Einstein Double Field Equations

$$G_{AB} = 8\pi G T_{AB}$$

Hereafter A, B are $\mathbf{O}(D,D)$ indices

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Prologue

– GR is based on Riemannian geometry, where the only geometric and gravitational field is the Riemannian metric, $g_{\mu\nu}$. Other fields are meant to be extra matters.

– On the other hand, string theory suggests us to put a two-form gauge potential, $B_{\mu\nu}$, and a scalar dilaton, ϕ, on an equal footing along with the metric:

- They form the closed string massless sector, being ubiquitous in all string theories,

$$\int \mathrm{d}^D x \, \sqrt{-g} e^{-2\phi} \left(R_g + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right) \qquad \text{where} \qquad H = \mathrm{d}B \,.$$

This action hides $\mathbf{O}(D, D)$ symmetry of T-duality which transforms g, B, ϕ into one another. Buscher 1987

– T-duality hints at a natural augmentation to General Relativity, in which the entire closed string massless sector constitutes the fundamental gravitational multiplet and the above action corresponds to a 'pure' gravity.

Double Field Theory (DFT), initiated by Sielgel 1993 & Hull-Zwiebach 2009-2010, turns out to provide a concrete realization for this idea of Stringy Gravity by manifesting $\mathbf{O}(D, D)$ T-duality.

– This talk sketches the geometric construction of Stringy Gravity, and in particular, introduces Einstein Double Field Equations, $G_{AB} = 8\pi G T_{AB}$, as the unifying single expression for all the equations of motion of the closed string massless sector.

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DFT as Stringy Gravity

Notation for $\mathbf{O}(D,D)$ and $\mathbf{Spin}(1,D{-}1)_L \times \mathbf{Spin}(D{-}1,1)_R$ Symmetries

Index	Representation	Metric (raising/lowering indices)
$A, B, \cdots, M, N, \cdots$	$\mathbf{O}(D,D)$ vector	$\mathcal{J}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$p, q, \cdots$	$\mathbf{Spin}(1,D{-}1)_L$ vector	$\eta_{pq} = \mathrm{diag}(-++\cdots+)$
$\alpha, \beta, \cdots$	$\mathbf{Spin}(1,D{-}1)_L$ spinor	$C_{\alpha\beta}, \qquad (\gamma^p)^T = C\gamma^p C^{-1}$
$\bar{p}, \bar{q}, \cdots$	$\mathbf{Spin}(D{-}1,1)_R$ vector	$\bar{\eta}_{\bar{p}\bar{q}} = \mathrm{diag}(+--\cdots-)$
$\bar{\alpha}, \bar{\beta}, \cdots$	$\mathbf{Spin}(D{-}1,1)_R$ spinor	$\bar{C}_{\bar{\alpha}\bar{\beta}}, \qquad (\bar{\gamma}^{\bar{p}})^T = \bar{C}\bar{\gamma}^{\bar{p}}\bar{C}^{-1}$

– Each symmetry rotates its own indices *exclusively*: spinors are $\mathbf{O}(D,D)$ singlet.

– The constant $\mathbf{O}(D,D)$ metric, $\mathcal{J}_{AB}$, decomposes the doubled coordinates into two parts,

$$x^A = (\tilde{x}_\mu, x^\nu)\,, \qquad\qquad \partial_A = (\tilde{\partial}^\mu, \partial_\nu)\,,$$

where μ, ν are D-dimensional curved indices.

– The twofold local Lorentz symmetries indicate two distinct locally inertial frames for the left-moving and the right-moving closed string sectors separately.

- **Closed string massless sector as ‘Stringy Graviton Fields’**

  The stringy graviton fields consist of the DFT dilaton, d, and DFT metric, $\mathcal{H}_{MN}$:

$$\mathcal{H}_{MN}=\mathcal{H}_{NM}\,,\qquad\qquad \mathcal{H}_{K}{}^{L}\mathcal{H}_{M}{}^{N}\mathcal{J}_{LN}=\mathcal{J}_{KM}\,.$$

  Combining $\mathcal{J}_{MN}$ and $\mathcal{H}_{MN}$, we acquire a pair of symmetric projection matrices,

$$P_{MN}=P_{NM}=\tfrac{1}{2}(\mathcal{J}_{MN}+\mathcal{H}_{MN})\,,\qquad P_{L}{}^{M}P_{M}{}^{N}=P_{L}{}^{N}\,,$$
$$\bar{P}_{MN}=\bar{P}_{NM}=\tfrac{1}{2}(\mathcal{J}_{MN}-\mathcal{H}_{MN})\,,\qquad \bar{P}_{L}{}^{M}\bar{P}_{M}{}^{N}=\bar{P}_{L}{}^{N}\,,$$

  which are orthogonal and complete,

$$P_{L}{}^{M}\bar{P}_{M}{}^{N}=0\,,\qquad\qquad P_{M}{}^{N}+\bar{P}_{M}{}^{N}=\delta_{M}{}^{N}\,.$$

  Further, taking the “square roots" of the projectors,

$$P_{MN}=V_{M}{}^{p}V_{N}{}^{q}\eta_{pq}\,,\qquad\qquad \bar{P}_{MN}=\bar{V}_{M}{}^{\bar{p}}\bar{V}_{N}{}^{\bar{q}}\bar{\eta}_{\bar{p}\bar{q}}\,,$$

  we get a pair of DFT vielbeins:

$$V_{Mp}V^{M}{}_{q}=\eta_{pq}\,,\qquad \bar{V}_{M\bar{p}}\bar{V}^{M}{}_{\bar{q}}=\bar{\eta}_{\bar{p}\bar{q}}\,,\qquad V_{Mp}\bar{V}^{M}{}_{\bar{q}}=0\,,\qquad V_{M}{}^{p}V_{Np}+\bar{V}_{M}{}^{\bar{p}}\bar{V}_{N\bar{p}}=\mathcal{J}_{MN}\,.$$

