A connection between integrable systems and gravity

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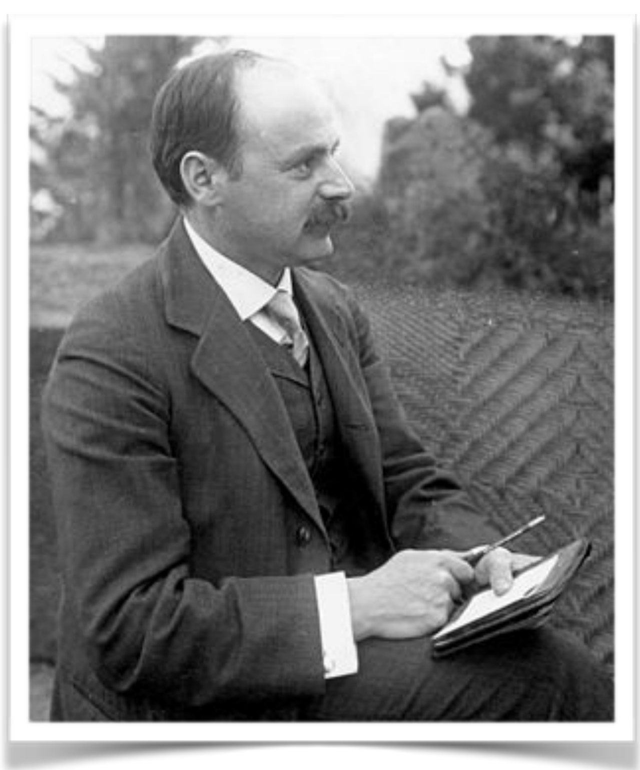
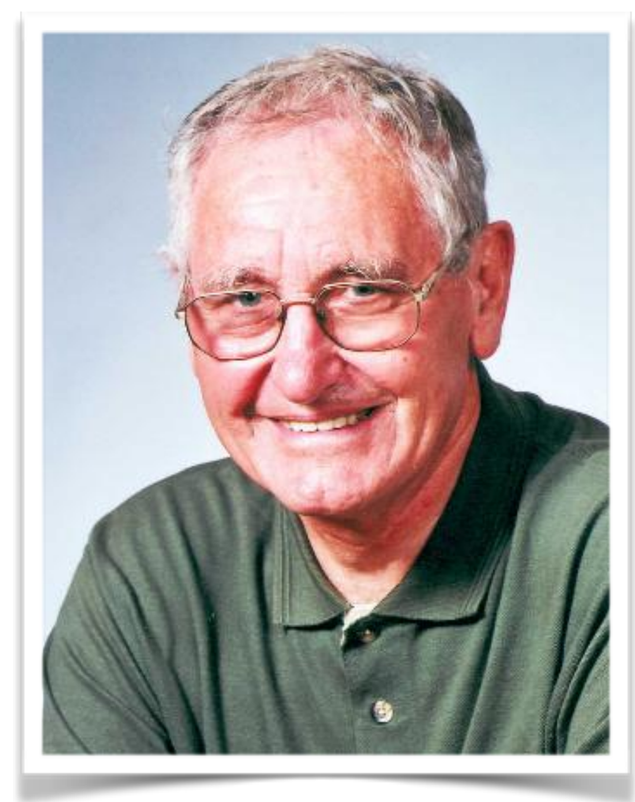
based on work with T. Araujo, I. Bakhmatov (APCTP), Ö. Kelekci (Ankara), J. Sakamoto, K. Yoshida (Kyoto), M. M. Sheikh-Jabbari (IPM Tehran), H. Yavartanoo (KITPC), M. Hong, Y. Kim (Postech)

Motivation

Finding exact solutions to Einstein gravity is hard.

$$R_{\mu\nu} = 0$$





Narrative

There is a gravity theory (“supergravity”).

For any* solution to this theory with an isometry group, there exists a deformation where the equations of motion reduce to the Classical Yang-Baxter Equation.

So for each “r-matrix” solution to the CYBE, we have a new solution.

In other words, we have a solution generating technique.

Outline

1. Review integrability and CYBE.
2. Introduce gravity theory.
3. Explain relation between CYBE & gravity.

Why integrability?

