Few-Nucleon Systems in the Bethe-Salpeter approach: three-nucleon system

S. Bondarenko, V. Burov, S. Yurev

Joint Institute for Nuclear Research, Dubna, Russia

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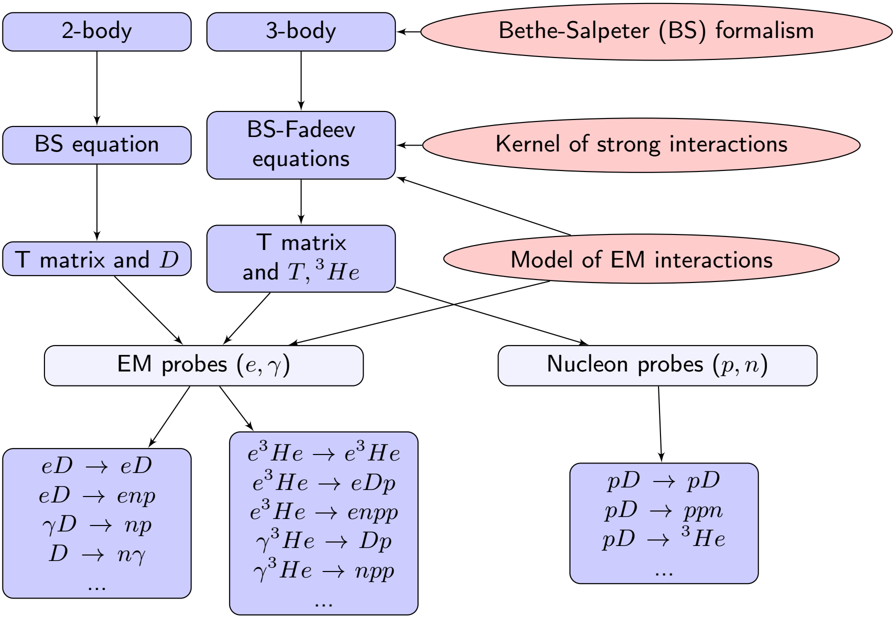
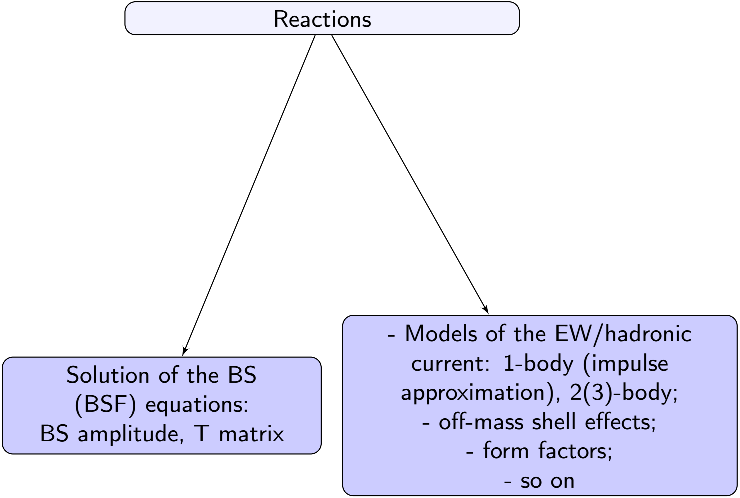
The Bethe-Salpeter approach is a powerful tool to investigate few-body compounds such as the deuteron, unbound neutron-proton (np) system, three-nucleon systems, elastic and nonelastic scattering. We have a great experience in working within such approach.


2-body
3-body
Bethe-Salpeter (BS) formalism
BS equation
BS-Fadeev equations
Kernel of strong interactions
T matrix and D
T matrix and $T, {}^3He$
Model of EM interactions
EM probes (e, γ)
Nucleon probes (p, n)
$eD \to eD$
$eD \to enp$
$\gamma D \to np$
$D \to n\gamma$
...
$e^3He \to e^3He$
$e^3He \to eDp$
$e^3He \to enpp$
$\gamma^3He \to Dp$
$\gamma^3He \to npp$
...
$pD \to pD$
$pD \to ppn$
$pD \to {}^3He$
...


Reactions
Solution of the BS (BSF) equations: BS amplitude, T matrix
- Models of the EW/hadronic current: 1-body (impulse approximation), 2(3)-body;
- off-mass shell effects;
- form factors;
- so on

Bethe-Salpeter equation for the nucleon-nucleon T matrix

$$T(p,p';P) = V(p,p';P) + \frac{i}{(2\pi)^4}\int d^4k\, V(p,k;P)\, G(k;P)\, T(k,p';P)$$

p', p - the relative four-momenta
P - the total four-momentum

$T(p,p';P)$ – two-nucleon t matrix
$V(p,p';P)$ – kernel of nucleon-nucleon interaction
$G(p;P)$ – free scalar two-particle propagator

$$G^{-1}(p;P) = \left[(P/2+p)^2 - m_N^2 + i\epsilon\right]\left[(P/2-p)^2 - m_N^2 + i\epsilon\right]$$

