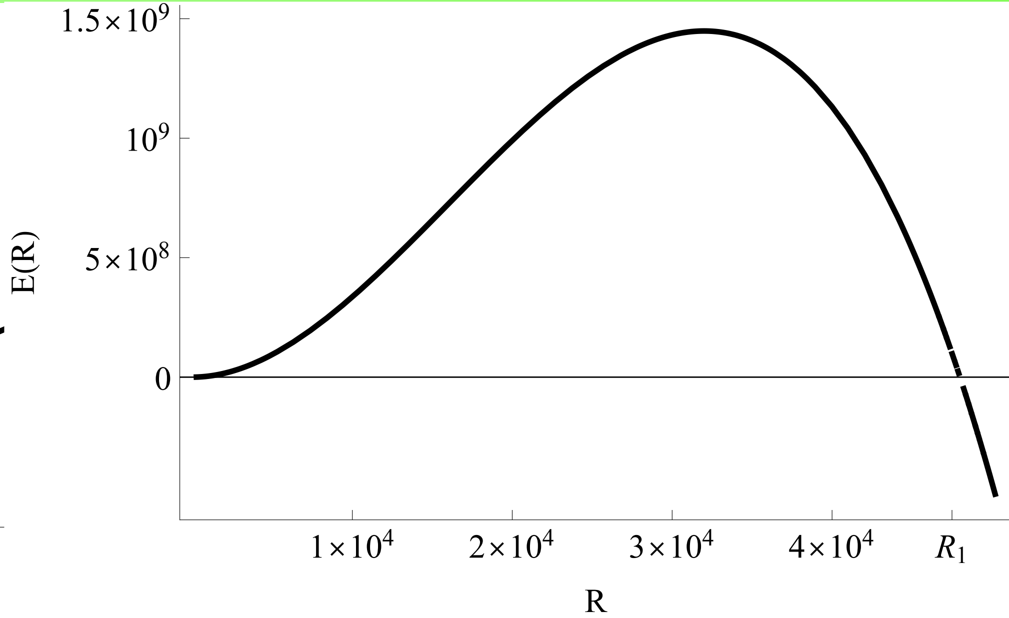
$$
\begin{aligned}
E_{\rm wall} &= \frac{2}{3\pi}\int_0^{\pi} df \sqrt{\left(2B\sin^2 f + \frac{\mathcal{I}\sin^4 f}{R^2} + R^2 V\right)} \\
&\quad \times\sqrt{(R^2+2B\sin^2 f)},
\end{aligned}
$$

$$
E_{\rm ext} = 0
$$

- This gives the form of the effective energy functional:

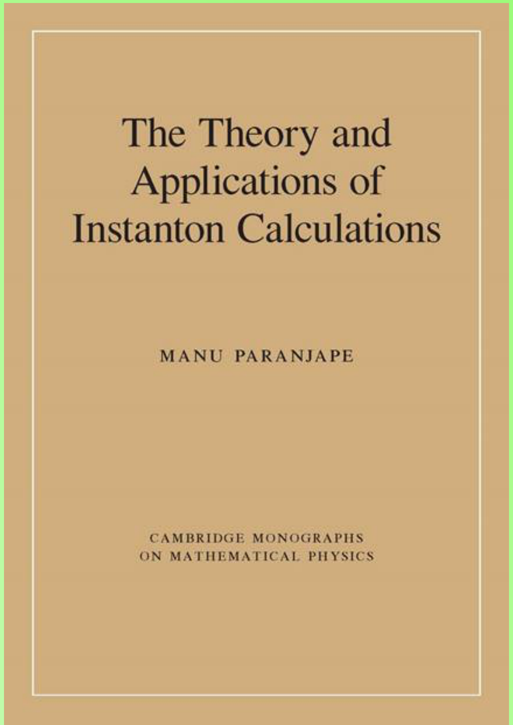
$$E = \alpha R^2 + \frac{\beta}{R} - \frac{1}{9\pi}\epsilon R^3$$





Tunnelling decay of the false Skyrmion

- The Skyrmion decays by its radius increasing by quantum tunnelling out through the barrier.
- We calculate the amplitude via the Euclidean path integral and the method of instantons.



