Transverse momentum distributions of hadrons in heavy-ion and *pp* collisions with the Tsallis statistics

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The Tsallis Statistics

Boltzmann-Gibbs Statistics	**Tsallis-1 Statistics**	**Tsallis-2 Statistics**	**Tsallis-3 Statistics**
$q=1$	$0<q<\infty$	$0<q<\infty$	$0<q<\infty$
$S=-\sum_i p_i \ln p_i$	$S=-\sum_i \frac{p_i - p_i^q}{1-q}$	$S=-\sum_i \frac{p_i - p_i^q}{1-q}$	$S=-\sum_i \frac{p_i - p_i^q}{1-q}$
$Tr(\hat{\rho})=\sum_i p_i = 1$	$Tr(\hat{\rho})=\sum_i p_i = 1$	$Tr(\hat{\rho})=\sum_i p_i = 1$	$Tr(\hat{\rho})=\sum_i p_i = 1$
$\langle A\rangle = Tr(\hat{\rho}\hat{A})$	$\langle A\rangle = Tr(\hat{\rho}\hat{A})$	$\langle A\rangle = Tr(\hat{\rho}^q\hat{A})$	$\langle A\rangle = \frac{Tr(\hat{\rho}^q\hat{A})}{Tr(\hat{\rho}^q)}$
$=\sum_i p_i A_i$	$=\sum_i p_i A_i$	$=\sum_i p_i^q A_i$	$=\frac{\sum_i p_i^q A_i}{\sum_i p_i^q}$

Standard expectation values

Generalized expectation values

p_i – probability of *i*-th microstate of system

- The expectation values of the Tsallis-2 statistics are non-normalized

C. Tsallis, J. Stat. Phys. 52 (1988) 479

C. Tsallis, R.S. Mendes, A.R. Plastino, Physica A 261 (1998) 534

Equilibrium Statistical Mechanics

- **Fundamental statistical ensemble** $(x^1,\dots,x^n)$

$$E = E(x^1,\dots,x^n), \qquad x^1 = S,\ x^2 = V,\ x^3 = N,\ \dots$$

$$dE = \sum_{k=1}^{n} u^k dx^k,$$

$$u^k = \frac{\partial E}{\partial x^k}, \qquad u^1 = T,\ u^2 = -p,\ u^3 = \mu,\ \dots$$

- **Statistical ensemble** $(u^1,\dots,u^m,x^{m+1},\dots,x^n)$

  - Legendre transform for the thermodynamic potential:

$$Y = Y(u^1,\dots,u^m,x^{m+1},\dots,x^n) = E - \sum_{k=1}^{m} u^k x^k$$

  - Statistical averages for the fluctuating quantities:

$$x^k = \sum_i p_i x_i^k, \qquad x_i^1 = S_i(p_i),\ x_i^2 = V_i,\ x_i^3 = N_i,\ \dots$$

$$E = \sum_i p_i E_i,$$

$$Y = \sum_i p_i Y_i, \qquad Y_i = E_i - \sum_{k=1}^{m} u^k x_i^k$$

A.S.P., In Recent Advances in Thermo and Fluid Dynamics, ed. Mod Gorji-Bandpy, InTech, Chapter 11, 2015, pp.303-331

Probability Distribution of Microstates

- Principle of the maximum entropy from the second law of thermodynamics (Constrained local extrema of the thermodynamic potential in the method of Lagrange multipliers)

$$\Phi = Y - \lambda\varphi, \qquad \varphi = \sum_i p_i - 1 = 0, \qquad \frac{\partial \Phi}{\partial p_i} = 0$$

- Probabilities of microstates and norm function:

$$p_i = F\left(\frac{1}{u^1}\left(\Lambda - E_i + \sum_{k=2}^{m} u^k x_i^k\right)\right)$$

$$1 = \sum_i F\left(\frac{1}{u^1}\left(\Lambda - E_i + \sum_{k=2}^{m} u^k x_i^k\right)\right) \qquad \rightarrow \qquad \Lambda = \Lambda(u^1, \ldots, u^m, x^{m+1}, \ldots, x^n)$$

- Thermodynamic quantities are partial derivatives of thermodynamic potential :

$$x^k = -\frac{\partial Y}{\partial u^k} = \sum_i p_i x_i^k \qquad (k = 1, \ldots, m)$$

$$u^k = \frac{\partial Y}{\partial x^k} = \sum_i p_i \frac{\partial E_i}{\partial x^k} \qquad (k = m+1, \ldots, n)$$

A.S.P., In Recent Advances in Thermo and Fluid Dynamics, ed. Mod Gorji-Bandpy, InTech, Chapter 11, 2015, pp.303-331

Microcanonical Ensemble *(E,V,N)*

$$\Phi = S - \lambda\varphi, \qquad \varphi = \sum_i p_i - 1 = 0, \qquad \frac{\partial \Phi}{\partial p_i} = 0 \qquad [Y = S]$$

Boltzmann-Gibbs Statistics

$q = 1$

Tsallis-1 Statistics

$0 < q < \infty$

Tsallis-2 Statistics

$0 < q < \infty$

Boltzmann-Gibbs	Tsallis-1	Tsallis-2
$Y = S = -\sum_i p_i \ln p_i$	$Y = S = -\sum_i \frac{p_i - p_i^q}{1-q}$	$Y = S = -\sum_i \frac{p_i - p_i^q}{1-q}$
$p_i = \frac{1}{W}$	$p_i = \frac{1}{W}$	$p_i = \frac{1}{W}$
$W = \sum_i \delta_{E_i,E}\delta_{N_i,N}\delta_{V_i,V}$	$W = \sum_i \delta_{E_i,E}\delta_{N_i,N}\delta_{V_i,V}$	$W = \sum_i \delta_{E_i,E}\delta_{N_i,N}\delta_{V_i,V}$

C. Tsallis, J. Stat. Phys. 52 (1988) 479

C. Tsallis, R.S. Mendes, A.R. Plastino, Physica A 261 (1998) 534

A.S.P., Phys.Lett. A 350 (2006) 331

Canonical Ensemble *(T,V,N)*

$$\Phi = F - \lambda\varphi, \qquad \varphi = \sum_i p_i - 1 = 0, \qquad \frac{\partial\Phi}{\partial p_i} = 0 \qquad [Y = F = E - TS]$$

Boltzmann-Gibbs Statistics $q = 1$	**Tsallis-1 Statistics** $0 < q < \infty$	**Tsallis-2 Statistics** $0 < q < \infty$
$F = T\sum_i p_i\left[\ln p_i + \frac{E_i}{T}\right]$	$F = T\sum_i p_i\left[\frac{1-p_i^{q-1}}{1-q} + \frac{E_i}{T}\right]$	$F = T\sum_i p_i^q\left[\frac{p_i^{1-q}-1}{1-q} + \frac{E_i}{T}\right]$
$p_i = \frac{1}{Z}\exp\left(-\frac{E_i}{T}\right)$	$p_i = \left[1 + \frac{q-1}{q}\frac{\Lambda - E_i}{T}\right]^{\frac{1}{q-1}}$	$p_i = \frac{1}{Z}\left[1-(1-q)\frac{E_i}{T}\right]^{\frac{1}{1-q}}$
$Z = \sum_i \exp\left(-\frac{E_i}{T}\right)$	$1 = \sum_i\left[1 + \frac{q-1}{q}\frac{\Lambda - E_i}{T}\right]^{\frac{1}{q-1}}$	$Z = \sum_i\left[1-(1-q)\frac{E_i}{T}\right]^{\frac{1}{1-q}}$
$\langle A\rangle = \sum_i p_i A_i$	$\langle A\rangle = \sum_i p_i A_i$	$\langle A\rangle = \sum_i p_i^q A_i$

C. Tsallis, J. Stat. Phys. 52 (1988) 479
C. Tsallis, R.S. Mendes, A.R. Plastino, Physica A 261 (1998) 534
A.S.P., Phys.Lett. A 360 (2006) 26

- The expectation values of the Tsallis-2 statistics are non-normalized

Grand Canonical Ensemble *(T,V,μ)*

$$\Phi = \Omega - \lambda\varphi, \qquad \varphi = \sum_i p_i - 1 = 0, \qquad \frac{\partial \Phi}{\partial p_i} = 0 \qquad \left[Y = \Omega = E - TS - \mu N\right]$$

Boltzmann-Gibbs Statistics

$q = 1$

$$\Omega = T\sum_i p_i \left[\ln p_i + \frac{E_i - \mu N_i}{T}\right]$$

$$p_i = \frac{1}{Z}\exp\left(-\frac{E_i - \mu N_i}{T}\right)$$

$$Z = \sum_i \exp\left(-\frac{E_i - \mu N_i}{T}\right)$$

$$\langle A \rangle = \sum_i p_i A_i$$

Tsallis-1 Statistics

$0 < q < \infty$

$$\Omega = T\sum_i p_i \left[\frac{1 - p_i^{q-1}}{1-q} + \frac{E_i - \mu N_i}{T}\right]$$

$$p_i = \left[1 + \frac{q-1}{q}\frac{\Lambda - E_i + \mu N_i}{T}\right]^{\frac{1}{q-1}}$$

$$1 = \sum_i \left[1 + \frac{q-1}{q}\frac{\Lambda - E_i + \mu N_i}{T}\right]^{\frac{1}{q-1}}$$

$$\langle A \rangle = \sum_i p_i A_i$$

Tsallis-2 Statistics

$0 < q < \infty$

$$\Omega = T\sum_i p_i^q \left[\frac{p_i^{1-q} - 1}{1-q} + \frac{E_i - \mu N_i}{T}\right]$$

$$p_i = \frac{1}{Z}\left[1 - (1-q)\frac{E_i - \mu N_i}{T}\right]^{\frac{1}{1-q}}$$

$$Z = \sum_i \left[1 - (1-q)\frac{E_i - \mu N_i}{T}\right]^{\frac{1}{1-q}}$$

$$\langle A \rangle = \sum_i p_i^q A_i$$

A.S.P., Eur. Phys. J. A 51 (2015) 108; Eur. Phys. J. A 53 (2017) 53

- The expectation values of the Tsallis-2 statistics are non-normalized

Transverse Momentum Distribution: Tsallis-2 Statistics

- **Ultrarelativistic Maxwell-Boltzmann Ideal Gas in the Grand Canonical Ensemble** $(m=0)$

$$(\text{Case} \;\; q>1)$$

$$\frac{d^2N}{dp_T dy} = \frac{V}{(2\pi)^3} \sum_\sigma \int_0^{2\pi} d\varphi\, p_T \varepsilon_{\vec{p}} \langle n_{\vec{p}\sigma}\rangle, \qquad \varepsilon_{\vec{p}} = p_T \cosh y \quad \text{for} \;\; m=0$$

$$\langle n_{\vec{p}\sigma}\rangle = \frac{1}{Z^q} \sum_{\{n_{\vec{p}\sigma}\}} n_{\vec{p}\sigma} \frac{1}{\prod_{\vec{p}\sigma} n_{\vec{p}\sigma}!} \left[1-(1-q)\frac{\sum_{\vec{p}\sigma} n_{\vec{p}\sigma}(\varepsilon_{\vec{p}}-\mu)}{T}\right]^{\frac{q}{1-q}} = \frac{1}{Z^q} \sum_{N=0}^{N_0} \frac{\tilde{\omega}^N}{N!} a_0(1) \left[1+(q-1)\frac{\varepsilon_{\vec{p}}-\mu(N+1)}{T}\right]^{\frac{q}{1-q}+3N}$$

$$Z = \sum_{\{n_{\vec{p}\sigma}\}} \frac{1}{\prod_{\vec{p}\sigma} n_{\vec{p}\sigma}!} \left[1-(1-q)\frac{\sum_{\vec{p}\sigma} n_{\vec{p}\sigma}(\varepsilon_{\vec{p}}-\mu)}{T}\right]^{\frac{1}{1-q}} = \sum_{N=0}^{N_0} \frac{\tilde{\omega}^N}{N!} a_0(0) \left[1+(1-q)\frac{\mu N}{T}\right]^{\frac{1}{1-q}+3N}$$

$$a_\eta(\xi) = \frac{\Gamma\left(\frac{1}{q-1}+\xi-3(N+\eta)\right)}{(q-1)^{3(N+\eta)}\,\Gamma\left(\frac{1}{q-1}+\xi\right)}, \qquad \tilde{\omega} = \frac{gVT^3}{\pi^2}$$

- The expectation values of the Tsallis-2 statistics are non-normalized
- N_0 is a cut-off parameter
- Terms with $N > N_0$ increase and became divergent

A.S.P., Eur. Phys. J. A 53 (2017) 53

Transverse Momentum Distribution: Tsallis-2 Statistics

➢ **Ultrarelativistic Maxwell-Boltzmann Ideal Gas in the Grand Canonical Ensemble** $(m=0)$

$$\left(\text{Case}\quad q>1\right)$$

✓ **Exact Solutions:** ***A.S.P., Eur. Phys. J. A 53 (2017) 53***

$$\frac{d^2N}{dp_T dy}=\frac{gV}{(2\pi)^2}p_T^2\cosh y\,\frac{1}{Z^q}\sum_{N=0}^{N_0}\frac{\tilde{\omega}^N}{N!}\frac{\Gamma\left(\frac{q}{q-1}-3N\right)}{(q-1)^{3N}\,\Gamma\left(\frac{q}{q-1}\right)}\left[1+(q-1)\frac{p_T\cosh y-\mu(N+1)}{T}\right]^{\frac{q}{1-q}+3N}$$

✓ **Zeroth term approximation** $\left(N_0=0\right)$

$$\frac{d^2N}{dp_T dy}=\frac{gVp_T^2\cosh y}{(2\pi)^2}\left[1+(q-1)\frac{p_T\cosh y-\mu}{T}\right]^{\frac{q}{1-q}}\qquad \text{for } m=0$$

- The transverse momentum distribution (TMD) of the Tsallis-2 statistics in the zeroth term approximation $\left(N_0=0\right)$ exactly coincides with ***the TMD of the Tsallis-factorized statistics*** introduced in ***[J. Cleymans, D. Worku, Eur. Phys. J. A 48 (2012) 160]***
- Thus ***the TMD of the Tsallis-factorized statistics*** corresponds to the Tsallis-2 statistics for which the statistical averages are non-normalized.