Partial-wave decomposition of the Bethe-Salpeter equation

In the center-mass system of two particles $P = (\sqrt{s}, \mathbf{0})$

$$T_{LL'}(p_0, |\mathbf{p}|, p_0', |\mathbf{p}'|; s) = V_{LL'}(p_0, |\mathbf{p}|, p_0', |\mathbf{p}'|; s) + \frac{i}{4\pi^3} \int dk_0 \, |\mathbf{k}|^2 \, d|\mathbf{k}|$$

$$\sum_{L''} V_{LL''}(p_0, |\mathbf{p}|, k_0, |\mathbf{k}|; s) \, G(k_0, |\mathbf{k}|; s) \, T_{L''L'}(k_0, |\mathbf{k}|, p_0', |\mathbf{p}'|; s)$$

Separable kernels of the NN interaction

The separable kernels of the nucleon-nucleon interaction are widely used in the calculations. The separable kernel as a *nonlocal* covariant interaction representing complex nature of the space-time continuum.

Separable rank-one *Ansatz* for the kernel

$$V_L(p_0', |\mathbf{p}'|; p_0, |\mathbf{p}|; s) = \lambda^{[L]}(s) g^{[L]}(p_0', |\mathbf{p}'|) g^{[L]}(p_0, |\mathbf{p}|)$$

Solution for the T matrix

$$T_L(p_0', |\mathbf{p}'|; p_0, |\mathbf{p}|; s) = \tau(s)\, g^{[L]}(p_0', |\mathbf{p}'|)\, g^{[L]}(p_0, |\mathbf{p}|)$$

with

$$\Big[\tau(s)\Big]^{-1} = \Big[\lambda^{[L]}(s)\Big]^{-1} + h(s),$$

$$h(s) = \sum_{coupled\, L} h_L(s) = -\frac{i}{4\pi^3} \int dk_0 \int |\mathbf{k}|^2\, d|\mathbf{k}| \sum_L [g^{[L]}(k_0, |\mathbf{k}|)]^2 S(k_0, |\mathbf{k}|; s)$$

$g^{[L]}$ - the model function, $\lambda^{[L'L]}(s)$ - a model parameter.

Separable kernel for Schrodinger equation with separable potential

Yoshio Yamaguchi "Two-Nucleon Problem When the Potential Is Nonlocal but Separable. I" Phys.Rev.95, 1628 (1954)
Yoshio Yamaguchi, Yoriko Yamaguchi "Two-Nucleon Problem When the Potential Is Nonlocal but Separable. II" Phys.Rev.95, 1635 (1954)

Nonlocal: $\langle \mathbf{r}|V|\mathbf{r}'\rangle \neq \delta^{(3)}(\mathbf{r}-\mathbf{r}')$
in configuration space

$$\langle \mathbf{r}|V|\mathbf{r}'\rangle = -(\lambda/m_N)v^*(\mathbf{r})v^*(\mathbf{r}')$$

in momentum space

$$\langle \mathbf{p}|V|\mathbf{p}'\rangle = (\lambda/m_N)g^*(\mathbf{p})g^*(\mathbf{p}')$$

for S-state: $g(p) = 1/(p^2+\beta^2)$
for D-state: $g(p) = p^2/(p^2+\beta^2)^2$
for the deuteron and scattering problem.

Separable nucleon-nucleon potential was widely uses for the two- and three-nucleon calculations in nonrelativistic nuclear physics

Willibald Plessas et al. Graz, Graz-II potentials, separable representation of the popular Bonn and Paris potentials

K. Schwarz, Willibald Plessas, L. Mathelitsch "Deuteron Form-factors And E D Polarization Observables For The Paris And Graz-II Potentials" Nuovo Cim. A76 (1983) 322-329.
J. Haidenbauer, Willibald Plessas "Separable Representation Of The Paris Nucleon Nucleon Potential" Phys.Rev. C30 (1984) 1822-1839.
Johann Haidenbauer, Y. Koike, Willibald Plessas "Separable representation of the Bonn nucleon-nucleon potential" Phys.Rev. C33 (1986) 439-446.