- The euclidean action is given by just two terms, the exterior contribution still vanishes: $S^E_{\text{int}} = \int d\tau E_{\text{int}}$

$$S^E_{\text{wall}} \approx \frac{1}{3\pi}\int d\tau dr \left[R^2(\dot{f}^2 + f'^2)\left(1 + 2B\sin^2 f\right) + \left(2BR^2\sin^2 f + \mathcal{I}\frac{\sin^4 f}{R^2} + R^2 V(f)\right)\right]$$

- One easily finds, through some tricks and techniques that are now not new:

$$S^E_{\text{sky}} = \int d\tau L_E = \int_{R_0}^{R_1} \frac{dR}{\dot{R}}\left(\frac{E_{\text{wall}}}{\gamma^E} + E_{\text{int}}\right)$$

- with $\gamma^E = \left(1 + \dot{R}^2\right)^{-1/2}$

- Euclidean time translation invariance yields:

$$\gamma^E E_{\rm wall} + E_{\rm int} \equiv E_0$$

- ie. $$\gamma^E = -\frac{E_{\rm int} - E_0}{E_{\rm wall}}$$

- which then allows us to isolate:

$$\dot{R} = \sqrt{\left(\frac{E_{\rm wall}}{E_{\rm int} - E_0}\right)^2 - 1}$$

- Then the tunnelling exponent will be given by the difference:

$$\begin{aligned}\tilde{S}^E_{\rm sky} &\equiv S^E_{\rm sky}\Big|_{R(\tau)_{\rm instanton}} - S^E_{\rm sky}\Big|_{R_0} \\ &= \int_{R_0}^{R_1} \frac{dR}{\dot{R}} \left(\frac{E_{\rm wall}}{\gamma^E} + E_{\rm int} - E_0\right)\end{aligned}$$

- ie.

$$\tilde{S}^E_{\rm sky} = -\int_{R_0}^{R_1} dR\,(E_{\rm int} - E_0)\sqrt{\left(\frac{E_{\rm wall}}{E_{\rm int} - E_0}\right)^2 - 1}$$

- We can compute this analytically in the approximation that the tunnelling to radius is much larger than the static metastable radius and keeping only the leading powers of R. This gives:

$$E_{\rm int} - E_0 \;\approx\; E_{\rm int} = -\frac{1}{3\pi}\frac{\epsilon}{3}R^3$$

$$E_{\rm wall} \;\approx\; \frac{R^2}{3\pi}\int_0^{\pi} dr_p\,(f'^2 + V(f)) \equiv \frac{R^2}{3\pi}\sigma$$

- with

$$\sigma \;\equiv\; \int_0^{\pi} dr\,(f'^2 + V(f)) \approx 2\int_0^{\pi} dr\, f'^2$$

$$=\; 2\int_0^{\pi} df\,\sqrt{V(f)} = 4m_2 + \mathcal{O}\left(\frac{m_1^2}{m_2^2}\right)$$

- Then the action for the instanton is:

$$\tilde{S}^E_{\rm sky} \approx -\int_0^{R_1} dR\, E_{\rm int}\sqrt{\left(\frac{E_{\rm wall}}{E_{\rm int}}\right)^2 - 1} = \frac{\epsilon}{144}R_1^4$$

- We determine R_1 from $\dot{R}|_{R=R_1} = 0$ which gives

$$\dot{R} = \sqrt{\left(\frac{\sigma R^2}{(\epsilon/3)R^3} - 1\right)^2} = 0$$

- with solution $R_1 = 3\sigma/\epsilon$
- Then we find:

$$\tilde{S}^E_{\rm sky} = \frac{\epsilon}{144}R_1^4 = \frac{9\sigma^4}{16\epsilon^3} \equiv 144\frac{m_2^4}{m_1^6}$$

Tunnelling amplitude

- The instanton method gives:

$$\Gamma = A' L^{(\#\text{zero modes}-1)} \left(\frac{\tilde{S}^E}{2\pi}\right)^{(\#\text{zero modes})/2} e^{-\tilde{S}^E}$$

- Comparing then with the simple (Skyrmion less) false vacuum decay, we have:

$$\frac{\Gamma^{\text{vac}}}{\mathcal{N}\Gamma^{\text{sky}}} = \frac{V A'^{\text{vac}} \left(\frac{\tilde{S}^E_{\text{vac}}}{2\pi}\right)^{4/2} \exp\left(-\tilde{S}^E_{\text{vac}}\right)}{\mathcal{N} A'^{\text{sky}} \left(\frac{\tilde{S}^E_{\text{sky}}}{2\pi}\right)^{1/2} \exp\left(-\tilde{S}^E_{\text{sky}}\right)}$$

$$= \frac{\sqrt{2} A'^{\text{vac}}}{(\mathcal{N}/V) A'^{\text{sky}}} \left(\frac{\tilde{S}^E_{\text{vac}}}{2\pi}\right)^{3/2} \exp\left(-\frac{\tilde{S}^E_{\text{vac}}}{2}\right)$$

- using $\tilde{S}^E_{\text{sky}} = \tilde{S}^E_{\text{vac}}/2 = 48 m_2^4/m_1^6$

Conclusions

- The induced decay rate can be substantially higher than the homogeneous false vacuum decay rate.
- The thin wall limit can be analytically computed.
- Skyrmions appear in particle (nuclear) physics but also condensed matter physics. Concrete examples of vacuum decay due to Skyrmions should be experimentally accessible.