"One can ask, what is good in 1+1 models, when our spacetime is 3+1 dimensional…

A. …can teach us about the realistic field-theoretical models in a nonperturbative way…

B. …numerous physical applications…

C. …useful in modern string theory…conformal field theory models are special massless limits…

D. …teaches us about new phenomena…

E. …delightful pastime…"

L. Faddeev

Classical Integrability

Hamiltonian dynamical system with 2d phase space M

$$(q_\mu, p_\mu), \quad \mu = 1, \ldots, d$$

$$\{q_\mu, q_\nu\} = \{p_\mu, p_\nu\} = 0, \quad \{q_\mu, p_\nu\} = \delta_{\mu\nu},$$

Liouville integrability: d independent conserved quantities

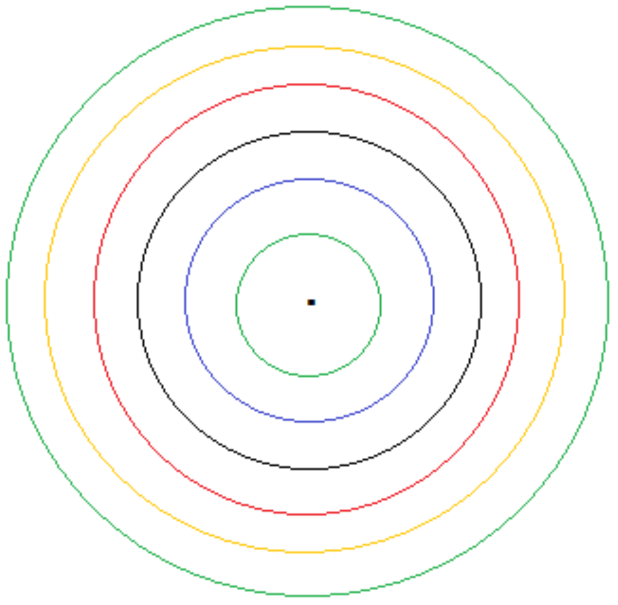
(involution) $$\{F_\mu, F_\nu\} = 0, \quad H = H(F_\mu)$$

EoMs can be solved by integration.

Example

Harmonic oscillator in 1D

$$H = \frac{1}{2}(p^2 + q^2)$$



$$M_f = \{(q,p) | q^2 + p^2 = 2F = R^2\}$$

Algebraic Methods

Suppose we can recast Hamilton's equations.

$$\frac{dL}{dt} = [M, L]$$

$$O_n \equiv \mathrm{tr} L^n, \qquad \frac{dO_n}{dt} = \sum_{i=0}^{n-1} \mathrm{tr} L^i [M, L] L^{n-1-i} = 0$$

Eigenvalues of L conserved in time. $$O_n = \sum_{\alpha} \lambda_{\alpha}^n$$

Example

Can define Lax pair for harmonic oscillator.

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix} = p\sigma_3 + \omega q \sigma_1, \quad M = \begin{pmatrix} 0 & -\frac{\omega}{2} \\ \frac{\omega}{2} & 0 \end{pmatrix} = -i\frac{\omega}{2}\sigma_2$$

$$\frac{dL}{dt} = [M, L] \quad \Rightarrow \quad \dot{p} = -\omega^2 q, \quad \dot{q} = p$$

Lie algebra $\qquad L, M \in \mathfrak{g}, \quad \mathfrak{g} = sl(2, \mathbb{C})$

Involution

Have Lax pair, conserved quantities, but not yet involution.

Theorem: Eigenvalues of L are in involution iff

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2],$$

$$L_1 \equiv L \times 1, \quad L_2 \equiv 1 \otimes L, \quad r_{12} \in \mathfrak{g} \otimes \mathfrak{g}$$