Classification of DFT backgrounds, 1707.03713 with Kevin Morand

The most general form of the DFT metric, $\mathcal{H}_{MN}=\mathcal{H}_{NM}$, $\mathcal{H}_{K}{}^{L}\mathcal{H}_{M}{}^{N}\mathcal{J}_{LN}=\mathcal{J}_{KM}$, is characterized by two non-negative integers, $(n,\bar{n})$, $0\leq n+\bar{n}\leq D$:

$$\mathcal{H}_{AB}=\begin{pmatrix} H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda}+Y_{i}^{\mu}X_{\lambda}^{i}-\bar{Y}_{\bar{\imath}}^{\mu}\bar{X}_{\lambda}^{\bar{\imath}} \\ B_{\kappa\rho}H^{\rho\nu}+X_{\kappa}^{i}Y_{i}^{\nu}-\bar{X}_{\kappa}^{\bar{\imath}}\bar{Y}_{\bar{\imath}}^{\nu} & K_{\kappa\lambda}-B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda}+2X_{(\kappa}^{i}B_{\lambda)\rho}Y_{i}^{\rho}-2\bar{X}_{(\kappa}^{\bar{\imath}}B_{\lambda)\rho}\bar{Y}_{\bar{\imath}}^{\rho} \end{pmatrix}$$

i) Symmetric and skew-symmetric fields: $H^{\mu\nu}=H^{\nu\mu}$, $K_{\mu\nu}=K_{\nu\mu}$, $B_{\mu\nu}=-B_{\nu\mu}$;

ii) Two kinds of eigenvectors having zero eigenvalue, with $i,j=1,2,\cdots,n$ & $\bar{\imath},\bar{\jmath}=1,2,\cdots,\bar{n}$,

$$H^{\mu\nu}X_{\nu}^{i}=0\,,\qquad H^{\mu\nu}\bar{X}_{\nu}^{\bar{\imath}}=0\,,\qquad K_{\mu\nu}Y_{j}^{\nu}=0\,,\qquad K_{\mu\nu}\bar{Y}_{\bar{\jmath}}^{\nu}=0\,;$$

iii) Completeness relation: $H^{\mu\rho}K_{\rho\nu}+Y_{i}^{\mu}X_{\nu}^{i}+\bar{Y}_{\bar{\imath}}^{\mu}\bar{X}_{\nu}^{\bar{\imath}}=\delta^{\mu}{}_{\nu}$.

- Orthonormality follows

$$Y_{i}^{\mu}X_{\mu}^{j}=\delta_{i}{}^{j}\,,\qquad \bar{Y}_{\bar{\imath}}^{\mu}\bar{X}_{\mu}^{\bar{\jmath}}=\delta_{\bar{\imath}}{}^{\bar{\jmath}}\,,\qquad Y_{i}^{\mu}\bar{X}_{\mu}^{\bar{\jmath}}=\bar{Y}_{\bar{\imath}}^{\mu}X_{\mu}^{j}=0\,.$$

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- It is instructive to note the $\mathbf{O}(D,D)$ invariant trace, $\mathcal{H}_{A}{}^{A}=2(n-\bar{n})$ and

$$\mathcal{H}_{AB}=\left(\begin{array}{cc}1&0\\B&1\end{array}\right)\left(\begin{array}{cc}H & Y_{i}(X^{i})^{T}-\bar{Y}_{\bar{\imath}}(\bar{X}^{\bar{\imath}})^{T}\\ X^{i}(Y_{i})^{T}-\bar{X}^{\bar{\imath}}(\bar{Y}_{\bar{\imath}})^{T} & K\end{array}\right)\left(\begin{array}{cc}1&-B\\0&1\end{array}\right).$$

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- $(n,\bar{n})=(0,0)$ corresponds to the Riemannian geometry or "Generalized Geometry":

$$\mathcal{H}_{MN}\equiv\begin{pmatrix}g^{-1} & -g^{-1}B\\ Bg^{-1} & g-Bg^{-1}B\end{pmatrix}$$

Giveon-Rabinovici-Veneziano '89, Duff '90

- **String becomes chiral over the n dimensions and anti-chiral over the $\bar{n}$ dimensions:**

$$X^i_\mu\,\partial_+x^\mu(\tau,\sigma)\equiv 0\,,\qquad\qquad \bar{X}^{\bar{\imath}}_\mu\,\partial_-x^\mu(\tau,\sigma)\equiv 0\,.$$

Examples include

- $(D,0)$ Siegel's chiral string (maximally non-Riemannian, $\mathcal{H}_{MN}=\mathcal{J}_{MN}$);
- $(1,1)$ Gomis-Ooguri non-relativistic string   Ko-Melby-Thompson-Meyer-JHP 2015 ;
- $(D{-}1,0)$ ultra-relativistic Carroll gravity;
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$$\hat{\mathcal{L}}_{\xi} T_{A_1\cdots A_n} := \xi^B\partial_B T_{A_1\cdots A_n} + \omega_T\,\partial_B\xi^B T_{A_1\cdots A_n} + \sum_{i=1}^{n}(\partial_{A_i}\xi_B - \partial_B\xi_{A_i})T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n}\,,$$

where ω_T is the weight, *e.g.* $\delta e^{-2d} = \partial_B(\xi^B e^{-2d})$, $\delta V_{Ap} = \xi^B\partial_B V_{Ap} + (\partial_A\xi_B - \partial_B\xi_A)V^B{}_p$.

- For consistency, so-called the ‘section condition’ should be imposed: $\partial_M\partial^M = 0$.

  From $\partial_M\partial^M = 2\partial_\mu\tilde{\partial}^\mu$, the section condition can be easily solved by letting $\tilde{\partial}^\mu = 0$.

  The general solutions are then generated by the **O**(D, D) rotation of it.