➢ **Ultrarelativistic Maxwell-Boltzmann Ideal Gas in the Grand Canonical Ensemble** $(m=0)$

$(\text{Case } \; q<1)$

A.S.P., Eur. Phys. J. A 53 (2017) 53; Eur. Phys. J. A 52 (2016) 355

$$\frac{d^2N}{dp_T dy} = \frac{V}{(2\pi)^3}\sum_\sigma \int_0^{2\pi} d\varphi\, p_T \varepsilon_{\vec{p}} \langle n_{\vec{p}\sigma}\rangle, \qquad \varepsilon_{\vec{p}} = p_T \cosh y \quad \text{for } \; m=0$$

$$\langle n_{\vec{p}\sigma}\rangle = \sum_{\{n_{\vec{p}\sigma}\}} n_{\vec{p}\sigma} \frac{1}{\prod_{\vec{p}\sigma} n_{\vec{p}\sigma}!}\left[1+\frac{q-1}{q}\frac{\Lambda - \sum_{\vec{p}\sigma} n_{\vec{p}\sigma}(\varepsilon_{\vec{p}}-\mu)}{T}\right]^{\frac{1}{q-1}} = \sum_{N=0}^{N_0}\frac{\tilde{\omega}^N}{N!} h_0(0)\left[1+\frac{q-1}{q}\frac{\Lambda-\varepsilon_{\vec{p}}+\mu(N+1)}{T}\right]^{\frac{1}{q-1}+3N}$$

$$1 = \sum_{\{n_{\vec{p}\sigma}\}} \frac{1}{\prod_{\vec{p}\sigma} n_{\vec{p}\sigma}!}\left[1+\frac{q-1}{q}\frac{\Lambda - \sum_{\vec{p}\sigma} n_{\vec{p}\sigma}(\varepsilon_{\vec{p}}-\mu)}{T}\right]^{\frac{1}{q-1}} = \sum_{N=0}^{N_0}\frac{\tilde{\omega}^N}{N!} h_0(0)\left[1+\frac{q-1}{q}\frac{\Lambda+\mu N}{T}\right]^{\frac{1}{q-1}+3N}$$

$$h_\eta(\xi) = \frac{\left(\frac{q}{1-q}\right)^{3(N+\eta)}\Gamma\left(\frac{1}{1-q}-\xi-3(N+\eta)\right)}{\Gamma\left(\frac{1}{1-q}-\xi\right)}, \qquad \tilde{\omega} = \frac{gVT^3}{\pi^2}$$

- The expectation values of the Tsallis-1 statistics are well normalized
- N_0 is a cut-off parameter
- Terms with $N > N_0$ increase and became divergent

Transverse Momentum Distribution: Tsallis-1 Statistics

➢ **Ultrarelativistic Maxwell-Boltzmann Ideal Gas in the Grand Canonical Ensemble** $(m=0)$

$(\text{Case } q<1)$

A.S.P., Eur. Phys. J. A 53 (2017) 53; Eur. Phys. J. A 52 (2016) 355

✓ **Exact Solutions:**

$$\frac{d^2N}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T^2 \cosh y \sum_{N=0}^{N_0} \frac{\tilde{\omega}^N}{N!} \frac{\Gamma\left(\frac{1}{1-q} - 3N\right)}{\left(\frac{1-q}{q}\right)^{3N} \Gamma\left(\frac{1}{1-q}\right)} \left[1 + \frac{q-1}{q} \frac{\Lambda - p_T \cosh y + \mu(N+1)}{T}\right]^{\frac{1}{q-1}+3N}$$

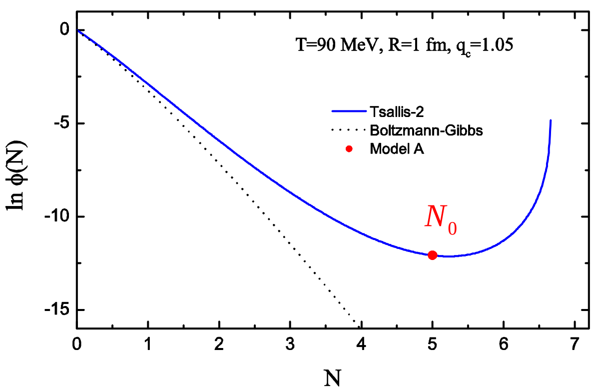
✓ **Zeroth term approximation** $(N_0 = 0)$

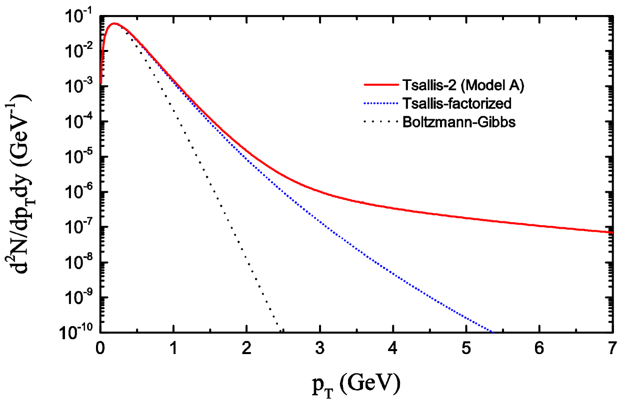
$$\frac{d^2N}{dp_T dy} = \frac{gVp_T^2 \cosh y}{(2\pi)^2} \left[1 - \frac{q-1}{q} \frac{p_T \cosh y - \mu}{T}\right]^{\frac{1}{q-1}} \qquad \text{for } m=0$$

- The transverse momentum distribution (TMD) of the Tsallis-1 statistics in the zeroth term approximation $(N_0 = 0)$ under the transformation $(q \to 1/q)$ exactly recovers the TMD of the Tsallis-2 statistics in the zeroth term approximation.
- Thus the TMD of the Tsallis-1 statistics in the zeroth term approximation under the transformation $(q \to 1/q)$ exactly coincides with ***the TMD of the Tsallis-factorized statistics*** introduced in ***[J. Cleymans, D. Worku, Eur. Phys. J. A 48 (2012) 160]***

The cut-off parameter of the Tsallis-2 statistics

- **Model A (from the minimum):** $\min(\ln\phi(N))\,|_{N=N_0}$



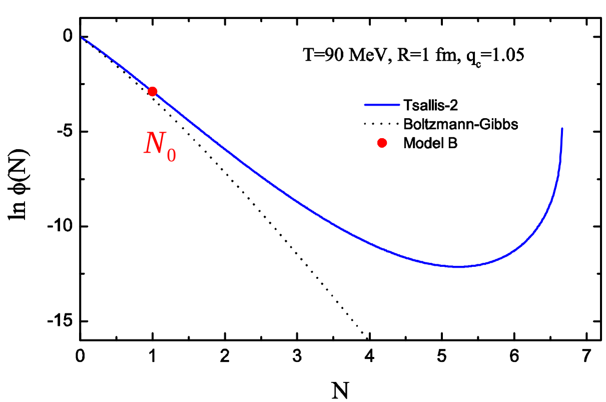


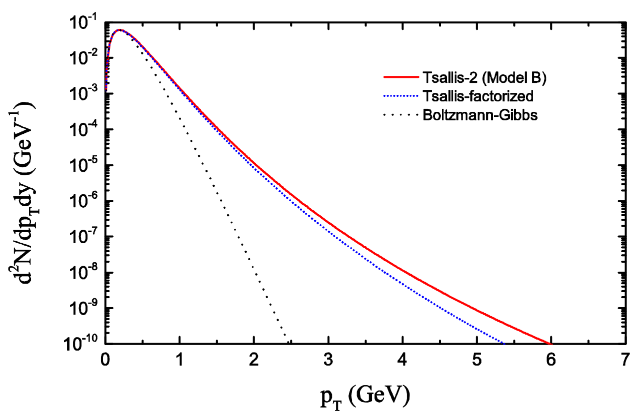
✓ Tsallis-2 statistics:

$$Z=\sum_{N=0}^{N_0}\phi(N)$$

$$\phi(N)=\frac{\tilde{\omega}^N}{N!}\frac{\Gamma\left(\frac{1}{q-1}-3N\right)}{(q-1)^{3N}\,\Gamma\left(\frac{1}{q-1}\right)}\times\left[1-(q-1)\frac{\mu N}{T}\right]^{\frac{1}{1-q}+3N}$$

- **Model B (from the inflection point):** $\left.\frac{\partial^2\ln\phi(N)}{\partial N^2}\right|_{N=N_0}=0$





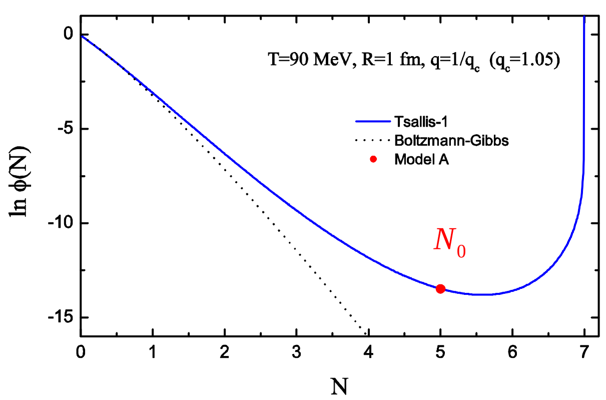
✓ Boltzmann-Gibbs statistics:

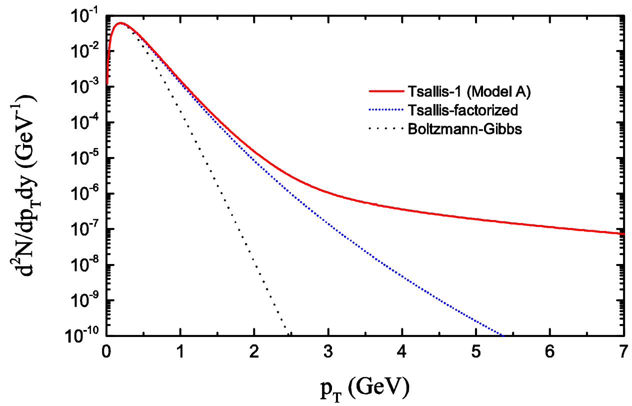
$$Z=\sum_{N=0}^{\infty}\phi(N)$$

$$\phi(N)=\frac{\tilde{\omega}^N}{N!}e^{\frac{\mu N}{T}}$$

The cut-off parameter of the Tsallis-1 statistics

➢ **Model A (from the minimum):** $$\min(\ln\phi(N))\,|_{N=N_0}$$





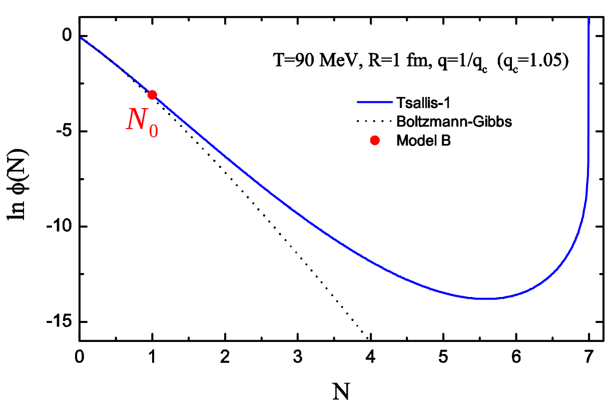
✓ Tsallis-1 statistics:

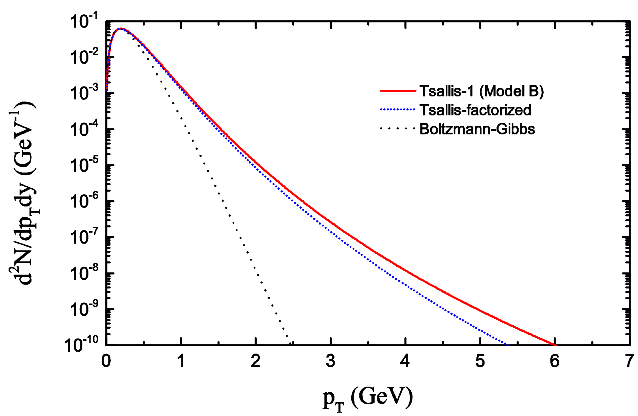
$$\sum_{N=0}^{N_0}\phi(N)=1$$

$$\phi(N)=\frac{\tilde{\omega}^N}{N!}\frac{\left(\frac{q}{1-q}\right)^{3N}\Gamma\left(\frac{1}{1-q}-3N\right)}{\Gamma\left(\frac{1}{1-q}\right)}$$

$$\times\left[1+\frac{q-1}{q}\frac{\Lambda+\mu N}{T}\right]^{\frac{1}{q-1}+3N}$$

➢ **Model B (from the inflection point):** $$\left.\frac{\partial^2\ln\phi(N)}{\partial N^2}\right|_{N=N_0}=0$$





✓ Boltzmann-Gibbs statistics:

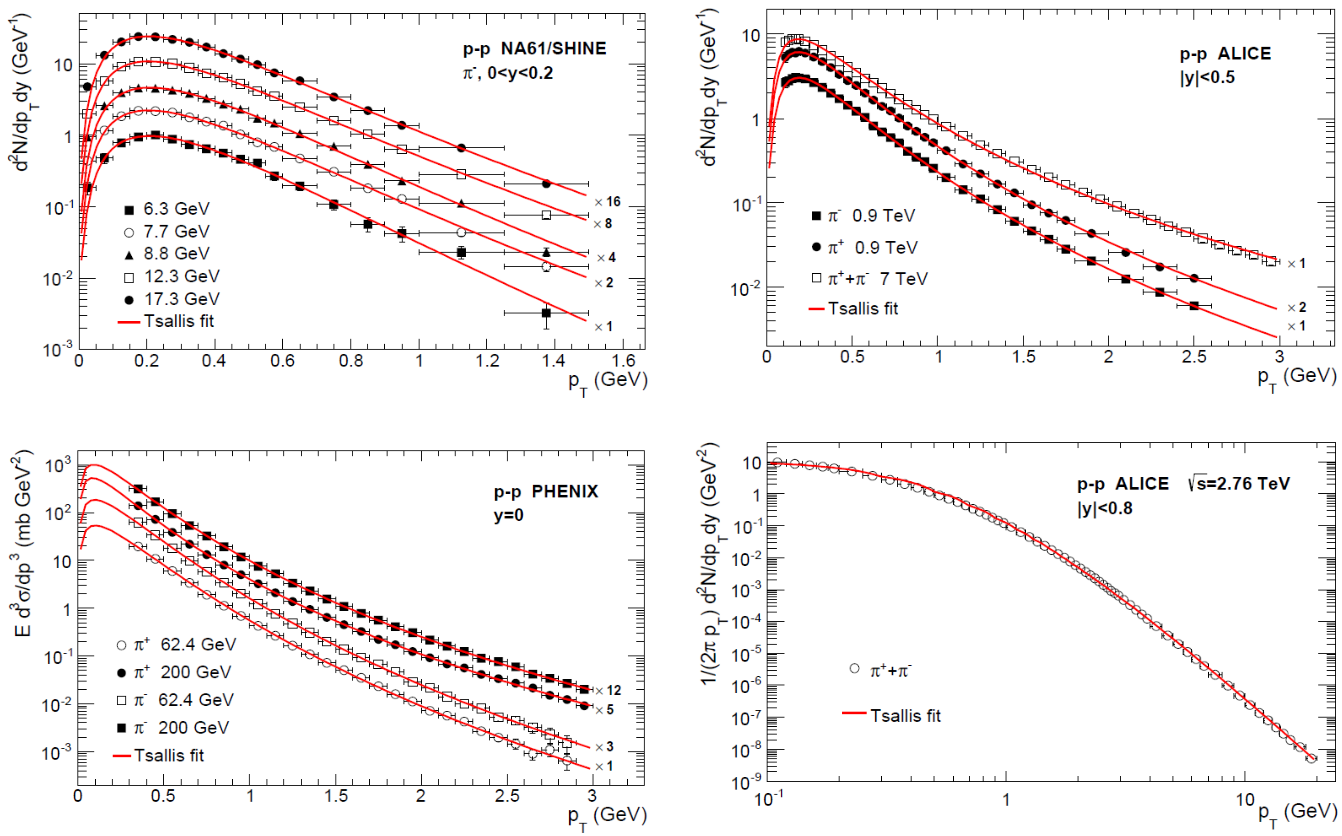
$$\sum_{N=0}^{N_0}\phi(N)=1$$

$$\phi(N)=\frac{\tilde{\omega}^N}{N!}e^{\frac{\Omega+\mu N}{T}}$$

Comparison of Model B of Tsallis statistics with the Tsallis-factorized statistics

➢ Charged pions in *pp* collisions:

A.S.P., Eur. Phys. J. A 52 (2016) 355



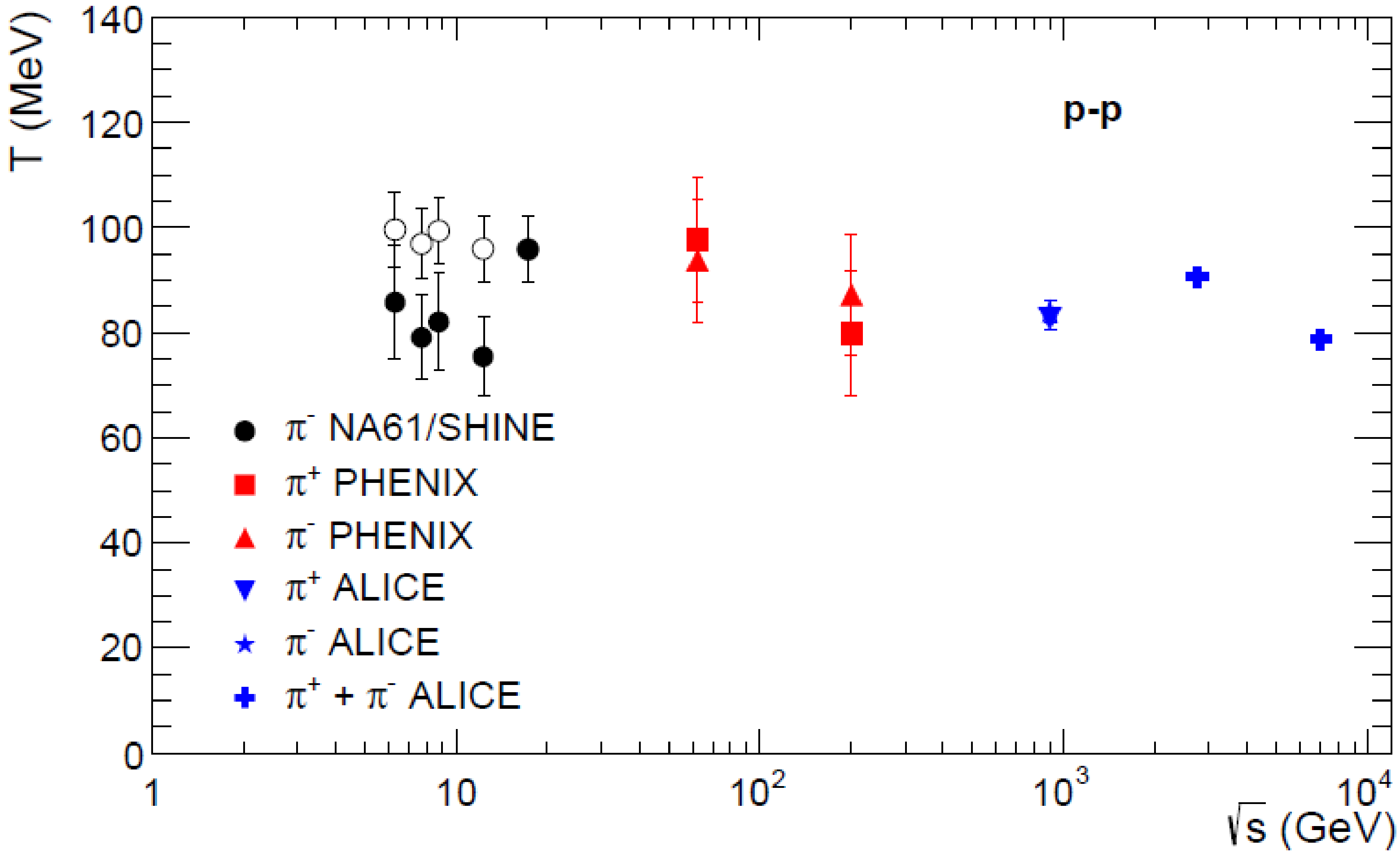
- Transverse momentum distributions (TMD) of charged pions produced in *pp* collisions at SPS, RHIC and LHC energies
- The yields were integrated in the experimental rapidity interval $y_0 \le y \le y_1$
- The solid curves are the fits of the experimental data to the ultrarelativistic (m=0) transverse momentum distributions of
  1.) ***Tsallis-1 statistics***
  2.) ***Tsallis-2 statistics***
  3.) ***Tsallis-factorized statistics***
- The curves are the same for all statistics but only the parameters are different.

Experimental Data:

NA61/SHINE, EPJC 74 (2014) 2794; PHENIX, PRC 83 (2011) 064903 ALICE, EPJC 71 (2011) 1655; ALICE, EPJC 75 (2015) 226; ALICE, PLB 736 (2014) 196

Temperature for Model B of Tsallis statistics and for Tsallis-factorized statistics