Jacobi satisfied if r_{12} constant and skew-symmetric

(CYBE) $$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

Classical Yang-Baxter Equation

$$[T_i, T_j] = f_{ij}{}^k T_k, \quad T_i \in \mathfrak{g}$$

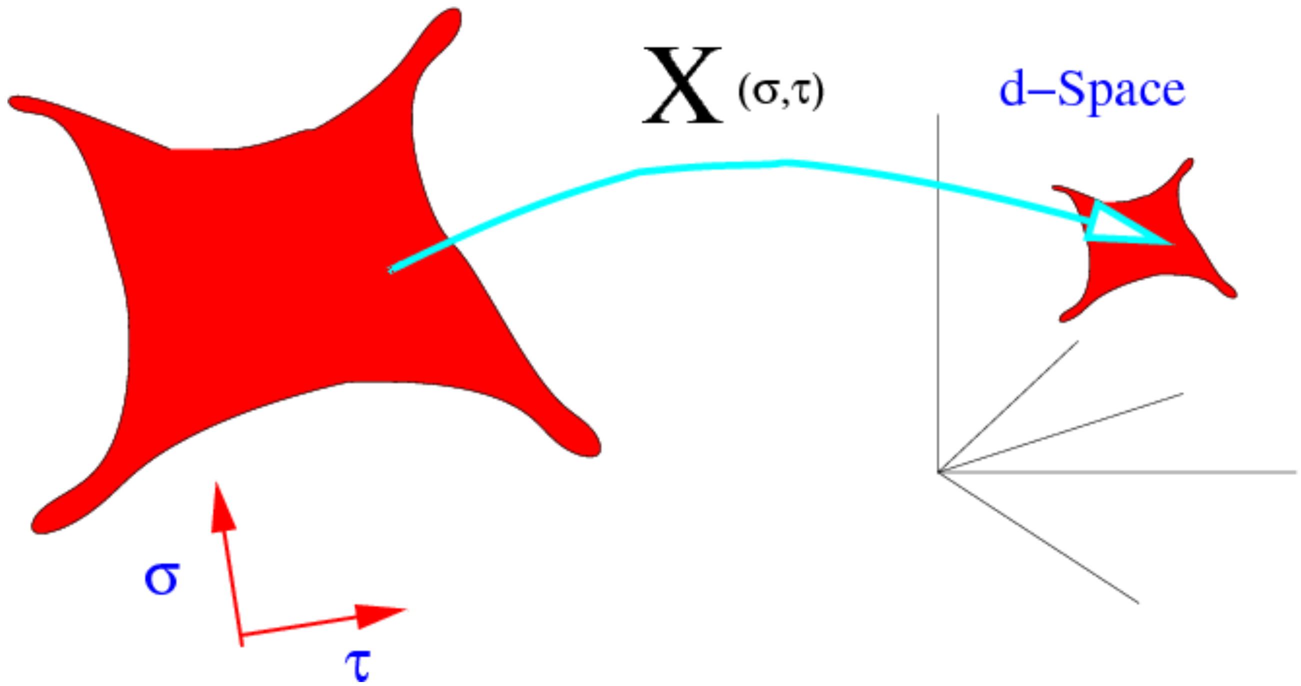
r-matrix $\quad r = r^{ij} T_i \wedge T_j, \quad r^{ij} = -r^{ji}$

(homogeneous) CYBE takes the form

$$f_{l_1 l_2}{}^i r^{l_1 j} r^{l_2 k} + f_{l_1 l_2}{}^j r^{l_1 k} r^{l_2 i} + f_{l_1 l_2}{}^k r^{l_1 i} r^{l_2 j} = 0$$

1. Classical limit of QYB $\quad \hbar \to 0$
2. Schouten bracket from diff. geometry (R-flux in DFT)

Cartoon



2D sigma-model | target-space geometry

one-loop beta-functions | "supergravity" equations

Gravity theory

Consider non-linear sigma-model with target manifold M

$$S = \frac{1}{2} \int \mathrm{d}^2 x \, g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j , \quad i = 1, \ldots, d$$

$$\beta_{ij} = \mu \frac{\partial}{\partial \mu} g_{ij} = -\frac{\hbar}{2\pi} R_{ij}$$

Friedan (1980)

UV finiteness $\quad R_{ij} = \nabla_{(i} v_{j)} , \quad (v_i = \partial_i \Phi)$

scale versus conformal symmetry

Gravity theory

Wess-Zumino term $$S_{WZ} = \frac{1}{2}\int \mathrm{d}^2x\sqrt{\gamma}\epsilon^{\mu\nu}b_{ij}(\phi)\partial_\mu\phi^i\partial_\nu\phi^j$$

connection with torsion $$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk} + H^i_{jk} \quad H = \mathrm{d}b$$

Curtright, Zachos (1984)

beta-functions for sigma-model with scale symmetry

$$\hat{R}_{ij} \equiv \hat{R}^k{}_{ikj} = \nabla_{(i}v_{j)} + H_{ij}{}^k v_k + \partial_{[i}\lambda_{j]}$$

Generalized Supergravity

Arutyunov, Hoare, Frolov, Roiban, Tseytlin (2015)

$$\beta_{B_{\mu\nu}} = -\frac{1}{2}\nabla^\rho H_{\rho\mu\nu} + X^\rho H_{\rho\mu\nu} + \nabla_\mu X_\nu - \nabla_\nu X_\mu,$$

$$\beta_{G_{\mu\nu}} = R_{\mu\nu} - \frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + \nabla_\mu X_\nu + \nabla_\nu X_\mu,$$

$$\beta_\Phi = R - \frac{1}{12}H^2 + 4\nabla_\mu X^\mu - 4X_\mu X^\mu,$$

$$X_\mu \equiv \partial_\mu \Phi + (g_{\nu\mu} + B_{\nu\mu})I^\nu$$

(we omit RR sector)

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Comment on GS: new or not?