- The section condition is mathematically equivalent to certain translational invariance:

$$\Phi_i(x) = \Phi_i(x+\Delta)\,, \qquad \Delta^M = \Phi_j\partial^M\Phi_k\,,$$

where $\Phi_i, \Phi_j, \Phi_k \in \{\, d, \mathcal{H}_{MN}\,, \xi^M\,, \partial_N d\,, \partial_L\mathcal{H}_{MN}, \cdots\}$, arbitrary functions appearing in DFT, and Δ^M is said to be derivative-index-valued.

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Doubled-yet-gauged spacetime

JHP 1304.5946

Doubled coordinates, $x^M=(\tilde{x}_\mu\,,x^\nu)$, are gauged through an equivalence relation,

$$x^M \;\sim\; x^M+\Delta^M(x)\,,$$

where Δ^M is derivative-index-valued.

Each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^D$.

- If we solve the section condition by letting $\tilde{\partial}^\mu\equiv 0$, and further choose $\Delta^M=c_\mu\partial^M x^\mu$, we note

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Doubled-yet-gauged spacetime

JHP 1304.5946

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it is possible to define **O**(D, D) & diffeomorphism covariant 'proper length' through a path integral,

$$\textbf{Proper Length} := -\ln\left[\int \mathcal{D}\mathcal{A} \,\exp\left(-\int \sqrt{Dx^M Dx^N \mathcal{H}_{MN}}\right)\right] ,$$

and construct associated sigma models such as for the point particle Ko-JHP-Suh '16, Blair '17, bosonic strings Hull '06, Lee-JHP '13, Arvanitakis-Blair '17, '18, κ-symmetric Green-Schwarz superstring JHP '16.

In particular, for the (0, 0) Riemannian DFT-metric, with $\tilde{\partial}^\mu \equiv 0$, after integrating out the auxiliary potential, $\mathcal{A}^M = A_\lambda \partial^M x^\lambda = (A_\mu , 0)$, one can recover all the conventional results, *e.g.*

$$Dx^M = (\mathrm{d}\tilde{x}_\mu - A_\mu , \mathrm{d}x^\nu) , \qquad \textbf{Proper Length} \Longrightarrow \int \sqrt{\mathrm{d}x^\mu \mathrm{d}x^\nu g_{\mu\nu}(x)} ,$$

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Covariant derivatives and curvatures in Stringy Gravity feature two stages:

‘**semi-covariance**’ and ‘**complete covariantization**’.

- **Semi-covariant derivative :** Jeon-Lee-JHP 2010, 2011

$$\nabla_C T_{A_1A_2\cdots A_n} := \partial_C T_{A_1A_2\cdots A_n} - \omega_T \Gamma^B{}_{BC} T_{A_1A_2\cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}\,,$$

for which the stringy Christoffel connection can be uniquely fixed,

$$\Gamma_{CAB}=2\big(P\partial_C P\bar{P}\big)_{[AB]}+2\Big(\bar{P}_{[A}{}^D\bar{P}_{B]}{}^E-P_{[A}{}^DP_{B]}{}^E\Big)\partial_D P_{EC}-\tfrac{4}{D-1}\Big(\bar{P}_{C[A}\bar{P}_{B]}{}^D+P_{C[A}P_{B]}{}^D\Big)\Big(\partial_D d+(P\partial^E P\bar{P})_{[ED]}\Big)$$

by demanding the compatibility, $\nabla_A P_{BC} = \nabla_A \bar{P}_{BC} = \nabla_A d = 0$, and some torsionless conditions.

$*$ There are no normal coordinates where Γ_{CAB} would vanish point-wise: Equivalence Principle is broken for string (*i.e.* extended object) but recoverable for particle.

- **Semi-covariant Riemann curvature :**

$$S_{ABCD} = S_{[AB][CD]} = S_{CDAB} := \tfrac{1}{2}\left(R_{ABCD} + R_{CDAB} - \Gamma^E{}_{AB}\Gamma_{ECD}\right)\,, \qquad S_{[ABC]D} = 0\,,$$

where R_{ABCD} denotes the ordinary "field strength": $R_{CDAB}=\partial_A\Gamma_{BCD}-\partial_B\Gamma_{ACD}+\Gamma_{AC}{}^E\Gamma_{BED}-\Gamma_{BC}{}^E\Gamma_{AED}$.

By construction, it varies as 'total derivative': $\delta S_{ABCD} = \nabla_{[A}\delta\Gamma_{B]CD} + \nabla_{[C}\delta\Gamma_{D]AB}$.

- **Semi-covariant 'Master' derivative :**

$$\mathcal{D}_A := \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A = \nabla_A + \Phi_A + \bar{\Phi}_A\,.$$

The two spin connections for the **Spin**$(1, D{-}1)_L \times$ **Spin**$(D{-}1, 1)_R$ local Lorentz symmetries are determined in terms of the stringy Christoffel connection by requiring the compatibility with DFT vielbeins,

$$\mathcal{D}_A V_{Bp} = \nabla_A V_{Bp} + \Phi_{Ap}{}^q V_{Bq} = 0\,, \qquad \mathcal{D}_A \bar{V}_{B\bar{p}} = \nabla_A \bar{V}_{B\bar{p}} + \bar{\Phi}_{A\bar{p}}{}^{\bar{q}} \bar{V}_{B\bar{q}} = 0\,.$$

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- **Complete covariantization**
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  - RR sector, $\mathcal{C}^\alpha{}_{\bar{\alpha}}$ **O**(D, D) *covariant extension of H-twisted cohomology*

$$\mathcal{D}_\pm \mathcal{C} := \gamma^p \mathcal{D}_p \mathcal{C} \pm \gamma^{(D+1)} \mathcal{D}_{\bar{p}} \mathcal{C} \bar{\gamma}^{\bar{p}} \,, \quad (\mathcal{D}_\pm)^2 = 0 \quad \Longrightarrow \quad \mathcal{F} := \mathcal{D}_+ \mathcal{C} \quad (\text{RR flux}) \,.$$

  - Yang-Mills,

$$\mathcal{F}_{p\bar{q}} := \mathcal{F}_{AB} V^A{}_p \bar{V}^B{}_{\bar{q}} \qquad \text{where} \qquad \mathcal{F}_{AB} := \nabla_A \mathcal{V}_B - \nabla_B \mathcal{V}_A - i [\mathcal{V}_A, \mathcal{V}_B] \,.$$

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Equipped with the semi-covariant derivatives, one can construct, e.g.