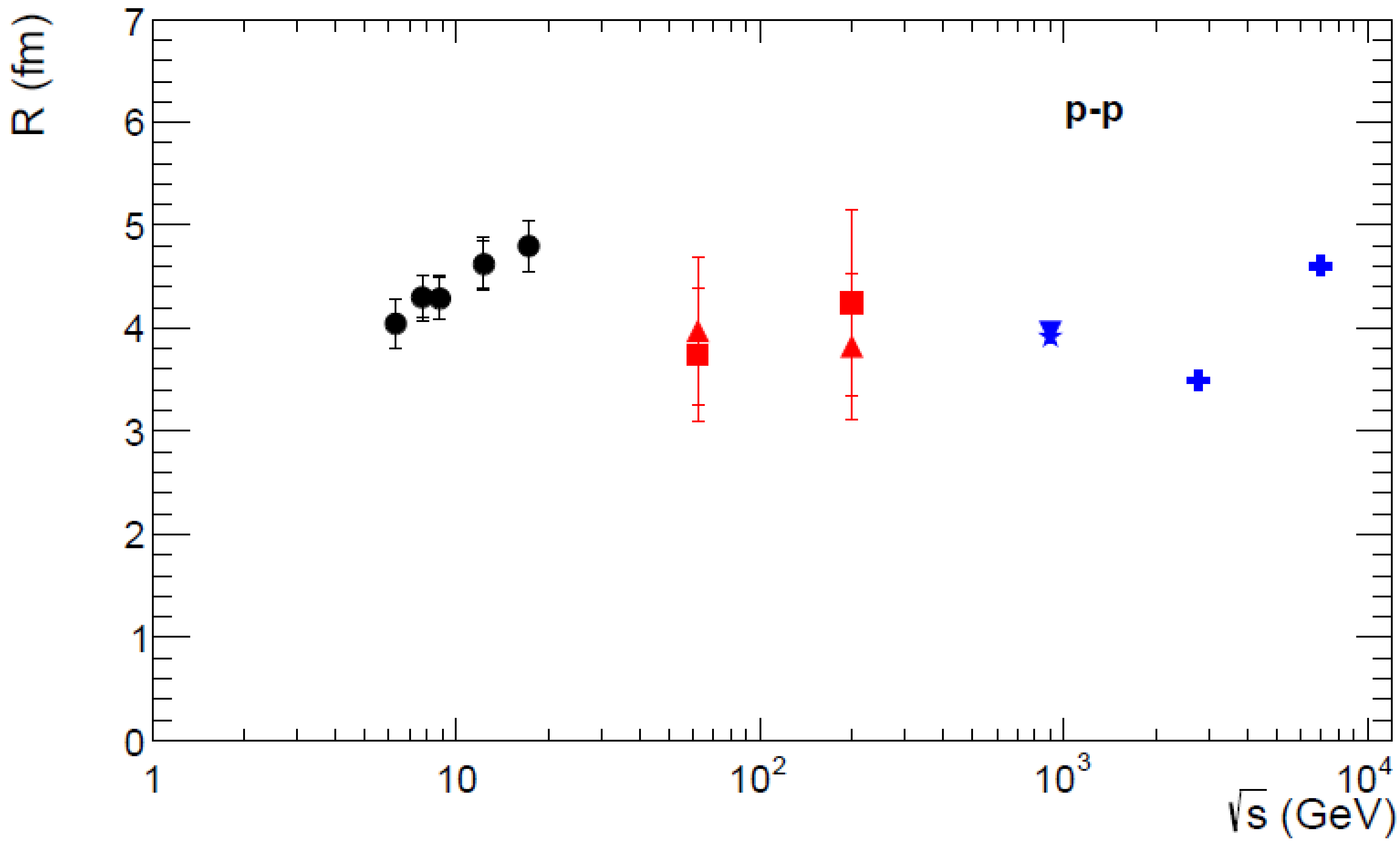
A.S.P., Eur. Phys. J. A 52 (2016) 355



- ✓ Solid points are the results of the fit by Model B of Tsallis statistics
- ✓ Open symbols are the results of the fit by Tsallis-factorized statistics

Radius for Model B of Tsallis statistics and for Tsallis-factorized statistics

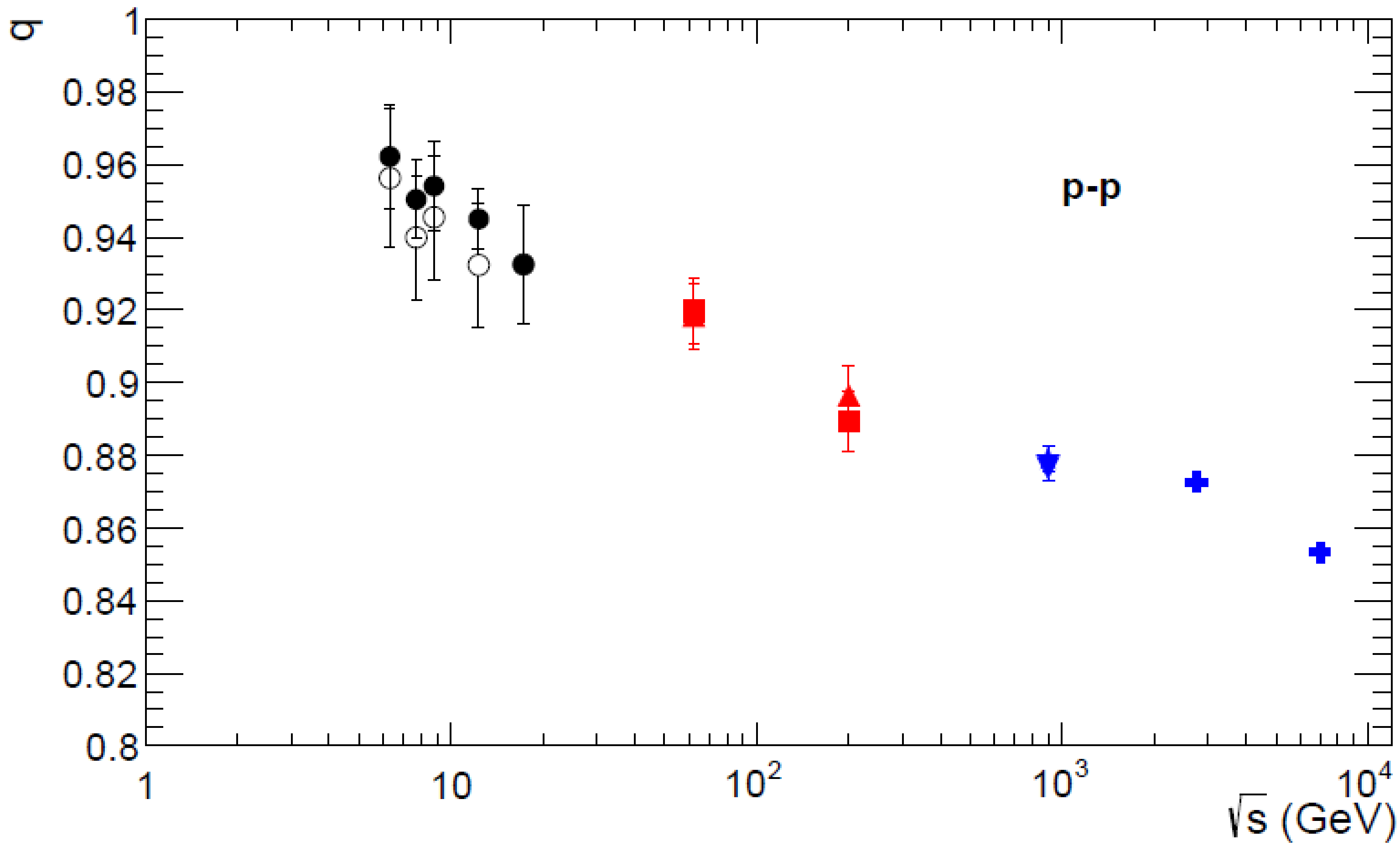
A.S.P., Eur. Phys. J. A 52 (2016) 355



- ✓ Solid points are the results of the fit by Model B of Tsallis statistics
- ✓ Open symbols are the results of the fit by Tsallis-factorized statistics

Entropic parameter for Model B of Tsallis statistics and for Tsallis-factorized statistics

A.S.P., Eur. Phys. J. A 52 (2016) 355



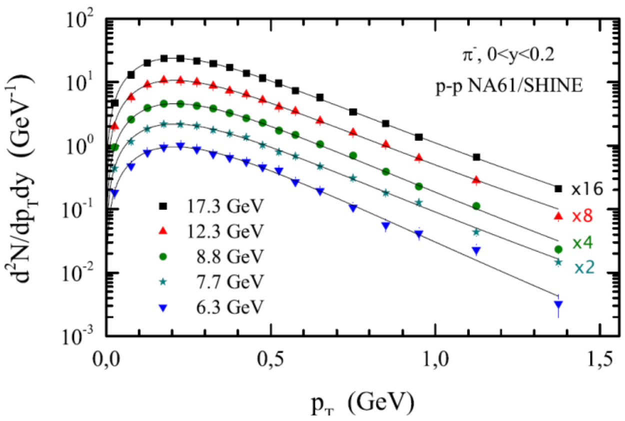
- ✓ Solid points are the results of the fit by Model B of Tsallis statistics
- ✓ Open symbols are the results of the fit by Tsallis-factorized statistics

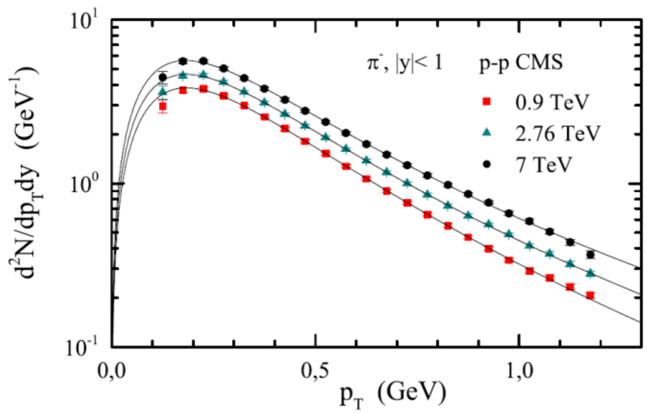
Applications of the Tsallis-factorized statistics

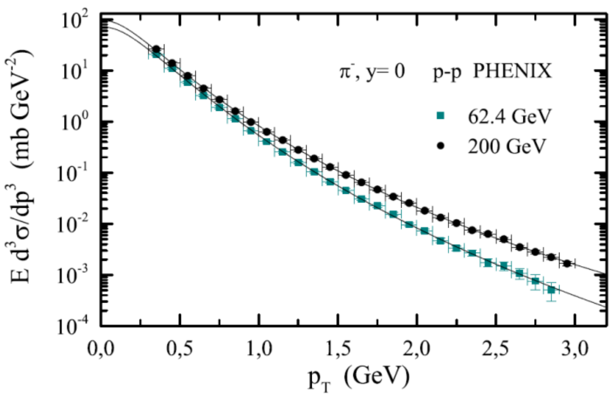
Identified hadrons in *pp* collisions

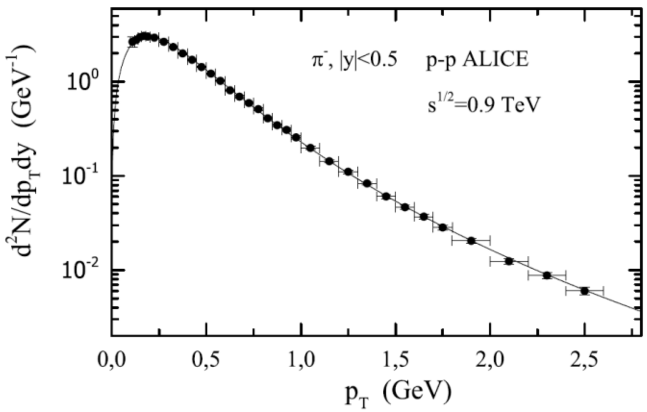
- **Negatively charged pions:**

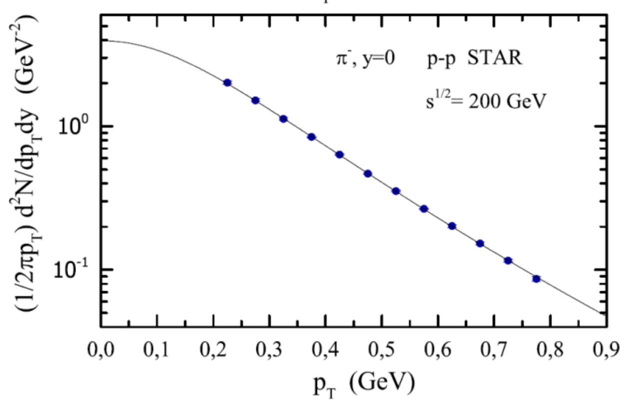
A.S.P., O.V. Teryaev, J. Cleymans, Eur. Phys. J. A 53 (2017) 102











- Transverse momentum distributions (TMD) of negatively charged pions produced in *pp* collisions at SPS, RHIC and LHC energies
- The yields were integrated in the experimental rapidity interval $y_0 \le y \le y_1$