$$g(p) = \sum_n p^{2m}/(p^2 + \beta_n^2)^n,$$

m corresponds to angular momentum

Lippmann-Schwinger equation → Bethe-Salpeter equation

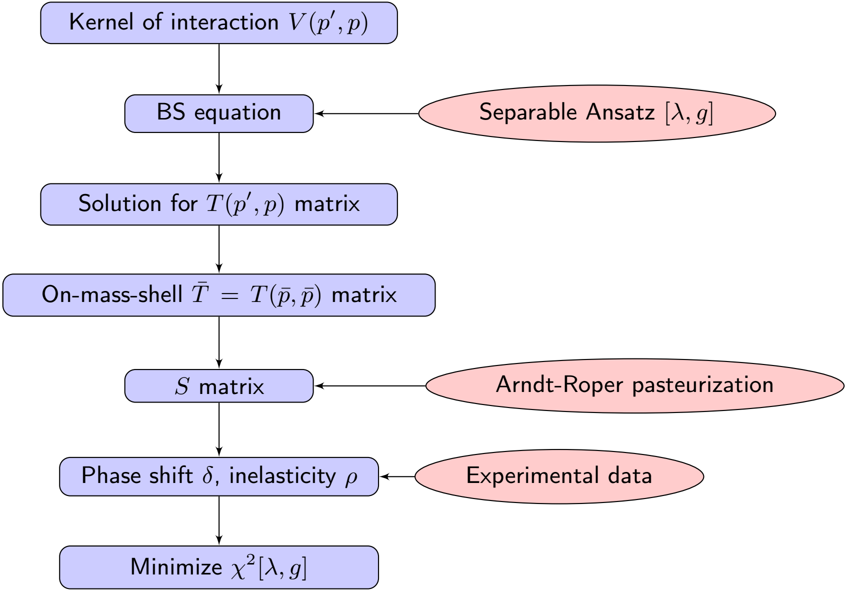
G. Rupp and J. A. Tjon "Relativistic contributions to the deuteron electromagnetic form factors" Phys. Rev. C41. 472 (1990)

$$\mathbf{p}^2 \to -p^2 = -p_0^2 + \mathbf{p}^2$$

$$g_p(p,P) = \frac{1}{-p^2+\beta^2} \xrightarrow{\text{c.m.}} \frac{1}{-p_0^2+\mathbf{p}^2+\beta^2+i\epsilon}$$

singularities: $p^0 = \pm\sqrt{\mathbf{p}^2+\beta^2} \mp i\epsilon$

This procedure works well for reactions with 2-body bound state but failed for unbound np-state


Kernel of interaction $V(p', p)$
BS equation
Separable Ansatz $[\lambda, g]$
Solution for $T(p', p)$ matrix
On-mass-shell $\bar{T} = T(\bar{p}, \bar{p})$ matrix
S matrix
Arndt-Roper pasteurization
Phase shift δ, inelasticity ρ
Experimental data
Minimize $\chi^2[\lambda, g]$

S matrix (Arndt-Roper parametrization)

$$S = \frac{1 - K_i + iK_r}{1 + K_i - iK_r} = \eta \exp(2i\delta)$$

$$K = K_r + iK_i$$

$$K_r = \tan\delta, \quad K_i = \tan^2\rho$$

δ - the phase shift, ρ - the inelasticity parameter.

$$\eta^2 = \frac{1 + K^2 - 2K_i}{1 + K^2 + 2K_i} = |S|^2 \sim \sigma_{np}$$

$$K^2 = K_r^2 + K_i^2$$

$$\delta = \frac{1}{2}\{\tan^{-1}[K_r/(1 - K_i)] + \tan^{-1}[K_r/(1 + K_i)]\}$$

If there are no inelastic channels: $(\rho = 0)$, $\delta = \delta_e$, $\eta = 1$ and $S = S_e = \exp(2i\delta_e)$.

Procedure $(J = 0 - 1)$

calculate the kernel parameters – $\lambda(s)$-matrix and parameter of the g-functions – to minimize the function χ^2:

$\chi^2 =$

$\sum_{i=1}^{n} (\delta^{\exp}(s_i) - \delta(s_i))^2/(\Delta\delta^{\exp}(s_i))^2$ – for all partial-wave states

$\sum_{i=1}^{n} (\rho^{\exp}(s_i) - \rho(s_i))^2/(\Delta\rho^{\exp}(s_i))^2$ – for all partial-wave states

$+(a_0^{\exp} - a_0)^2/(\Delta a_0^{\exp})^2$ – for the ${}^1S_0^+$ and ${}^3S_1^+$ partial-wave states

$+(E_d^{\exp} - E_d)^2/(\Delta E_d^{\exp})^2$ – for the ${}^3S_1^+$-${}^3D_1^+$ partial-wave states

$\{+ ...\}$

δ - the phase shifts, a_0, r_0 - the low-energy parameters (the scattering length, the effective range), E_d - the deuteron binding energy

Covariant generalization of the *Yamaguchi*-functions

functions for $g^{[L]}(p_0, p)$:

$$g^{[S]}(p_0, |\mathbf{p}|) = \frac{1}{p_0^2 - \mathbf{p}^2 - \beta_0^2 + i0}$$

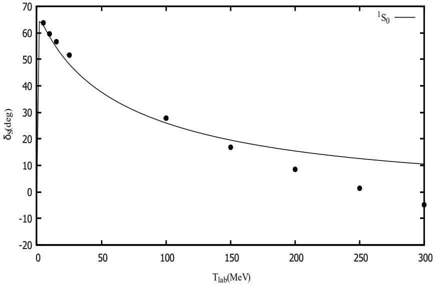
$$g^{[P]}(p_0, |\mathbf{p}|) = \frac{\sqrt{|-p_0^2 + \mathbf{p}^2|}}{(p_0^2 - \mathbf{p}^2 - \beta_1^2 + i0)^2}$$

$$g^{[D]}(p_0, |\mathbf{p}|) = \frac{C(p_0^2 - \mathbf{p}^2)}{(p_0^2 - \mathbf{p}^2 - \beta_2^2 + i0)^2}$$