- $D = 10$ Maximally Supersymmetric Double Field Theory — Jeon-Lee-JHP-Suh 2012

$$\mathcal{L}_{\rm type\,II} = e^{-2d}\Big[\, \tfrac{1}{8}S_{(0)} + \tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}}) + i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q} + i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}_{p}\rho - i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho' \\ -i\bar{\psi}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho - i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}_{q}\psi_{\bar{p}} + i\bar{\psi}'^{p}\mathcal{D}_{p}\rho' + i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}_{\bar{q}}\psi'_{p}\,\Big]$$

  which unifies IIA and IIB SUGRAs, thanks to the twofold spin groups.

- Minimal coupling to the Standard Model — Kangsin Choi & JHP 2015 [PRL]

$$\mathcal{L}_{\rm SM} = e^{-2d}\left[\begin{array}{l} \frac{1}{16\pi G_N}S_{(0)} \\ +\sum_{\mathcal{V}} P^{AB}\bar{P}^{CD}{\rm Tr}(\mathcal{F}_{AC}\mathcal{F}_{BD}) + \sum_{\psi}\bar{\psi}\gamma^{a}\mathcal{D}_{a}\psi + \sum_{\psi'}\bar{\psi}'\bar{\gamma}^{\bar{a}}\mathcal{D}_{\bar{a}}\psi' \\ -\mathcal{H}^{AB}(\mathcal{D}_A\phi)^{\dagger}\mathcal{D}_B\phi - V(\phi) + y_d\,\bar{q}\cdot\phi\, d + y_u\,\bar{q}\cdot\tilde{\phi}\,u + y_e\,\bar{l}'\cdot\phi\, e' \end{array}\right]$$

Every single term above is completely covariant, w.r.t. O(D, D), diffeomorphisms, and twofold local Lorentz symmetries, Spin$(1, D{-}1)_L \times$ Spin$(D{-}1, 1)_R$.

Derivation of the Einstein Double Field Equations

Henceforth, we consider a general action for Stringy Gravity coupled to matter fields, Υ_a,

$$\int_\Sigma e^{-2d}\Big[\,\tfrac{1}{16\pi G}S_{(0)} + L_{\rm matter}\,\Big]\,,$$

where $S_{(0)}$ is the stringy scalar curvature and $L_{\rm matter}$ is the matter Lagrangian equipped with the completely covariantized master derivatives, $\mathcal{D}_M$. The integral is taken over a section, Σ.

We seek the variation of the action induced by all the fields, d, V_{Ap}, $\bar{V}_{Ap}$, Υ_a.

Firstly, the pure Stringy Gravity term transforms, up to total derivatives ($\simeq$), as

$$\delta\left(e^{-2d}S_{(0)}\right) \simeq 4e^{-2d}\left(\bar{V}^{B\bar{q}}\delta V_B{}^p S_{p\bar{q}} - \tfrac{1}{2}\delta d\, S_{(0)}\right)$$

Secondly, the matter Lagrangian transforms as

$$\delta\Big(e^{-2d}L_{\rm matter}\Big) \simeq e^{-2d}\left(-2\bar{V}^{A\bar{q}}\delta V_A{}^p K_{p\bar{q}} + \delta d\, T_{(0)} + \delta\Upsilon_a\frac{\delta L_{\rm matter}}{\delta\Upsilon_a}\right)$$

where we have been naturally led to define

$$K_{p\bar{q}} := \frac{1}{2}\left(V_{Ap}\frac{\delta L_{\rm matter}}{\delta\bar{V}_A{}^{\bar{q}}} - \bar{V}_{A\bar{q}}\frac{\delta L_{\rm matter}}{\delta V_A{}^p}\right)\,, \qquad T_{(0)} := e^{2d}\times\frac{\delta\left(e^{-2d}L_{\rm matter}\right)}{\delta d}\,.$$

In particular, when $L_{\rm matter}$ is bosonic (free of vielbeins), the former reduces to

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- Combining the two results, the variation of the action reads

$$\delta\int_{\Sigma}e^{-2d}\left[\tfrac{1}{16\pi G}S_{(0)}+L_{\rm matter}\right]$$
$$=\int_{\Sigma}e^{-2d}\left[\tfrac{1}{4\pi G}\bar{V}^{A\bar{q}}\delta V_{A}{}^{p}(S_{p\bar{q}}-8\pi GK_{p\bar{q}})-\tfrac{1}{8\pi G}\delta d(S_{(0)}-8\pi GT_{(0)})+\delta\Upsilon_{a}\frac{\delta L_{\rm matter}}{\delta\Upsilon_{a}}\right]$$

Hence, the equations of motion are exhaustively,

$$S_{p\bar{q}}=8\pi GK_{p\bar{q}}\,,\qquad\qquad S_{(0)}=8\pi GT_{(0)}\,,\qquad\qquad \frac{\delta L_{\rm matter}}{\delta\Upsilon_{a}}=0\,.$$

- Specifically when the variation is generated by diffeomorphisms, we have $\delta_{\xi}\Upsilon_{a}=\hat{\mathcal{L}}_{\xi}\Upsilon_{a}$ and

$$\delta_{\xi}d=-\tfrac{1}{2}e^{2d}\hat{\mathcal{L}}_{\xi}\left(e^{-2d}\right)=-\tfrac{1}{2}\mathcal{D}_{A}\xi^{A}\,,\qquad \bar{V}^{A\bar{q}}\delta_{\xi}V_{A}{}^{p}=\bar{V}^{A\bar{q}}\hat{\mathcal{L}}_{\xi}V_{A}{}^{p}=2\mathcal{D}_{[A}\xi_{B]}\bar{V}^{A\bar{q}}V^{Bp}\,.$$