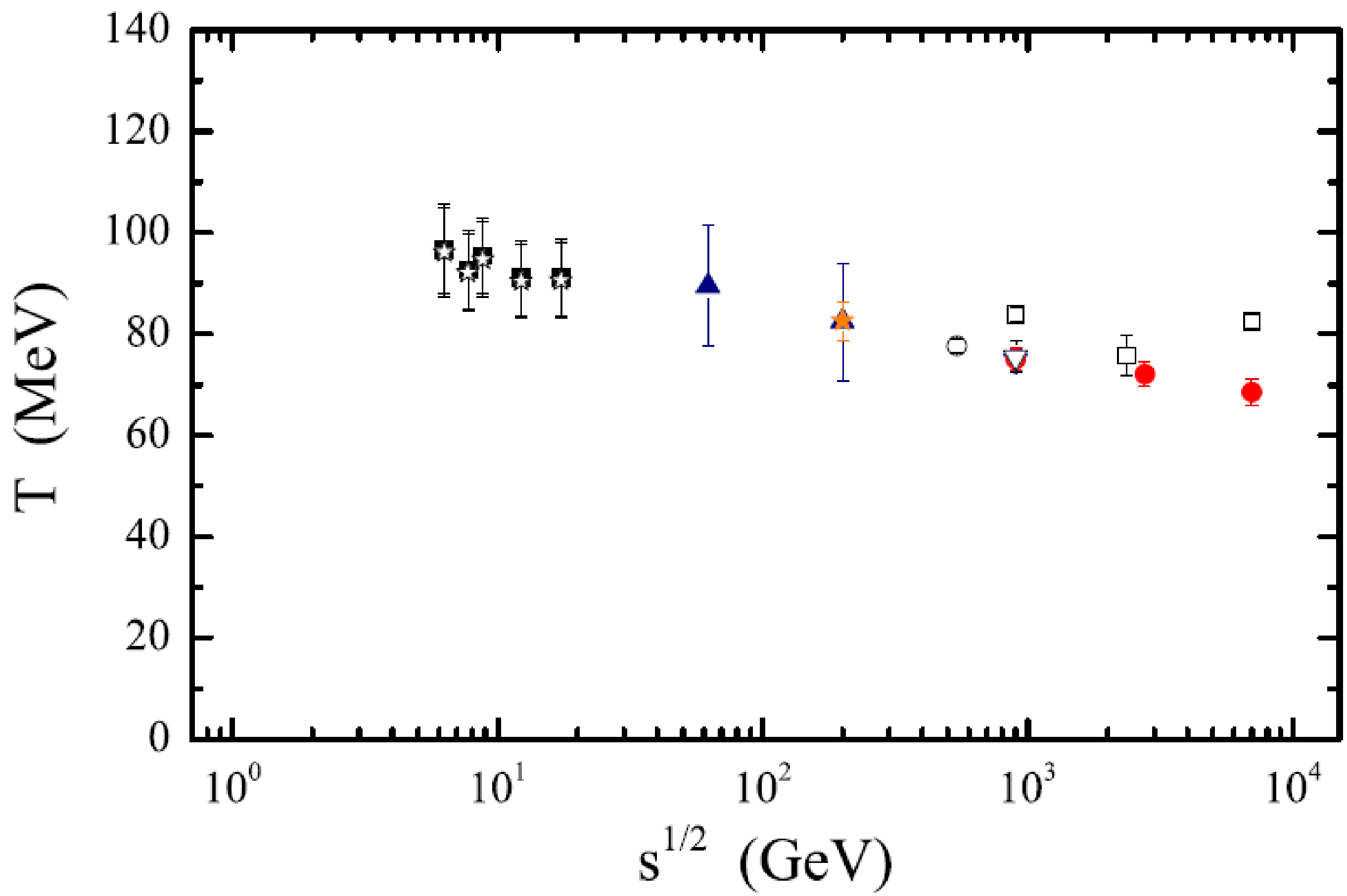
Experimental Data:
NA61/SHINE, EPJC 74 (2014) 2794;
PHENIX, PRD 83 (2011) 052004;
PHENIX, PRC 83 (2011) 064903;
CMS, JHEP 02 (2010) 041;
CMS, PRL 105 (2010) 022002;
CMS, EPJC 72 (2012) 2164

- The solid curves are the fits of the data to **the Tsallis-factorized distribution**

$$\left.\frac{d^2N}{dp_T dy}\right|_{y_0}^{y_1} = gV\int_{y_0}^{y_1} dy\, \frac{p_T m_T \cosh y}{(2\pi)^2}\left[1-(1-q)\frac{m_T \cosh y-\mu}{T}\right]^{\frac{q}{1-q}}$$

Temperature parameter for the Tsallis-factorized statistics

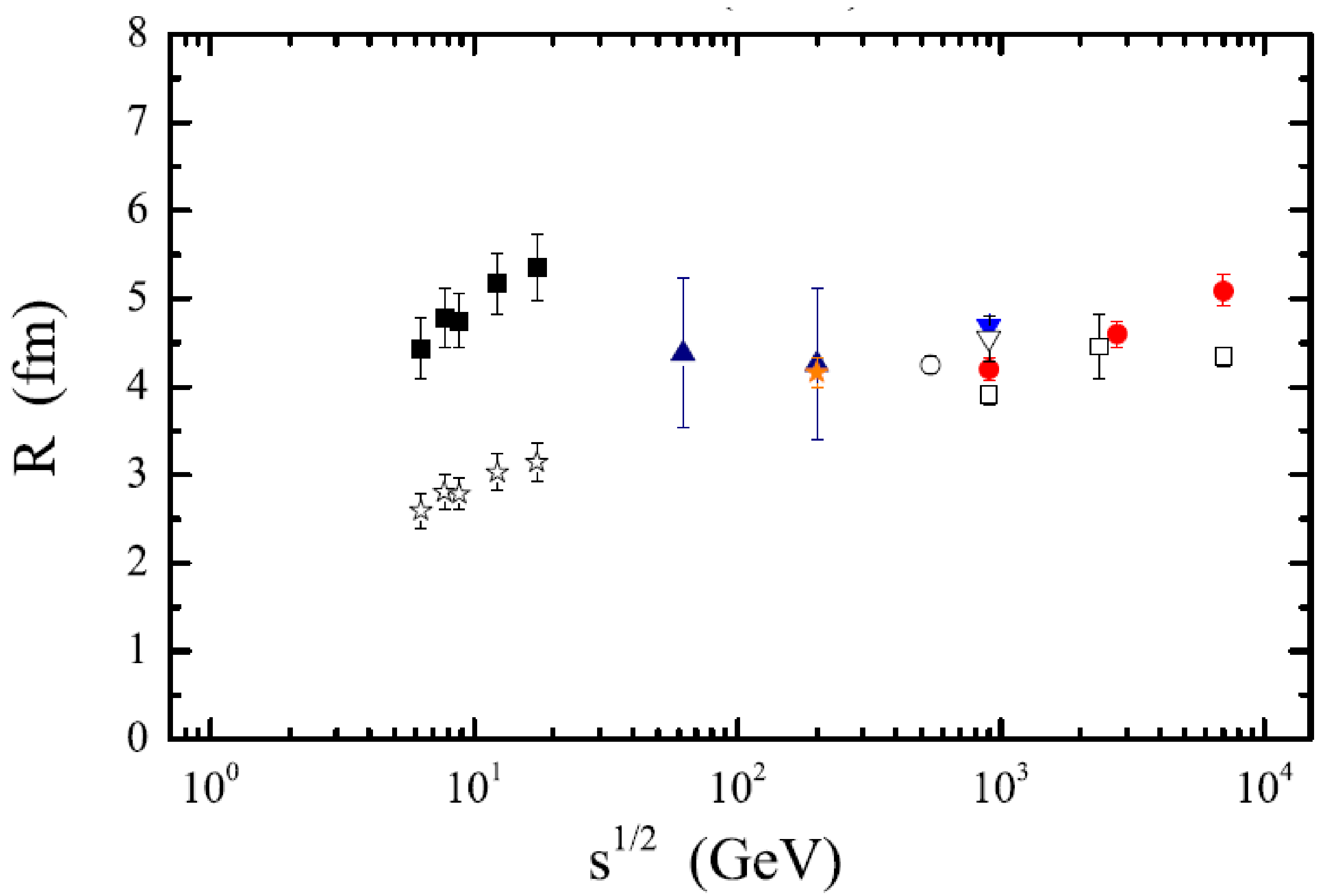
A.S.P., O.V. Teryaev, J. Cleymans, Eur. Phys. J. A 53 (2017) 102



- ✓ Open squares, triangles and circles - charged hadron yields from [***J. Cleymans, G.I. Lykasov, A.S.P., A.S. Sorin, O.V. Teryaev, D. Worku, Phys. Lett.B 723 (2013) 351***]
- ✓ Open stars -- the fit at y = 0 for the data of NA61/SHINE

Radius parameter for the Tsallis-factorized statistics

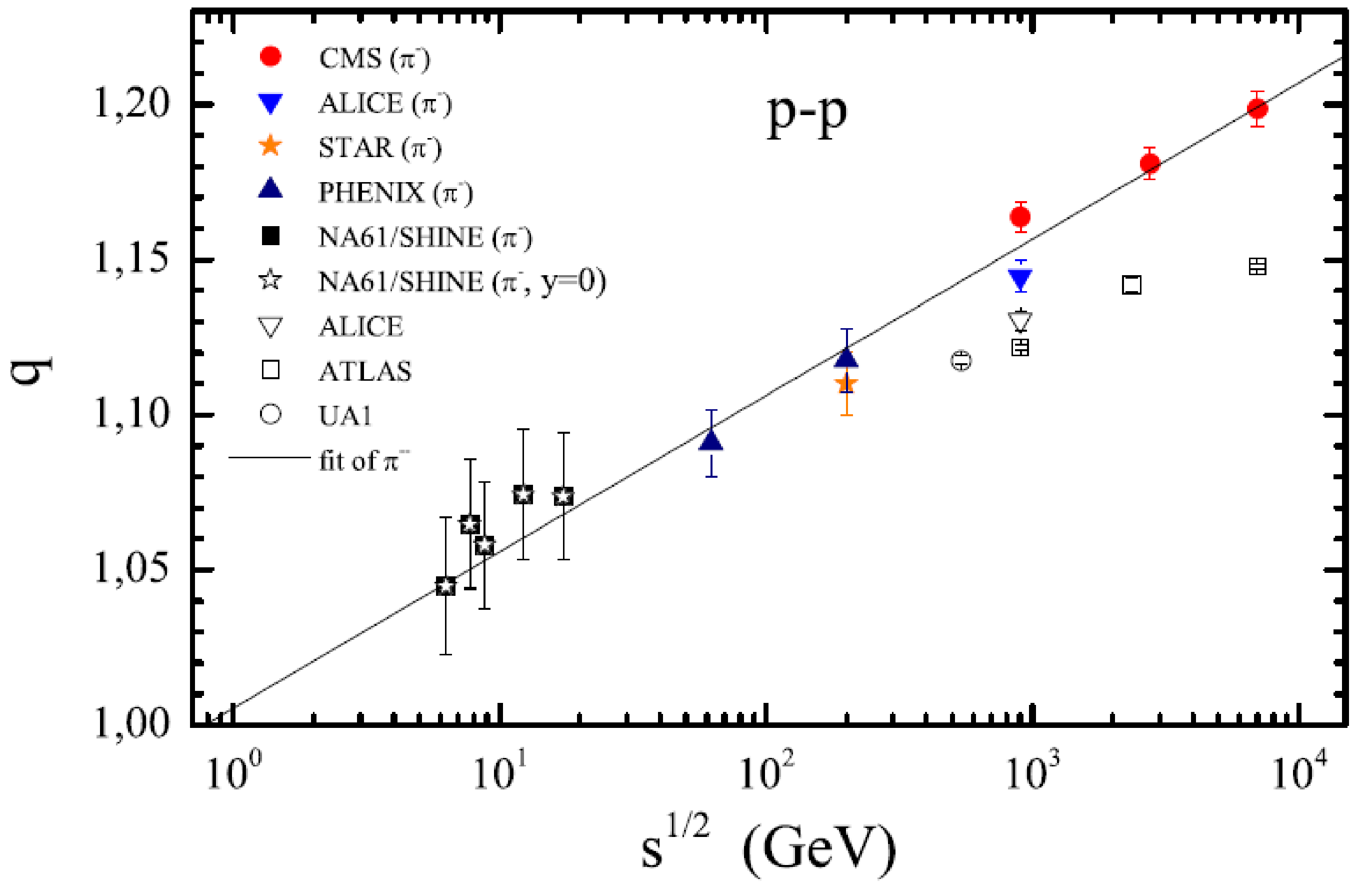
A.S.P., O.V. Teryaev, J. Cleymans, Eur. Phys. J. A 53 (2017) 102



- ✓ Open squares, triangles and circles - charged hadron yields from [***J. Cleymans, G.I. Lykasov, A.S.P., A.S. Sorin, O.V. Teryaev, D. Worku, Phys. Lett.B 723 (2013) 351***]
- ✓ Open stars -- the fit at $y = 0$ for the data of NA61/SHINE