Results for ${}^1S_0^+$ channel

Table: Parameters

	Exp.	${}^1S_0$
λ (GeV^4)		-1.12087
β_0 (GeV)		0.228302
a_L (fm)	-23.748	-23.753
r_L (fm)	2.75	2.75

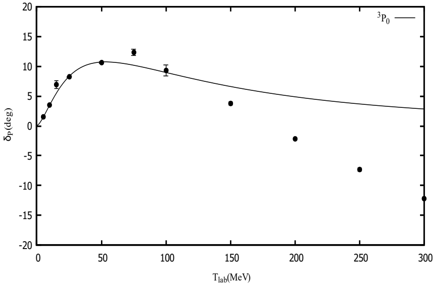


Phase shifts

Results for $^3P_0$, $^1P_1$ and $^3P_1$ channels

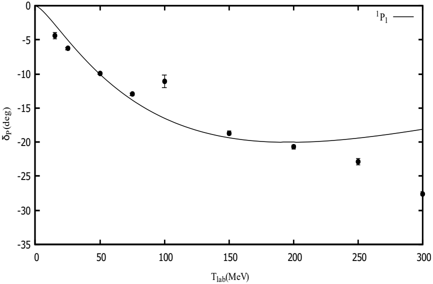
Table: Parameters

	$^3P_0$	$^3P_1$	$^1P_1$
λ (GeV^6)	0.0428572	-5.83051	-3.68029
β_1 (GeV)	0.19904	0.48273	0.44127

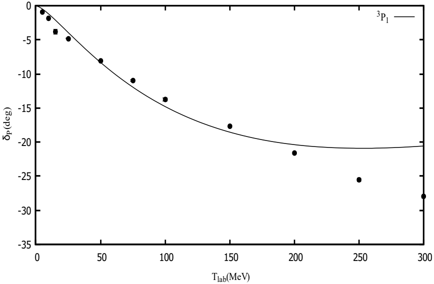


$^3P_0$ phase shifts

Results for $^3P_0$, $^1P_1$ and $^3P_1$ channels



$^1P_1$ phase shifts

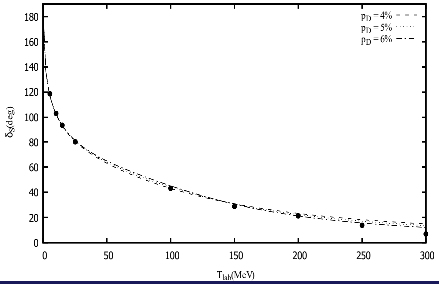


$^3P_1$ phase shifts

Results for ${}^3S_1^+ -{}^3D_1^+$ channels

Table: Parameters

	Exp.	${}^3S_1 -{}^3D_1$ ($p_d = 4\%$)	${}^3S_1 -{}^3D_1$ ($p_d = 5\%$)	${}^3S_1 -{}^3D_1$ ($p_d = 6\%$)
λ (GeV^4)		-1.83756	-1.57495	-1.34207
β_0 (GeV)		0.251248	0.246713	0.242291
C_2		1.71475	2.52745	3.46353
β_2 (GeV)		0.294096	0.324494	0.350217
a_L (fm)	5.424	5.454	5.454	5.453
r_L (fm)	1.756	1.81	1.81	1.80



The relativistic three-particle equation for T matrix

is considered in the Fadeev form with the following assumptions:

- no three-particles interaction $V_{123} = \sum_{i \neq j} V_{ij}$
- two-particles interaction is separable
- nucleon propagators are chosen in a scalar form
- the only strong interactions are considered (not EM), so $^3He \equiv T$

Bethe-Salpeter-Fadeev equation

$$\begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} - \begin{bmatrix} 0 & T_1G_1 & T_1G_1 \\ T_2G_2 & 0 & T_2G_2 \\ T_3G_3 & T_3G_3 & 0 \end{bmatrix} \begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix},$$

where full three-particles T matrix $T = \sum_i T^{(i)}$, G_i is the free two-particles (j and n) Green function (ijn is cyclic permutation of (1,2,3)):

$$G_i(k_j, k_n) = 1/(k_j^2 - m_N^2 + i\epsilon)/(k_n^2 - m_N^2 + i\epsilon),$$

and T_i is the two-particles T matrix which can be written as following

$$T_i(k_1, k_2, k_3; k_1', k_2', k_3') = (2\pi)^4 \delta^{(4)}(k_i - k_i') T_i(k_j, k_n; k_j', k_n').$$

with $s_i = (k_j + k_n)^2 = (k_j' + k_n')^2$.