Substituting these, the diffeomorphic invariance of the action implies

$$0=\int_{\Sigma}e^{-2d}\left[\tfrac{1}{8\pi G}\xi^{B}\mathcal{D}^{A}\left\{4V_{[A}{}^{p}\bar{V}_{B]}{}^{\bar{q}}(S_{p\bar{q}}-8\pi GK_{p\bar{q}})-\tfrac{1}{2}\mathcal{J}_{AB}(S_{(0)}-8\pi GT_{(0)})\right\}+\delta_{\xi}\Upsilon_{a}\frac{\delta L_{\rm matter}}{\delta\Upsilon_{a}}\right]$$

which leads to the definitions of the off-shell conserved stringy Einstein curvature ,

$$G_{AB}:=4V_{[A}{}^{p}\bar{V}_{B]}{}^{\bar{q}}S_{p\bar{q}}-\tfrac{1}{2}\mathcal{J}_{AB}S_{(0)}\,,\qquad\qquad \mathcal{D}_{A}G^{AB}=0\qquad\text{(off-shell)}\,,$$

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and the on-shell conserved stringy Energy-Momentum tensor ,

$$T_{AB}:=4V_{[A}{}^{p}\bar{V}_{B]}{}^{\bar{q}}K_{p\bar{q}}-\tfrac{1}{2}\mathcal{J}_{AB}T_{(0)}\,,\qquad\qquad \mathcal{D}_{A}T^{AB}=0\qquad\text{(on-shell)}\,.$$

- Combining the two results, the variation of the action reads

$$\delta\int_{\Sigma}e^{-2d}\Big[\tfrac{1}{16\pi G}S_{(0)}+L_{\rm matter}\Big]$$
$$=\int_{\Sigma}e^{-2d}\Big[\tfrac{1}{4\pi G}\bar{V}^{A\bar{q}}\delta V_{A}{}^{p}(S_{p\bar{q}}-8\pi GK_{p\bar{q}})-\tfrac{1}{8\pi G}\delta d(S_{(0)}-8\pi GT_{(0)})+\delta\Upsilon_{a}\frac{\delta L_{\rm matter}}{\delta\Upsilon_{a}}\Big]$$

Hence, the equations of motion are exhaustively,

$$S_{p\bar{q}}=8\pi GK_{p\bar{q}}\,,\qquad S_{(0)}=8\pi GT_{(0)}\,,\qquad \frac{\delta L_{\rm matter}}{\delta\Upsilon_{a}}=0\,.$$

- Specifically when the variation is generated by diffeomorphisms, we have $\delta_{\xi}\Upsilon_{a}=\hat{\mathcal{L}}_{\xi}\Upsilon_{a}$ and

$$\delta_{\xi}d=-\tfrac{1}{2}e^{2d}\hat{\mathcal{L}}_{\xi}\left(e^{-2d}\right)=-\tfrac{1}{2}\mathcal{D}_{A}\xi^{A}\,,\qquad \bar{V}^{A\bar{q}}\delta_{\xi}V_{A}{}^{p}=\bar{V}^{A\bar{q}}\hat{\mathcal{L}}_{\xi}V_{A}{}^{p}=2\mathcal{D}_{[A}\xi_{B]}\bar{V}^{A\bar{q}}V^{Bp}\,.$$

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$$0=\int_{\Sigma}e^{-2d}\Big[\tfrac{1}{8\pi G}\xi^{B}\mathcal{D}^{A}\Big\{4V_{[A}{}^{p}\bar{V}_{B]}{}^{\bar{q}}(S_{p\bar{q}}-8\pi GK_{p\bar{q}})-\tfrac{1}{2}\mathcal{J}_{AB}(S_{(0)}-8\pi GT_{(0)})\Big\}+\delta_{\xi}\Upsilon_{a}\frac{\delta L_{\rm matter}}{\delta\Upsilon_{a}}\Big]$$

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- Since G_{AB} and T_{AB} each have D^2+1 independent components as *c.f.* $\{g_{\mu\nu},\, B_{\mu\nu},\, \phi\}$

$$V^A{}_p \bar{V}^B{}_{\bar{q}} G_{AB} = 2S_{p\bar{q}}\,, \qquad G^A{}_A = -DS_{(0)}\,, \qquad V^A{}_p \bar{V}^B{}_{\bar{q}} T_{AB} = 2K_{p\bar{q}}\,, \qquad T^A{}_A = -DT_{(0)}\,,$$

the equations of motion of the DFT vielbeins and dilaton can be unified into a single expression:

Einstein Double Field Equations

$$G_{AB} = 8\pi G T_{AB}$$

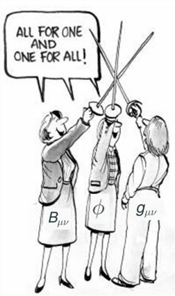
which is naturally consistent with the central idea that Stringy Gravity treats the entire closed string massless sector as geometrical stringy graviton fields.