T-duality is a symmetry of string sigma-model.

Buscher (1988)

Can apply Buscher procedure to non-Abelian isometries.

de la Ossa, Quevedo (1992)

A Problem with Non–Abelian Duality

M. Gasperini R. Ricci G. Veneziano

(Submitted on 24 Aug 1993 (v1), last revised 7 Sep 1993 (this version, v2))

We investigate duality transformations in a class of backgrounds with non–Abelian isometries, i.e. Bianchi–type (homogeneous) cosmologies in arbitrary dimensions. Simple duality transformations for the metric and the antisymmetric tensor field, generalizing those known from the Abelian isometry (Bianchi I) case, are obtained using either a Lagrangian or a Hamiltonian approach. Applying these prescriptions to a specific conformally invariant $\s$ –model, we show that no dilaton transformation leads to a new conformal background. Some possible ways out of the problem are suggested.

Anomaly

Non-semisimple groups: there is a gauge-gravity anomaly.

Alvarez, Alvarez-Gaumé, Lozano (1994)

NATD sigma-model

$$S = \frac{1}{2\pi}\int \mathrm{d}^2 z \Big(F_{ij}\partial x^i \bar{\partial} x^j + (2\Phi + \ln\det N)\partial\bar{\partial}\sigma$$

$$+ (\partial\lambda_a - \partial x^i F^L_{ia} + \mathrm{tr} T_a \partial\sigma) N^{ab} (\bar{\partial}\lambda_b + F^R_{bj}\bar{\partial} x^j - \mathrm{tr} T_b \bar{\partial}\sigma)\Big),$$

$$N = (E(x) + \lambda_c f^c)^{-1}, \quad (f^c)_{ab} = f^c_{ab}$$

Elitzur, Giveon, Rabinovici, Schwimmer, Veneziano (1994)

Resolution

EGRSV showed that Bianchi V NA T-dual satisfies **new equations** that include anomaly.

NATD of Bianchi V is also solution to GS.

Fernandez-Melgarejo, Sakamoto, Sakatani, Yoshida (2017)

We showed

1. Killing vector is trace of structure constant $I^i = f_{ji}^{\ \ j}$

2. variation of anomaly term recovers EoMs of GS

Hong, Kim, Ó C (2018)

“APS-phobic” Paper

Classical Yang-Baxter Equation from Supergravity

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[4] *School of Physics, Institute for Research in Fundamental Sciences (IPM), P.O.Box 19395-5531, Tehran, Iran*

We promote the open-closed string map, originally formulated by Seiberg & Witten, to a solution generating prescription in generalized supergravity. The approach hinges on a knowledge of an

Assume metric and scalar $$(G, \Phi)$$

Generate metric, B-field $$(G^{-1} + \Theta)^{-1} = g + B$$

Seiberg, Witten (1999)

T-duality invariant $$e^{-2\phi}\sqrt{-g} = e^{-2\Phi}\sqrt{-G}$$

Add a little magic $$I^\mu = \nabla_\nu^{(G)} \Theta^{\nu\mu}$$

What is the input?

For simple examples (2D) can solve for deformation.

But assuming it is bi-Killing, nice things happen.

$$\Theta^{\mu\nu} = r^{ij} K_i^\mu K_j^\nu \quad r^{ij} = -r^{ji}$$

$$I^\mu = \frac{1}{2} r^{ij} f_{ij}{}^k K_k^\mu$$

When vectors commute, we recover TsT transformations.

Lunin, Maldacena (2004)

Example 1

Flat spacetime in 3D

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

Killing vectors

$$T_1 = \partial_t, \quad T_2 = \partial_x, \quad T_3 = \partial_y,$$
$$T_4 = t\partial_x + x\partial_t, \quad T_5 = t\partial_y + y\partial_t, \quad T_6 = x\partial_y - y\partial_x$$

commutation relations

$$[T_1, T_4] = T_2, \quad [T_1, T_5] = T_3, \quad [T_2, T_4] = T_1, \quad [T_2, T_6] = T_3, \quad [T_3, T_5] = T_1,$$
$$[T_3, T_6] = -T_2, \quad [T_4, T_5] = T_6, \quad [T_4, T_6] = T_5, \quad [T_5, T_6] = -T_4$$