Bethe-Salpeter-Fadeev equation

Introducing the equal-mass Jacobi momenta

$$p_i = \frac{1}{2}(k_j - k_n), \quad q_i = \frac{1}{3}K - k_i, \quad K = k_1 + k_2 + k_3.$$

one can separate the conserved total momentum

$$T^{(i)}(k_1, k_2, k_3; k_1', k_2', k_3') = (2\pi)^4 \delta^{(4)}(K - K') T^{(i)}(p_i, q_i; p_i', q_i'; s),$$

with $s = K^2$

Amplitude of three-particle state as a projection of T matrix to the bound state:

$$\Psi^{(i)}(p_i, q_i; s) = \langle p_i, q_i | T^{(i)} | M_B \rangle,$$

with $\sqrt{s} = M_B = 3m_N - E_t$.

Bethe-Salpeter-Fadeev equation

Orbital momentum of triton

$$\boldsymbol{L} = \boldsymbol{l} + \boldsymbol{\lambda}$$

$\boldsymbol{l}$ – orbital momentum of NN-pair
$\boldsymbol{\lambda}$ – orbital momentum of 3d particle
Using separable Ansatz for two-particles T matrix one-rank

$$\Psi_{LM}(p,q;s) = \sum_{a\lambda} \Psi^{(a)}_{\lambda L}(p_0,|\mathbf{p}|,q_0,|\mathbf{q}|;s)\mathcal{Y}^{(a)}_{\lambda LM}(\hat{\mathbf{p}},\hat{\mathbf{q}})$$

$$\mathcal{Y}^{(a)}_{\lambda LM}(\hat{\mathbf{p}},\hat{\mathbf{q}}) = \sum_{m\mu} C^{LM}_{lm\lambda\mu} Y_{lm}(\hat{\mathbf{p}}) Y_{\lambda\mu}(\hat{\mathbf{q}}),$$

where $a \equiv {}^{2s+1}l_j$ is two-nucleon states of the NN-pair

Partial-wave three-nucleon functions

$$\Psi^{(a)}_{\lambda L}(p_0,|\mathbf{p}|,q_0,|\mathbf{q}|;s) = g^{(a)}(p_0,|\mathbf{p}|)\tau^{(a)}[(\frac{2}{3}\sqrt{s}+q_0)^2-\mathbf{q}^2]\,\Phi^{(a)}_{\lambda L}(q_0,|\mathbf{q}|;s)$$

System of the integral equations

$$\Phi^{(a)}_{\lambda L}(q_0,|\mathbf{q}|;s) = \frac{i}{4\pi^3}\sum_{a'\lambda'}\int_{-\infty}^{\infty}dq_0'\int_0^{\infty}\mathbf{q'}^2 d|\mathbf{q}'|\,Z^{(aa')}_{\lambda\lambda'}(q_0,q;q_0',|\mathbf{q}'|;s)$$

$$\frac{\tau^{(a')}[(\frac{2}{3}\sqrt{s}+q_0')^2-\mathbf{q'}^2]}{(\frac{1}{3}\sqrt{s}-q_0')^2-\mathbf{q'}^2-m^2+i\epsilon}\Phi^{(a')}_{\lambda' L}(q_0',|\mathbf{q}'|;s)$$

with effective kernels of equation

$$Z^{(aa')}_{\lambda\lambda'}(q_0,|\mathbf{q}|;q_0',|\mathbf{q}'|;s) = C_{(aa')}\int d\cos\vartheta_{\mathbf{q}\mathbf{q}'}\,K^{(aa')}_{\lambda\lambda' L}(|\mathbf{q}|,|\mathbf{q}'|,\cos\vartheta_{\mathbf{q}\mathbf{q}'})$$

$$\frac{g^{(a)}(-q_0/2-q_0',|\mathbf{q}/2+\mathbf{q}'|)g^{(a')}(q_0+q_0'/2,|\mathbf{q}+\mathbf{q}'/2|)}{(\frac{1}{3}\sqrt{s}+q_0+q_0')^2-(\mathbf{q}+\mathbf{q}')^2-m_N^2+i\epsilon}$$

Angular functions in general case

$$K^{(aa')}_{\lambda\lambda' L}(|\mathbf{q}|, |\mathbf{q}'|, \cos\vartheta_{\mathbf{q}\mathbf{q}'}) = (4\pi)^{3/2} \frac{\sqrt{2\lambda+1}}{2L+1} (-1)^{l'}$$

$$\sum_{mm'} C^{Lm}_{lm\lambda 0} C^{Lm}_{l'm'\lambda' m-m'} Y^*_{lm}(\vartheta, 0) Y_{l'm'}(\vartheta', 0) Y_{\lambda' m-m'}(\vartheta_{\mathbf{q}\mathbf{q}'}, 0)$$

where

$$\cos\vartheta = (\frac{|\mathbf{q}|}{2} + |\mathbf{q}'| \cos\vartheta_{\mathbf{q}\mathbf{q}'})/|\frac{\mathbf{q}}{2} + \mathbf{q}'|$$

$$\cos\vartheta' = (|\mathbf{q}| + \frac{|\mathbf{q}'|}{2} \cos\vartheta_{\mathbf{q}\mathbf{q}'})/|\mathbf{q} + \frac{\mathbf{q}'}{2}|$$