Entropic parameter for the Tsallis-factorized statistics

A.S.P., O.V. Teryaev, J. Cleymans, Eur. Phys. J. A 53 (2017) 102



- ✓ Open squares, triangles and circles - charged hadron yields from [***J. Cleymans, G.I. Lykasov, A.S.P., A.S. Sorin, O.V. Teryaev, D. Worku, Phys. Lett.B 723 (2013) 351***]
- ✓ Open stars -- the fit at $y = 0$ for the data of NA61/SHINE

Heavy-ion collisions: SPS CERN

➢ **Transverse mass spectra of identified hadrons with Tsallis-factorized statistics**

Data: NA49, Phys. Rev. C 66 (2002) 054902; Phys. Rev. C 77 (2008) 024903

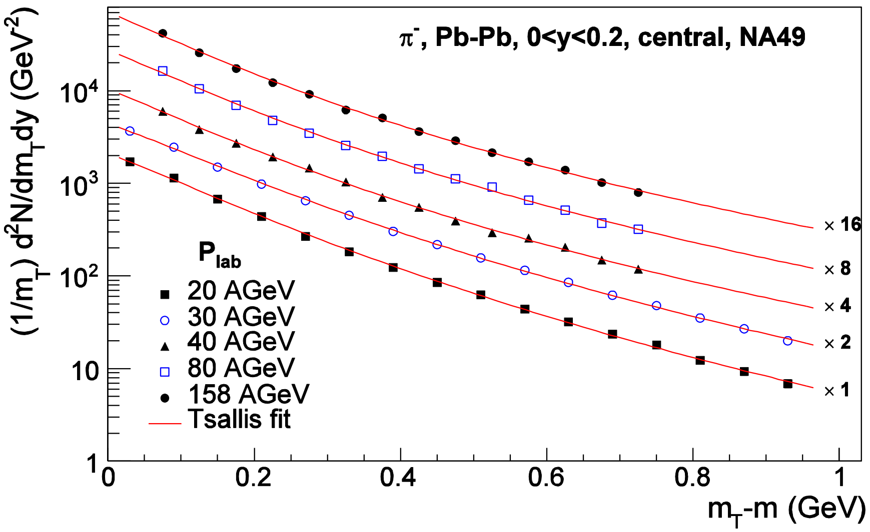
- mT – distribution in the Tsallis-factorized statistics:

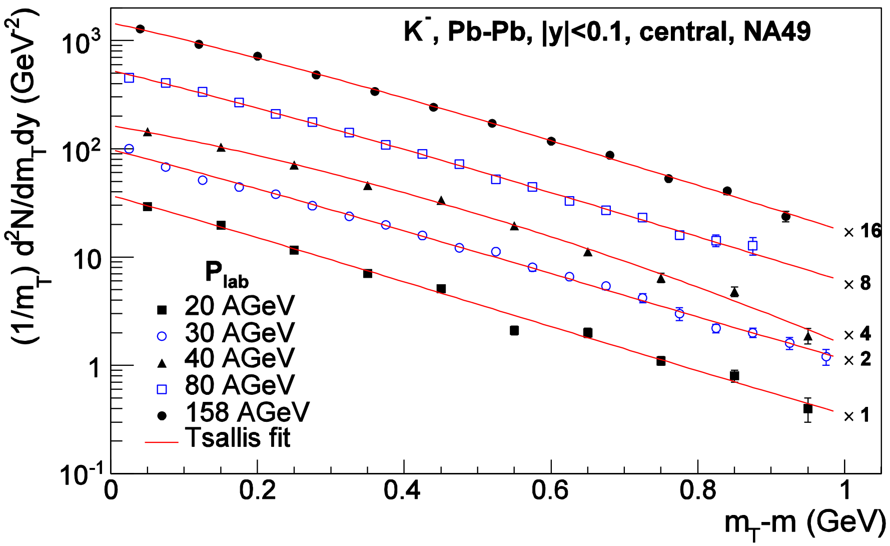
$$\frac{1}{m_T}\frac{d^2N}{dm_T dy}\bigg|_{y_0}^{y_1} = gV\int_{y_0}^{y_1} dy \frac{m_T \cosh y}{(2\pi)^2} \times \left[1-(1-q)\frac{m_T\cosh y-\mu}{T}\right]^{\frac{q}{1-q}}$$

- π^-, K^- – mesons

- Central PbPb collisions in the energy range $\sqrt{s_{NN}} = 6.3-17.3$ GeV

- The data of π^- are very well described by the Tsallis-factorized statistics
- The data of K^- measured by NA49 Collaboration in the energy range 6.3-12.3 GeV contain irregularities and they should be corrected by NICA experiment
- The data of K^- at 17.3 GeV fits very well the Tsallis-factorized distribution





Heavy-ion collisions: SPS CERN

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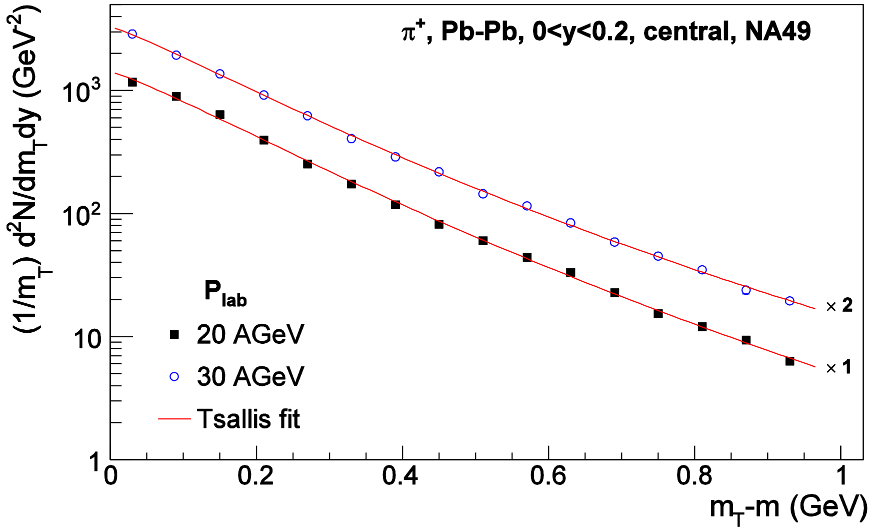
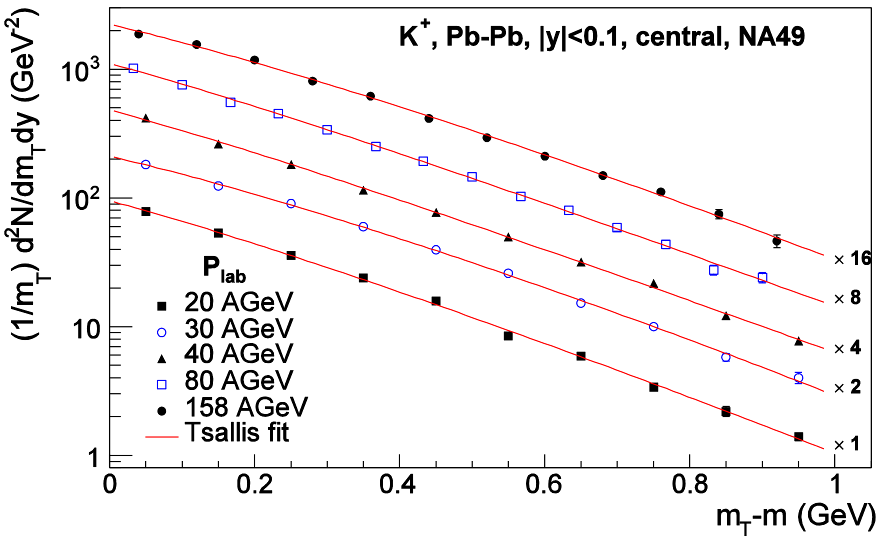
- $\pi^+, K^+ -$ mesons
- Central PbPb collisions in the energy range $\sqrt{s_{NN}} = 6.3-17.3\text{ GeV}$
- The NA49 data for π^+, K^+ are very well described by the Tsallis-factorized statistics in the all its energy range





Heavy-ion collisions: RHIC BNL

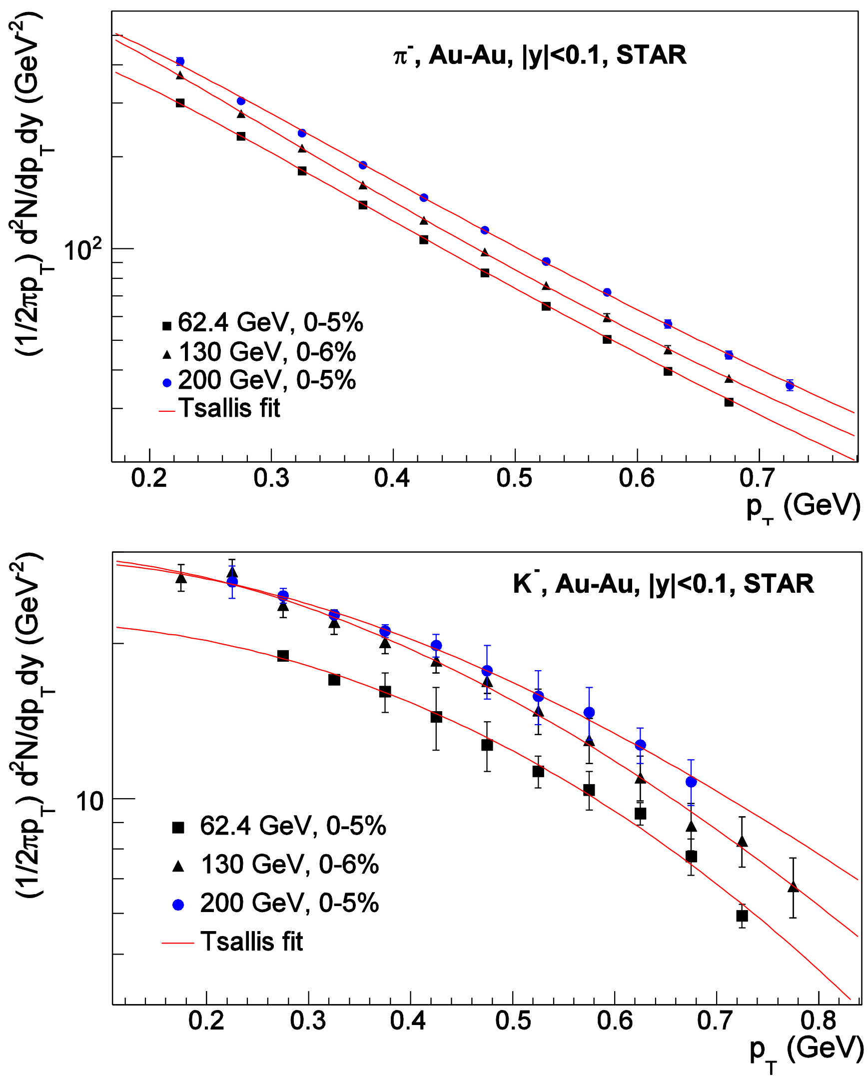
➢ **Transverse momentum distribution with Tsallis-factorized statistics**

- pT – distribution in the Tsallis-factorized statistics:

$$\frac{1}{2\pi p_T}\frac{d^2N}{dp_T dy}\bigg|_{y_0}^{y_1} = gV\int_{y_0}^{y_1} dy\, \frac{m_T \cosh y}{(2\pi)^3} \times \left[1-(1-q)\frac{m_T \cosh y - \mu}{T}\right]^{\frac{q}{1-q}}$$

- π^-, K^- – mesons
- Central AuAu collisions in the energy range $\sqrt{s_{NN}} = 62.4 - 200$ GeV
- The data of π^- are very well described by the Tsallis-factorized statistics
- The data of K^- measured by STAR Collaboration 62.4 and 130 GeV contain irregularities which should be corrected by another experiment.
- The data of K^- at 200 GeV fits very well the Tsallis-factorized distribution