Example 1

Now consider the candidate r-matrix

$$r = \alpha T_4 \wedge T_5 + \beta T_5 \wedge T_6 + \gamma T_6 \wedge T_4$$

Impose CYBE: $$r^{45} = \alpha, \quad r^{56} = \beta, \quad r^{64} = \gamma$$

$$\alpha^2 = \beta^2 + \gamma^2$$

Rewrite r-matrix as deformation parameter

$$\Theta = \alpha(t\partial_x + x\partial_t) \wedge (t\partial_y + y\partial_t) + \beta(t\partial_y + y\partial_t) \wedge (x\partial_y - y\partial_x) + \gamma(x\partial_y - y\partial_x) \wedge (t\partial_x + x\partial_t)$$

Example 1

$$\Theta^{tx} = -y\Delta, \quad \Theta^{ty} = x\Delta, \quad \Theta^{xy} = t\Delta,$$
$$\Delta \equiv \alpha t + \beta y - \gamma x$$

Killing vector: $I = \alpha(x\partial_y - y\partial_x) - \beta(t\partial_x + x\partial_t) - \gamma(y\partial_t + t\partial_y)$

Deformed solution:

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{1}{[1+\Delta^2(t^2-x^2-y^2)]}\Big[-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}z^2$$
$$-\Delta^2\left[(t\mathrm{d}t - x\mathrm{d}x)^2 + (t\mathrm{d}t - y\mathrm{d}y)^2 + (x\mathrm{d}x + y\mathrm{d}y)^2\right]\Big],$$
$$B = -\frac{1}{[1+\Delta^2(t^2-x^2-y^2)]}\left(y\mathrm{d}t\wedge\mathrm{d}x + t\mathrm{d}x\wedge\mathrm{d}y + x\mathrm{d}y\wedge\mathrm{d}t\right),$$
$$\phi = -\frac{1}{2}\log\left[1+\Delta^2(t^2-x^2-y^2)\right]$$

Example 2

AdS_2 x S^2 (2D example)

$$ds^2 = \frac{(-dt^2 + dz^2)}{z^2} + d\zeta^2 + \sin^2\zeta d\chi^2$$

Ansatz $$\Theta^{tz} = \Theta_1(t,z), \quad \Theta^{\zeta\chi} = \Theta_2(\zeta,\chi)$$

Solution $$\Theta_1 = c_1 tz + c_2 z(t^2 - z^2) + c_3 z$$
$$\Theta_2 = c_4 \cos\chi + c_5 \sin\chi + c_6 \cot\zeta$$

Constraint $$0 = -c_1^2 + 4c_2c_3 = c_4^2 + c_5^2 + c_6^2$$

Example 2

No deformation of S^2 - reflects simply CYBE for su(2).

$$T_1 = -t\partial_t - z\partial_z, \quad T_2 = -\partial_t, \quad T_3 = -(t^2 + z^2)\partial_t - 2tz\partial_z$$

r-matrix solution to the CYBE for sl(2)

$$r = -c_3 T_1 \wedge T_2 + \frac{c_1}{2} T_2 \wedge T_3 - c_2 T_3 \wedge T_1$$

deformation specified by EoMs

Perturbative Proof

Expand in NC parameter, plug into GS EOMs.

$$g_{\mu\nu} = G_{\mu\nu} + \Theta_{\mu}{}^{\alpha}\Theta_{\alpha\nu} + \mathcal{O}(\Theta^4),$$
$$B_{\mu\nu} = -\Theta_{\mu\nu} - \Theta_{\mu\alpha}\Theta^{\alpha\beta}\Theta_{\beta\nu} + \mathcal{O}(\Theta^5),$$
$$\phi = \Phi + \frac{1}{4}\Theta_{\rho\sigma}\Theta^{\rho\sigma} + \mathcal{O}(\Theta^4)$$

$$K_i^{\alpha} K_k^{\beta} \nabla_{\alpha} K_{\beta m} \left(f_{l_1 l_2}{}^{m} r^{il_1} r^{kl_2} + f_{l_1 l_2}{}^{k} r^{ml_1} r^{il_2} + f_{l_1 l_2}{}^{i} r^{kl_1} r^{ml_2} \right) + \left(\Theta^{\beta\gamma}\Theta^{\alpha\lambda} + \Theta^{\alpha\beta}\Theta^{\gamma\lambda} + \Theta^{\gamma\alpha}\Theta^{\beta\lambda} \right) R_{\beta\gamma\alpha\lambda} = 0.$$

Bakhmatov, Ó C, Sheikh-Jabbari, Yavartanoo (2018)

Summary

- "Supergravity" is equivalent to the CYBE through open-closed string map of Seiberg-Witten.
- Easy to now deform geometries.
- Can we classify solutions to CYBE (math problem) using gravity?
- Can we prove claim?
- Do higher-order corrections capture the QYB?