Consider the ground state of triton: $L = 0 \to \lambda = l$

Angular functions

$$K^{(aa')}_{ll'0} = (4\pi)^{3/2}(-1)^{l+l'} Y^{*}_{l0}(\vartheta, 0) A_{l'}(\vartheta', \vartheta_{\mathbf{qq'}})$$

$$A_{l'}(\vartheta', \vartheta_{\mathbf{qq'}}) = \sum_{m'} C^{00}_{l'm'l'-m'} Y_{l'm'}(\vartheta', 0) Y_{l'-m'}(\vartheta_{\mathbf{qq'}}, 0)$$

Spin-isospin dependence

$[(a) = {}^1S_0, {}^3S_1, {}^3D_1, {}^3P_0, {}^1P_1, {}^3P_1]$

$$C_{(aa')} = \frac{1}{4} \begin{pmatrix} 1 & -3 & -3 & \sqrt{3} & -\sqrt{3} & \sqrt{3} \\ -3 & 1 & 1 & \sqrt{3} & -\sqrt{3} & \sqrt{3} \\ -3 & 1 & 1 & \sqrt{3} & -\sqrt{3} & \sqrt{3} \\ \sqrt{3} & \sqrt{3} & \sqrt{3} & -1 & -3 & -1 \\ -\sqrt{3} & -\sqrt{3} & -\sqrt{3} & -3 & -1 & -3 \\ \sqrt{3} & \sqrt{3} & \sqrt{3} & -1 & -3 & -1 \end{pmatrix}$$

Singularities

Poles from one-particle propagator

$$q^{0\prime}_{1,2} = \frac{1}{3}\sqrt{s} \mp [E_{|\mathbf{q}'|} - i\epsilon]$$

Poles from propagator in Z-function

$$q^{0\prime}_{3,4} = -\frac{1}{3}\sqrt{s} - q^0 \pm [E_{|\mathbf{q}'+\mathbf{q}|} - i\epsilon]$$

Poles from Yamaguchi-functions

$$q^{0\prime}_{5,6} = -2q^0 \pm 2[E_{|\frac{1}{2}\mathbf{q}'+\mathbf{q}|,\beta} - i\epsilon]$$

and

$$q^{0\prime}_{7,8} = -\frac{1}{2}q^0 \pm \frac{1}{2}[E_{|\mathbf{q}'+\frac{1}{2}\mathbf{q}|,\beta} - i\epsilon]$$

Cuts from two-particle propagator τ

$$q^{0\prime}_{9,10} = \pm\sqrt{q'^2 + 4m^2} - \frac{2}{3}\sqrt{s} \qquad \text{and} \qquad \pm\infty$$

Poles from two-particle propagator τ

$$q^{0\prime}_{11,12} = \pm\sqrt{q'^2 + 4M_d^2} - \frac{2}{3}\sqrt{s}$$

Method of solution

To obtain the triton binding energy the iteration method is used:

$$\lim_{n\to\infty} \frac{\Phi_n(s)}{\Phi_{n-1}(s)}\Big|_{s=M_B^2} = 1$$

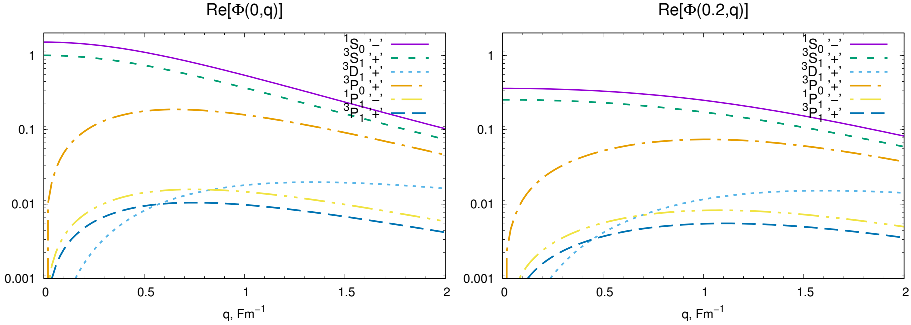
The Gaussian quadrature with $N_1 \times N_2[q_4 \times |\mathbf{q}|]$