Einstein Double Field Equations

$$G_{AB} = 8\pi G T_{AB}$$

- Restricting to the (0, 0) Riemannian backgrounds, the EDFE decompose into

$$\begin{aligned} R_{\mu\nu} + 2\bigtriangledown_{\mu}(\partial_{\nu}\phi) - \tfrac{1}{4}H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma} &= 8\pi G K_{(\mu\nu)}\,, \\ \bigtriangledown^{\rho}\left(e^{-2\phi}H_{\rho\mu\nu}\right) &= 16\pi G e^{-2\phi}K_{[\mu\nu]}\,, \\ R + 4\Box\phi - 4\partial_{\mu}\phi\partial^{\mu}\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} &= 8\pi G T_{(0)}\,. \end{aligned}$$



- For other non-Riemannian cases, $(n, \bar{n}) \neq (0, 0)$, EDFE govern the dynamics of the 'chiral' gravities, such as Newton-Cartan, Carroll, and Gomis-Ooguri, *etc.*

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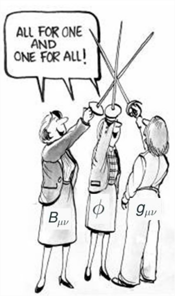

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Examples: $T_{AB} := 4V_{[A}{}^{p}\bar{V}_{B]}{}^{\bar{q}}K_{p\bar{q}} - \frac{1}{2}\mathcal{J}_{AB}T_{(0)}$

- Pure Stringy Gravity with cosmological constant,

$$\tfrac{1}{16\pi G}e^{-2d}\left(S_{(0)} - 2\Lambda_{\rm DFT}\right)\,,\qquad K_{p\bar{q}} = 0\,,\qquad T_{(0)} = \tfrac{1}{4\pi G}\Lambda_{\rm DFT}\,.$$

- RR sector,

$$L_{\rm RR} = \tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}})\,,\qquad K_{p\bar{q}} = -\tfrac{1}{4}{\rm Tr}(\gamma_p\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}})\,,\qquad T_{(0)} = 0\,.$$

- Spinor field,

$$L_\psi = \bar{\psi}\gamma^p\mathcal{D}_p\psi + m_\psi\bar{\psi}\psi\,,\qquad K_{p\bar{q}} = -\tfrac{1}{4}(\bar{\psi}\gamma_p\mathcal{D}_{\bar{q}}\psi - \mathcal{D}_{\bar{q}}\bar{\psi}\gamma_p\psi)\,,\qquad T_{(0)} \equiv 0\,.$$

- Green-Schwarz superstring (κ-symmetric, doubled-yet-gauged),

$$e^{-2d}L_{\rm string} = \tfrac{1}{4\pi\alpha'}\int{\rm d}^2\sigma\left[-\tfrac{1}{2}\sqrt{-h}h^{ij}\Pi_i^M\Pi_j^N\mathcal{H}_{MN} - \epsilon^{ij}D_iy^M(\mathcal{A}_{jM} - i\Sigma_{jM})\right]\delta^D(x - y(\sigma))\,,$$

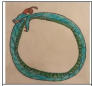
$$K_{p\bar{q}}(x) = \tfrac{1}{4\pi\alpha'}\int{\rm d}^2\sigma\sqrt{-h}h^{ij}(\Pi_i^MV_{Mp})(\Pi_j^N\bar{V}_{N\bar{q}})\,e^{2d}\delta^D(x - y(\sigma))\,,\qquad T_{(0)} = 0\,,$$

where $\Sigma_i^M = \bar{\theta}\gamma^M\partial_i\theta + \bar{\theta}'\bar{\gamma}^M\partial_i\theta'$ and $\Pi_i^M = \partial_iy^M - \mathcal{A}_i^M - i\Sigma_i^M$.

Gravitational effect

- The regular spherical solution to the $D = 4$ Einstein Double Field Equations shows that Stringy Gravity modifies GR (Schwarzschild geometry), in particular at “short" dimensionless scales, R/MG, *i.e.* distance normalized by mass times Newton constant.

  This might shed new light upon the dark matter/energy problems, as they arise essentially from “short distance" observations:

	Electron ($R \simeq 0$)	Proton	Hydrogen Atom	Billiard Ball	Earth	Solar System ($1\mathrm{AU}/M_{\odot}G$)	Milky Way (visible)	Galaxy Cluster	Universe ($M \propto R^3$)
$R/(MG)$	0^+	7.1×10^{38}	2.0×10^{43}	2.4×10^{26}	1.4×10^{9}	1.0×10^{8}	1.5×10^{6}	$\sim 10^{5}$	0^+

- Furthermore, it would be intriguing to view the B-field and DFT dilaton d as ‘dark gravitons’, since they decouple from the geodesic motion of point particles, which should be defined in string frame.

Concluding Remark

- It has been said that string theory is a piece of 21st century physics that happened to fall into the 20th century.
- String theory predicts its own gravity, *i.e.* Stringy Gravity, rather than GR.
- Stringy Gravity may be the 21st century theory of gravity, which is possibly formulated in 'doubled-yet-gauged' spacetime and deserves further explorations.

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Thank you

One must be prepared to follow up the consequence of theory, and feel that one just has to accept the consequences no matter where they lead.

– Paul Dirac –

Einstein Double Field Equations

Stephen Angus, Kyoungho Cho, and Jeong-Hyuck Park

Department of Physics, Sogang University, 35 Baekbeom-ro, Mapo-gu, Seoul 04107, KOREA

Core idea: string theory predicts its own gravity rather than GR

In General Relativity the metric $g_{\mu\nu}$ is the only geometric and gravitational field, whereas in string theory the closed-string massless sector comprises a two-form potential $B_{\mu\nu}$ and the string dilaton ϕ in addition to the metric $g_{\mu\nu}$. Furthermore, these three fields transform into each other under T-duality. This hints at a natural augmentation of GR: upon treating the whole closed string massless sector as stringy graviton fields, Double Field Theory [1, 2] may evolve into 'Stringy Gravity'. Equipped with an $\mathbf{O}(D,D)$ covariant differential geometry beyond Riemann [3], we spell out the definitions of the stringy Einstein curvature tensor and the stringy Energy-Momentum tensor. Equating them, all the equations of motion of the closed string massless sector are unified into a single expression [4],

$$G_{AB} = 8\pi G T_{AB} \tag{1}$$

which we dub the Einstein Double Field Equations.