Data: STAR, Phys. Rev. C 79 (2009) 034909



Heavy-ion collisions: RHIC BNL

- ## Transverse momentum distribution with Tsallis-factorized statistics

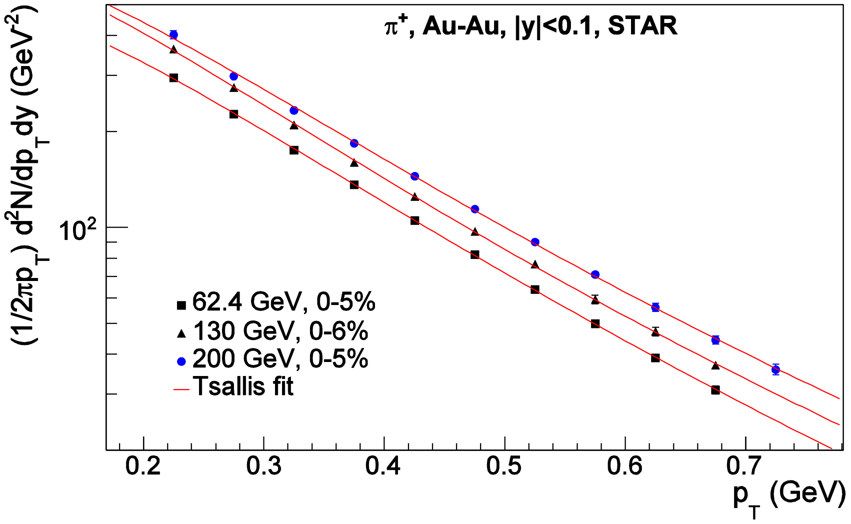
- pT – distribution in the Tsallis-factorized statistics:

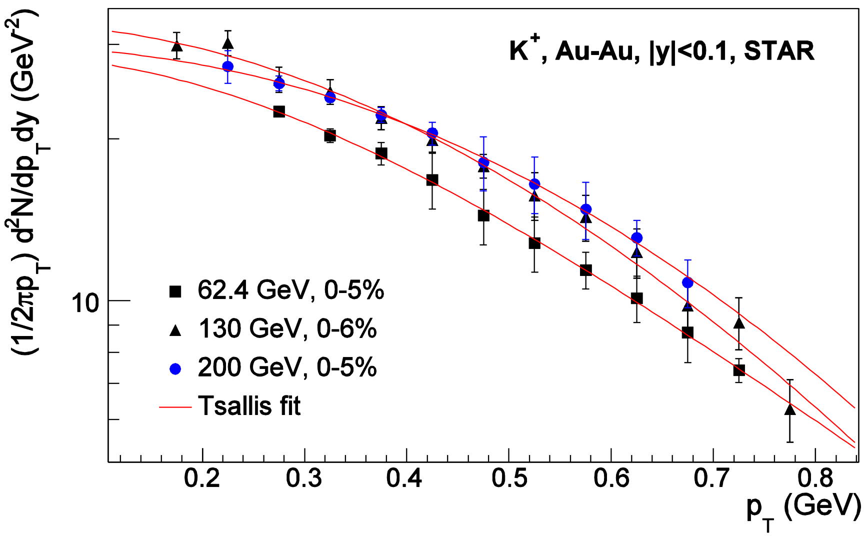
$$\frac{1}{2\pi p_T}\frac{d^2N}{dp_T dy}\bigg|_{y_0}^{y_1} = gV\int_{y_0}^{y_1} dy\,\frac{m_T\cosh y}{(2\pi)^3}$$

$$\times\left[1-(1-q)\frac{m_T\cosh y-\mu}{T}\right]^{\frac{q}{1-q}}$$

- $\pi^+, K^+ -$ mesons
- Central AuAu collisions in the energy range $\sqrt{s_{NN}} = 62.4 - 200\ \mathrm{GeV}$
- The data of π^+ are very well described by the Tsallis-factorized statistics
- The data of K^+ measured by STAR Collaboration at 130 GeV contain irregularities which should be corrected by another experiment.
- The data of K^+ at 62.4 and 200 GeV fits very well the Tsallis-factorized distribution

Data: STAR, Phys. Rev. C 79 (2009) 034909





Heavy-ion collisions: LHC CERN

➢ Transverse momentum distribution with Tsallis-factorized statistics

- pT – distribution in the Tsallis-factorized statistics:

$$\frac{1}{2\pi p_T}\frac{d^2N}{dp_T dy}\bigg|_{y_0}^{y_1} = gV\int_{y_0}^{y_1} dy \frac{m_T \cosh y}{(2\pi)^3}$$

$$\times\left[1-(1-q)\frac{m_T \cosh y-\mu}{T}\right]^{\frac{q}{1-q}}$$

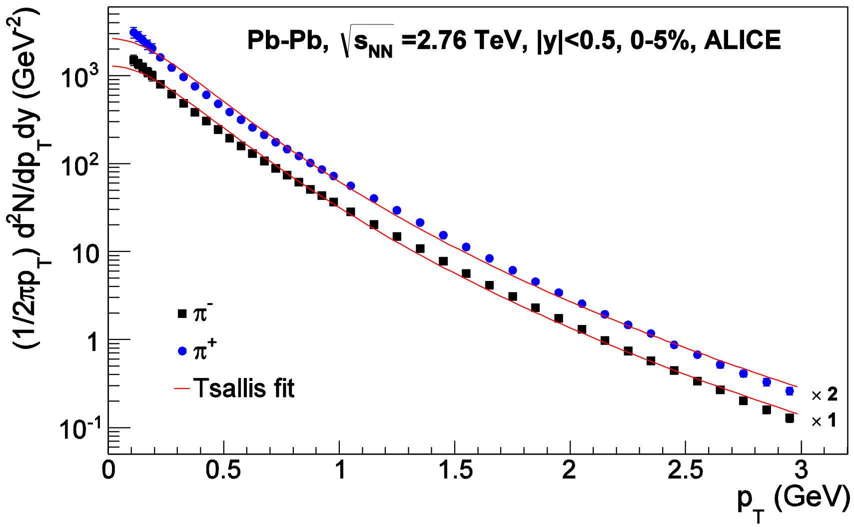
- $\pi^{\pm}, K^{\pm}-$ mesons

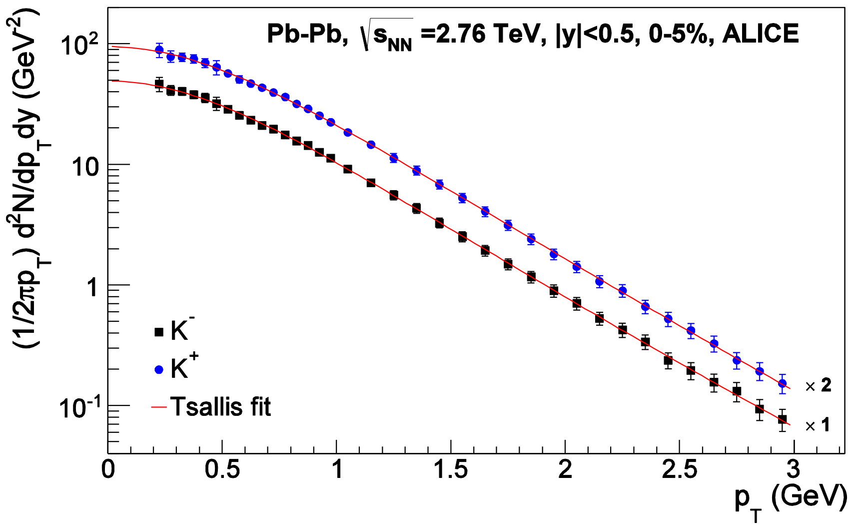
- Central PbPb collisions at 2.76 TeV

- The data of $K^{\pm}$ at 2.76 TeV are very well described by the Tsallis-factorized statistics

- The data of $\pi^{\pm}$ at 2.76 TeV can not be described by the Tsallis-factorized statistics for low pT momenta

Data: ALICE, Phys. Rev. C 88 (2013) 044910

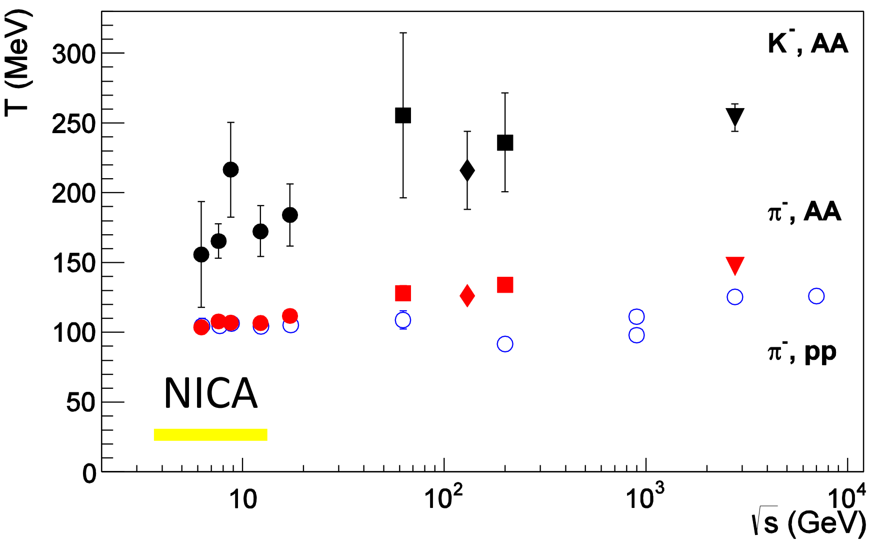


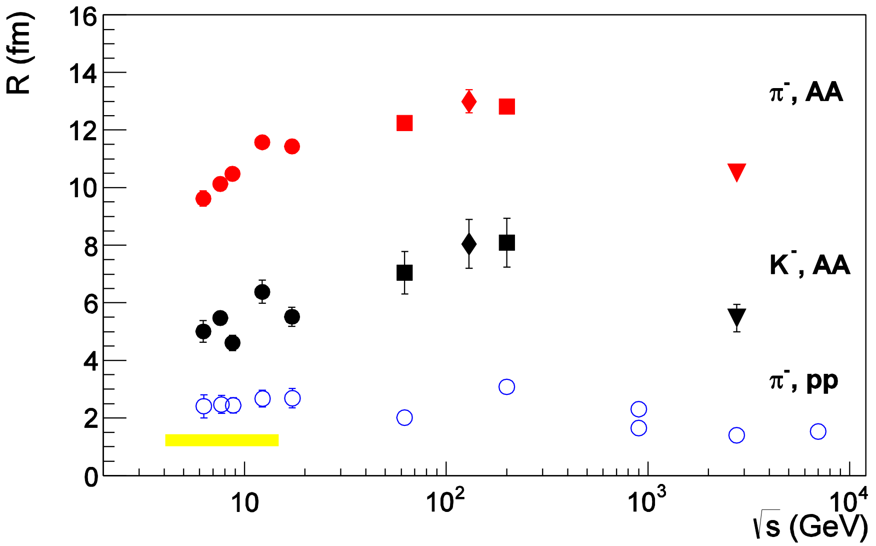


Parameters of the Tsallis-factorized statistics in AA and pp collisions

- **Temperature and volume for K- and p- mesons**

- The experimental transverse momentum distributions from heavy-ion collisions clearly show that K- and pi- mesons have different temperatures T and are emitted from different volumes V.
- The temperature of K- kaons in AA collisions is higher than the temperature of pi- pions.
- However, K- kaons in AA collisions are emitted from the smaller volume than pi- pions.
- The volume for pi- pions in AA collision corresponds to the geometrical volume of two nuclei.
- And the volume for pi- pions in pp collision corresponds to the geometrical volume of two protons.
- The temperatures for pi- pions from AA and pp collisions are close to each other in comparison with the temperature of K-