$$q_4 = (1+x)/(1-x)$$

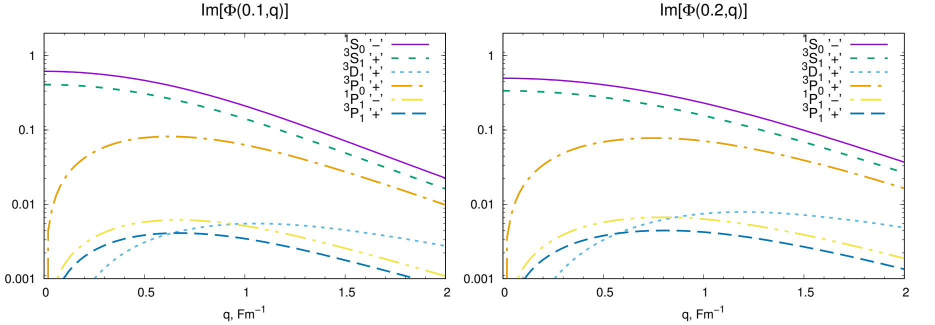
$$|\mathbf{q}| = (1+y)/(1-y)$$

The convergence was investigated and $N_1 = 96$, $N_2 = 15$ was used in calculations

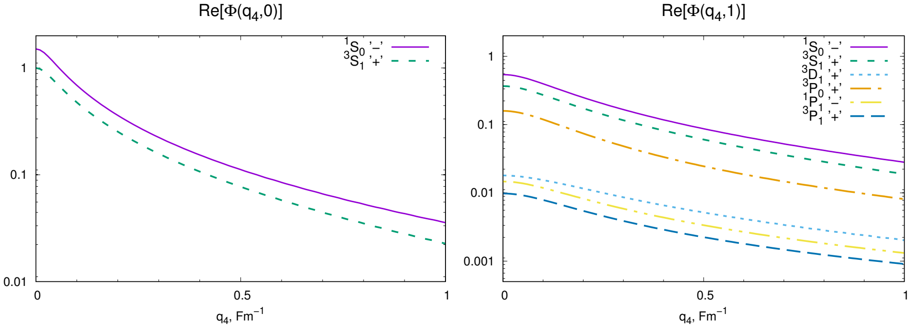
Real part at $q_4 = 0$ Fm^{-1} and $q_4 = 0.2$ Fm^{-1}



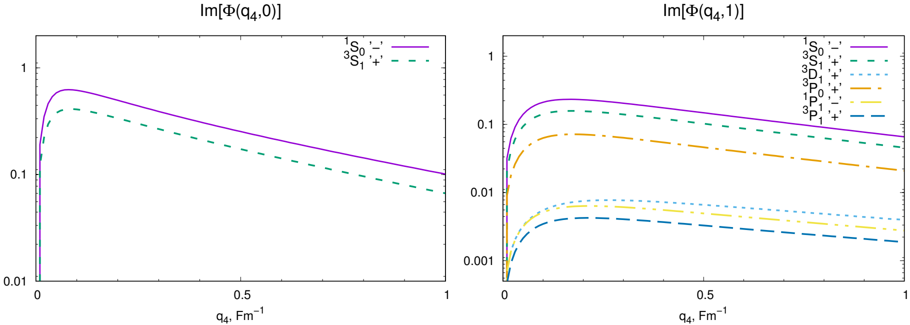
Imaginary part at $q_4 = 0.1$ Fm^{-1} and $q_4 = 0.2$ Fm^{-1}



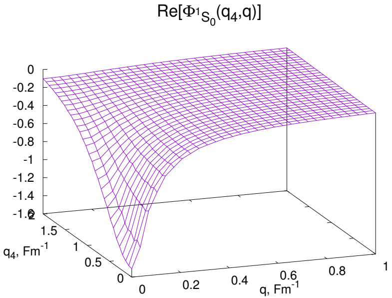
Real part at $q = 0$ Fm^{-1} and $q = 1$ Fm^{-1}

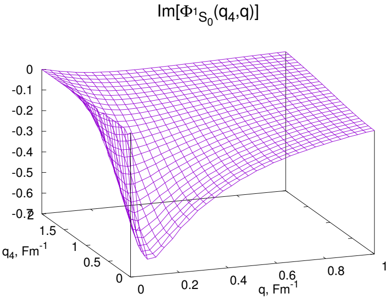


Imaginary part at $q = 0$ Fm^{-1} and $q = 1$ Fm^{-1}

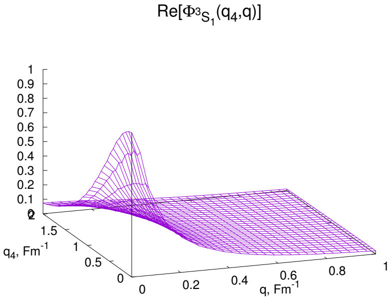


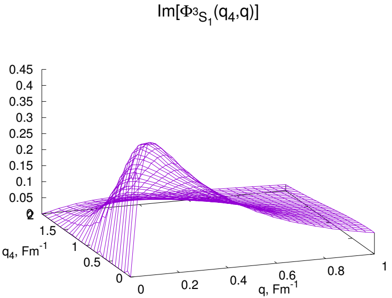
Real and Imaginary part of $\Phi_{^1S_0}(q_4, q)$