Double Field Theory as Stringy Gravity

• Built-in symmetries & Notation:

– $\mathbf{O}(D,D)$ T-duality

– DFT diffeomorphisms (ordinary diffeomorphisms plus B-field gauge symmetry)

– Twofold local Lorentz symmetries, $\mathbf{Spin}(1,D{-}1)\times\mathbf{Spin}(D{-}1,1)$

⇒ Two locally inertial frames exist separately for the left and the right modes.

Index	Representation	Metric (raising/lowering indices)
$A,B,\cdots,M,N,\cdots$	$\mathbf{O}(D,D)$ vector	$\mathcal{J}_{AB}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$p,q,\cdots$	$\mathbf{Spin}(1,D{-}1)$ vector	$\eta_{pq}=\mathrm{diag}(-++\cdots+)$
$\alpha,\beta,\cdots$	$\mathbf{Spin}(1,D{-}1)$ spinor	$C_{\alpha\beta},\quad (\gamma^p)^T=C\gamma^pC^{-1}$
$\bar{p},\bar{q},\cdots$	$\mathbf{Spin}(D{-}1,1)$ vector	$\bar{\eta}_{\bar{p}\bar{q}}=\mathrm{diag}(+--\cdots-)$
$\bar{\alpha},\bar{\beta},\cdots$	$\mathbf{Spin}(D{-}1,1)$ spinor	$\bar{C}_{\bar{\alpha}\bar{\beta}},\quad (\bar{\gamma}^{\bar{p}})^T=\bar{C}\bar{\gamma}^{\bar{p}}\bar{C}^{-1}$

The $\mathbf{O}(D,D)$ metric $\mathcal{J}_{AB}$ divides doubled coordinates into two: $x^A=(\tilde{x}_\mu,x^\nu)$, $\partial_A=(\tilde{\partial}^\mu,\partial_\nu)$.

• Doubled-yet-gauged spacetime:
The doubled coordinates are 'gauged' through a certain equivalence relation, $x^A\sim x^A+\Delta^A$, such that each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^D$ [5]. This implies a section condition, $\partial_A\partial^A=0$, which can be conveniently solved by setting $\tilde{\partial}^\mu\equiv 0$.

• Stringy graviton fields (closed-string massless sector), $\{d, V_{Mp}, \bar{V}_{N\bar{q}}\}$:
Defining properties of the DFT-metric,

$$\mathcal{H}_{MN}=\mathcal{H}_{NM}\,,\qquad \mathcal{H}_K{}^L\mathcal{H}_M{}^N\mathcal{J}_{LN}=\mathcal{J}_{KM}\,, \tag{2}$$

set a pair of symmetric and orthogonal projectors,

$$P_{MN}=P_{NM}=\tfrac{1}{2}(\mathcal{J}_{MN}+\mathcal{H}_{MN})\,,\qquad P_L{}^MP_M{}^N=P_L{}^N\,,$$
$$\bar{P}_{MN}=\bar{P}_{NM}=\tfrac{1}{2}(\mathcal{J}_{MN}-\mathcal{H}_{MN})\,,\qquad \bar{P}_L{}^M\bar{P}_M{}^N=\bar{P}_L{}^N\,,\qquad P_L{}^M\bar{P}_M{}^N=0\,.$$

Further, taking the "square roots" of the projectors, we acquire a pair of DFT vielbeins,

$$P_{MN}=V_M{}^pV_N{}^q\eta_{pq}\,,\qquad \bar{P}_{MN}=\bar{V}_M{}^{\bar{p}}\bar{V}_N{}^{\bar{q}}\bar{\eta}_{\bar{p}\bar{q}}\,,$$

satisfying their own defining properties,

$$V_{Mp}V^M{}_q=\eta_{pq}\,,\quad \bar{V}_{M\bar{p}}\bar{V}^M{}_{\bar{q}}=\bar{\eta}_{\bar{p}\bar{q}}\,,\quad V_{Mp}\bar{V}^M{}_{\bar{q}}=0\,,\quad V_M{}^pV_{Np}+\bar{V}_M{}^{\bar{p}}\bar{V}_{N\bar{p}}=\mathcal{J}_{MN}\,.$$

The most general solutions to (2) can be classified by two non-negative integers $(n,\bar{n})$ [6],

$$\mathcal{H}_{MN}=\begin{pmatrix} H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda}+Y_i^\mu X_\lambda^i-\bar{Y}_{\bar{\imath}}^\mu\bar{X}_\lambda^{\bar{\imath}} \\ B_{\kappa\rho}H^{\rho\nu}+X_\kappa^iY_i^\nu-\bar{X}_\kappa^{\bar{\imath}}\bar{Y}_{\bar{\imath}}^\nu & K_{\kappa\lambda}-B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda}+2X^i_{(\kappa}B_{\lambda)\rho}Y_i^\rho-2\bar{X}^{\bar{\imath}}_{(\kappa}B_{\lambda)\rho}\bar{Y}_{\bar{\imath}}^\rho \end{pmatrix}$$

where $1\leq i\leq n$, $1\leq \bar{\imath},i\leq\bar{n}$ and

$$H^{\mu\nu}X_\nu^i=0\,,\quad H^{\mu\nu}\bar{X}_\nu^{\bar{\imath}}=0\,,\quad K_{\mu\nu}Y_i^\nu=0\,,\quad K_{\mu\nu}\bar{Y}_{\bar{\imath}}^\nu=0\,,\quad H^{\mu\rho}K_{\rho\nu}+Y_i^\mu X_\nu^i+\bar{Y}_{\bar{\imath}}^\mu\bar{X}_\nu^{\bar{\imath}}=\delta^\mu{}_\nu\,.$$

Strings become chiral and anti-chiral over n and $\bar{n}$ directions: $X_\mu^i\partial_+x^\mu=0$, $\bar{X}_\mu^{\bar{\imath}}\partial_-x^\mu=0$. Examples include $(0,0)$ Riemannian geometry as $K_{\mu\nu}=g_{\mu\nu}$, $H^{\mu\nu}=g^{\mu\nu}$, $(1,1)$ Gomis-Ooguri non-relativistic background, $(1,0)$ Newton-Cartan gravity, and $(D-1,0)$ Carroll gravity.