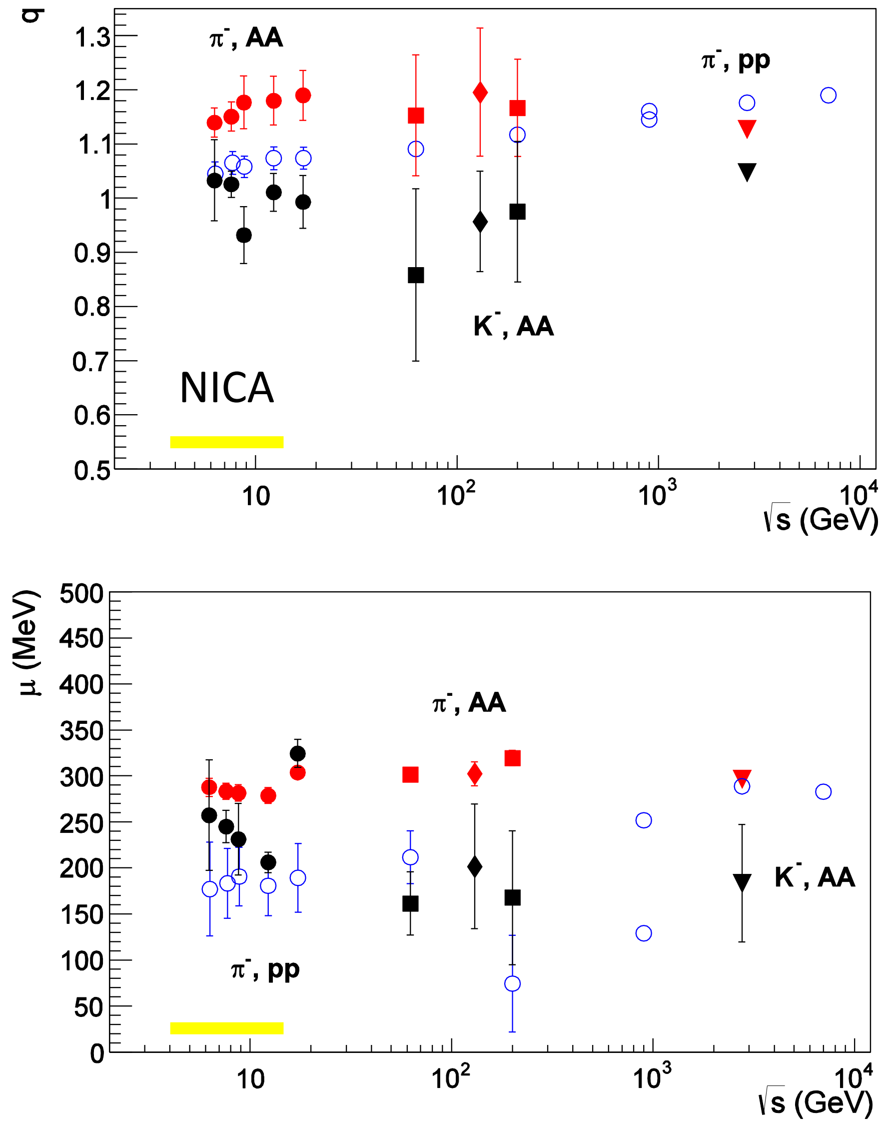




Parameters of the Tsallis-factorized statistics in AA and pp collisions

- **Parameter q and particle chemical potential m for K- and p- mesons**

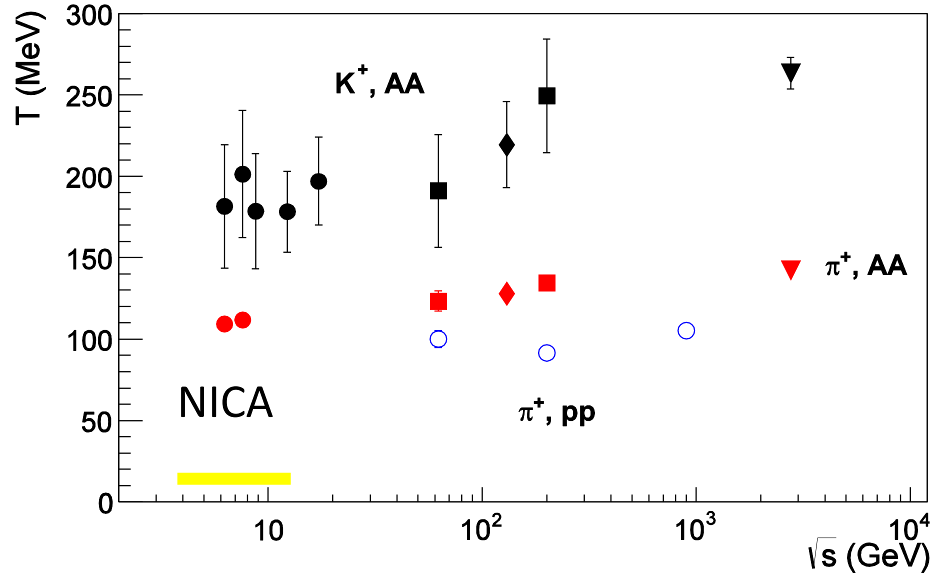
- The value q=1 corresponds to the Boltzmann-Gibbs statistics (exponential function).
- The deviation of the value of the parameter q from unity indicates on the measure of deviation of the power-law distribution from the Gibbs exponential function.
- The deviations from Boltzmann-Gibbs statistics are monotonically growing with beam energy for pi- pions in pp collisions.
- The transverse momentum distribution of pi- pions in AA collisions deviates essentially from the Gibbs exponent.
- The distribution of K- kaons in AA collisions is close to the Gibbs exponent at low energies.
- The introduction of the non-vanishing particle chemical potential allows to correctly describe the values of the volume of the system.

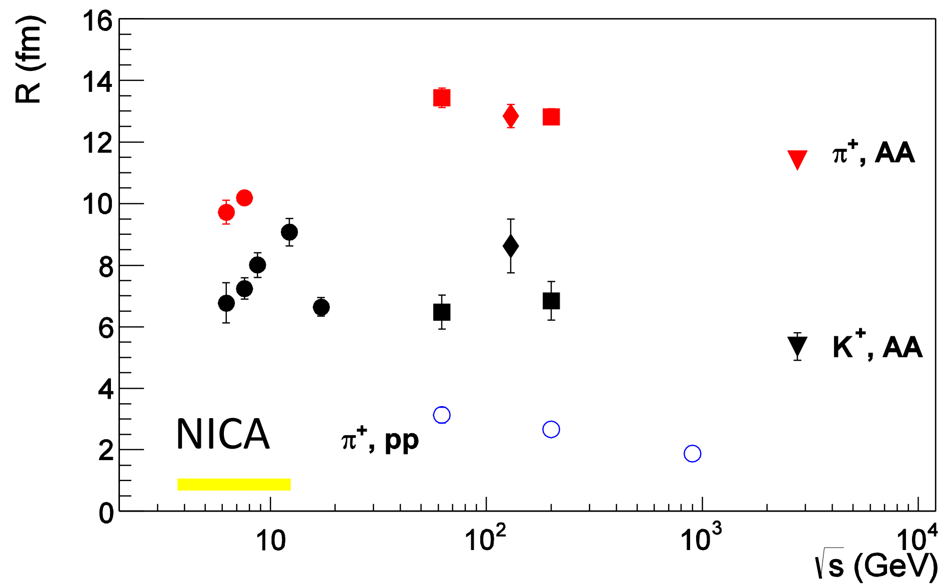


Parameters of the Tsallis-factorized statistics in AA and pp collisions

- **Temperature and volume for K+ and p+ mesons**

- At NICA energies the temperature and volume for K+ and pi+ have some structures as a function of energy.
- The temperature of K+ kaons in AA collisions is higher than the temperature of pi+ pions.
- However, K+ kaons in AA collisions are emitted from the smaller volume than pi+ pions.
- The volume for pi+ pions in AA collision corresponds to the geometrical volume of two nuclei.
- And the volume for pi+ pions in pp collision corresponds to the geometrical volume of two protons.
- The temperatures for pi+ pions from AA and pp collisions are close to each other in comparison with the temperature of K+ kaon

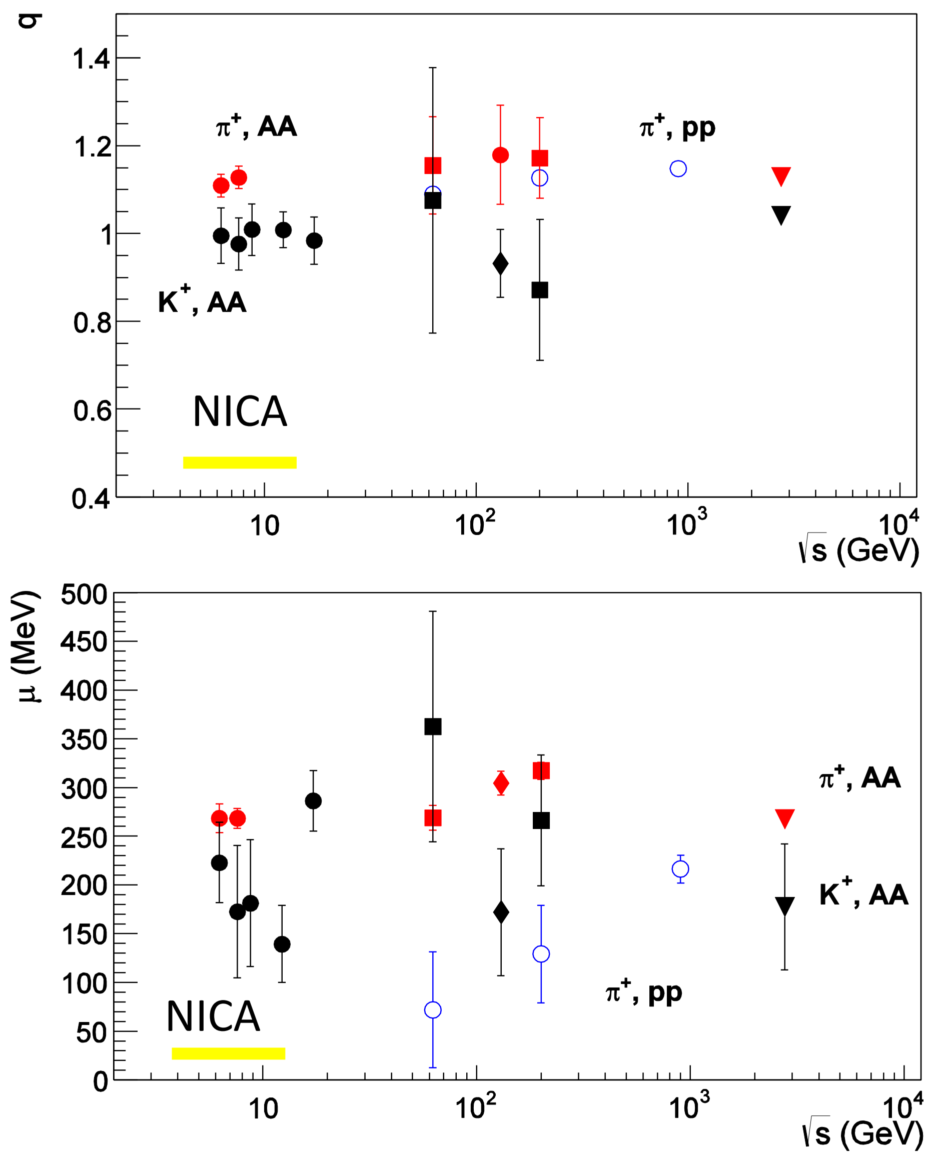




Parameters of the Tsallis-factorized statistics in AA and pp collisions

- **Parameter q and particle chemical potential m for K+ and p+ mesons**

- The deviations from Boltzmann-Gibbs statistics are monotonically growing with beam energy for pi+ pions in pp collisions.
- The transverse momentum distribution of pi+ pions in AA collisions deviates essentially from the Gibbs exponent.
- The distribution of K+ kaons in AA collisions is close to the Gibbs exponent at low energies.
- The introduction of the non-vanishing particle chemical potential allows to correctly describe the values of the volume of the system.
- The zero particle chemical potential leads to unphysical values of volume in AA and pp collisions



Conclusions

1. We have obtained that the Tsallis statistics (Tsallis-1 statistics at $q<1$ and Tsallis-2 statistics at $q>1$) is divergent.
2. It is convergent only in the case of $q=1$ which corresponds to the standard Boltzmann-Gibbs statistics.
3. However, we have found that a few terms in a series expansion of quantities in the Tsallis statistics at $q \neq 1$ are convergent and they describe very well the experimental data on the transverse momentum distributions (TMD) of hadrons in the *pp* collisions at high energies (the standard Boltzmann-Gibbs statistics fails to describe these experimental data).
4. The analytical exact expressions for the ultrarelativistic TMD of the Tsallis-1 and Tsallis-2 statistics were obtained.
5. We have demonstrated that the ultrarelativistic TMD of the usual Tsallis-factorized statistics is equivalent to the TMD of the Tsallis-2 statistics in the zeroth term approximation. But the statistical averages of the Tsallis-2 statistics are non-normalized.
6. We have demonstrated that the ultrarelativistic TMD of the Tsallis-factorized statistics recovers the TMD of the Tsallis-1 statistics in the zeroth term approximation under the transformation of the parameter q to $1/q$.
7. The TMD of the Model B of the Tsallis statistics (the cut-off from the inflection point) differs from TMD of Tsallis-factorized statistics only at low energies of NICA and NA61/SHINE.
8. The Tsallis-factorized statistics was successfully applied to describe the experimental data on the TMD of hadrons created in the heavy-ion and *pp* collisions at high energies

Thank you for your attention!