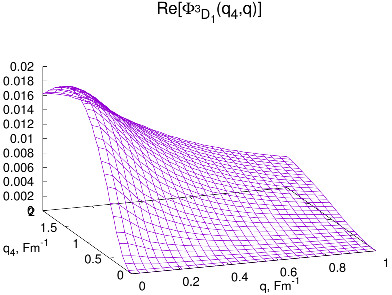


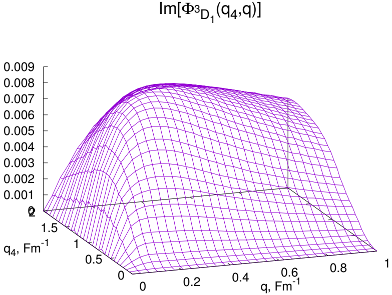
Real and Imaginary part of $\Phi_{^3S_1}(q_4, q)$



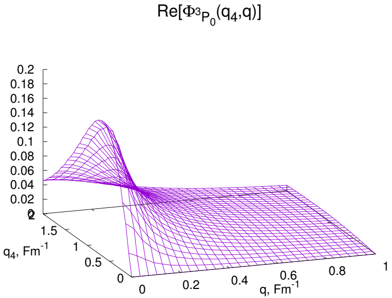


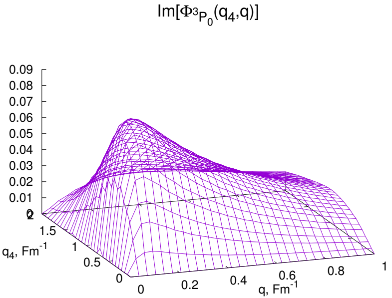
Real and Imaginary part of $\Phi_{^3D_1}(q_4, q)$



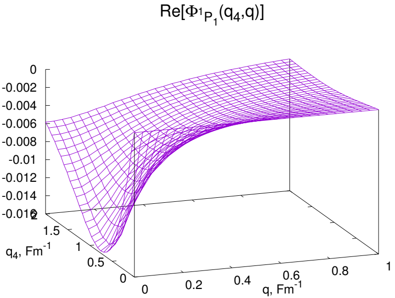


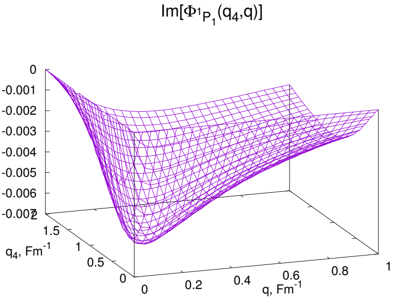
Real and Imaginary part of $\Phi_{^3P_0}(q_4, q)$



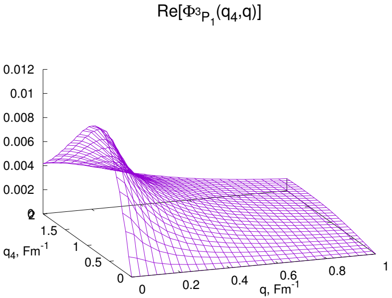


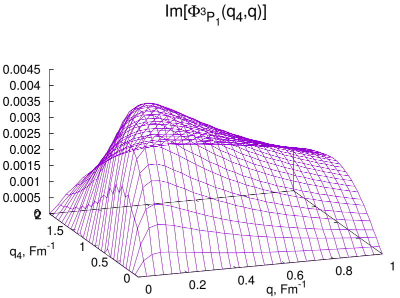
Real and Imaginary part of $\Phi_{^1P_1}(q_4, q)$





Real and Imaginary part of $\Phi_{^3P_1}(q_4, q)$





Triton binding energy (MeV)

p_D	${}^1S_0 - {}^3S_1$	${}^3D_1$	${}^3P_0$	${}^1P_1$	${}^3P_1$
4	9.221	9.294	9.314	9.287	9.271
5	8.819	8.909	8.928	8.903	8.889
6	8.442	8.545	8.562	8.540	8.527
		Exp.	8.48		

- the main contribution is from S-states
- the D-state contribution is about 0.8 – 1.2 % depending on D-wave (pseudo)probability in deuteron
- the P-state contributions are alternating and give about -0.2%

Summary

The BS approach

- is a full covariant relativistic description of few-body systems (two and three);
- allows to describe the properties of the deuteron, NN phases and inelasticities with separable potential
- gives very reasonable explanation of structure functions, form factors and tensor polarization of deuteron in the elastic eD-scattering
- gives possibility to investigate electro- and photodisintegration of the deuteron
- allows investigate behavior of the ratios of structure functions of the light nuclei to the structure functions of the free nucleon
- gives possibility to extend formalism to three-body systems and study reactions with them (^3He EM form factors)