• Covariant derivative:
The 'master' covariant derivative, $\mathcal{D}_A=\partial_A+\Gamma_A+\Phi_A+\bar{\Phi}_A$, is characterized by compatibility:

$$\mathcal{D}_Ad=\mathcal{D}_AV_{Bp}=\mathcal{D}_A\bar{V}_{B\bar{p}}=0\,,\qquad \mathcal{D}_A\mathcal{J}_{BC}=\mathcal{D}_A\eta_{pq}=\mathcal{D}_A\bar{\eta}_{\bar{p}\bar{q}}=\mathcal{D}_AC_{\alpha\beta}=\mathcal{D}_A\bar{C}_{\bar{\alpha}\bar{\beta}}=0\,.$$

The stringy Christoffel symbols are [3]

$$\begin{aligned}\Gamma_{CAB}=&\,2\left(P\partial_CP\bar{P}\right)_{[AB]}+2\left(\bar{P}_{[A}{}^D\bar{P}_{B]}{}^E-P_{[A}{}^DP_{B]}{}^E\right)\partial_DP_{EC}\\ &-4\left(\tfrac{1}{P_M{}^M-1}P_{C[A}P_{B]}{}^D+\tfrac{1}{\bar{P}_M{}^M-1}\bar{P}_{C[A}\bar{P}_{B]}{}^D\right)\left(\partial_Dd+(P\partial^EP\bar{P})_{[ED]}\right)\,,\end{aligned}$$

and the spin connections are $\Phi_{Apq}=V^B{}_p(\partial_AV_{Bq}+\Gamma_{AB}{}^CV_{Cq})$, $\bar{\Phi}_{A\bar{p}\bar{q}}=\bar{V}^B{}_{\bar{p}}(\partial_A\bar{V}_{B\bar{q}}+\Gamma_{AB}{}^C\bar{V}_{C\bar{q}})$.
In Stringy Gravity, there are no normal coordinates where Γ_{CAB} would vanish point-wise: the Equivalence Principle holds for point particles but is generically broken for strings (i.e. extended objects).

• Scalar and 'Ricci' curvatures:
The semi-covariant Riemann curvature in Stringy Gravity is defined by

$$S_{ABCD}:=\tfrac{1}{2}\left(R_{ABCD}+R_{CDAB}-\Gamma^E{}_{AB}\Gamma_{ECD}\right)\,,$$

where $R_{CDAB}=\partial_A\Gamma_{BCD}-\partial_B\Gamma_{ACD}+\Gamma_{ACE}\Gamma_B{}^E{}_D-\Gamma_{BCE}\Gamma_A{}^E{}_D$ (the "field strength" of Γ_{CAB}). The completely covariant 'Ricci' and scalar curvatures are, with $S_{AB}=S_{ACB}{}^C$,

$$S_{p\bar{q}}:=V^A{}_p\bar{V}^B{}_{\bar{q}}S_{AB}\,,\qquad S_{(0)}:=\left(P^{AC}P^{BD}-\bar{P}^{AC}\bar{P}^{CD}\right)S_{ABCD}\,.$$

While $e^{-2d}S_{(0)}$ corresponds to the original DFT Lagrangian density [1, 2], or the 'pure' Stringy Gravity, the master covariant derivative fixes its minimal coupling to extra matter fields, *e.g.* type II maximally supersymmetric DFT [7] or the Standard Model [8].

Derivation of Einstein Double Field Equations

Variation of the action for Stringy Gravity coupled to generic matter fields, Υ_a, gives

$$\begin{aligned}&\delta\int e^{-2d}\Big[\tfrac{1}{16\pi G}S_{(0)}+L_{\mathrm{matter}}\Big]\\ =&\int e^{-2d}\Big[\tfrac{1}{4\pi G}\bar{V}^{A\bar{q}}\delta V_A{}^p(S_{p\bar{q}}-8\pi GK_{p\bar{q}})-\tfrac{1}{8\pi G}\delta d(S_{(0)}-8\pi GT_{(0)})+\delta\Upsilon_a\frac{\delta L_{\mathrm{matter}}}{\delta\Upsilon_a}\Big]\\ =&\int e^{-2d}\Big[\tfrac{1}{8\pi G}\xi^B\mathcal{D}^A\left\{G_{AB}-8\pi GT_{AB}\right\}+(\hat{\mathcal{L}}_\xi\Upsilon_a)\frac{\delta L_{\mathrm{matter}}}{\delta\Upsilon_a}\Big]\,,\end{aligned}$$

where the second line is for generic variations and the third line is specifically for diffeomorphic transformations. We are naturally led to define

$$K_{p\bar{q}}:=\frac{1}{2}\left(V_{Ap}\frac{\delta L_{\mathrm{matter}}}{\delta\bar{V}_A{}^{\bar{q}}}-\bar{V}_{A\bar{q}}\frac{\delta L_{\mathrm{matter}}}{\delta V_A{}^p}\right)\,,\qquad T_{(0)}:=e^{2d}\times\frac{\delta\left(e^{-2d}L_{\mathrm{matter}}\right)}{\delta d}\,,$$

and subsequently the stringy *Einstein curvature*, G_{AB}, and *Energy Momentum tensor*, T_{AB},

$$G_{AB}=4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}S_{p\bar{q}}-\tfrac{1}{2}\mathcal{J}_{AB}S_{(0)}\,,\qquad \mathcal{D}_AG^{AB}=0\qquad\text{(off-shell)}\,,$$
$$T_{AB}:=4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}K_{p\bar{q}}-\tfrac{1}{2}\mathcal{J}_{AB}T_{(0)}\,,\qquad \mathcal{D}_AT^{AB}=0\qquad\text{(on-shell)}\,.$$

The equations of motion of the stringy graviton fields are thus unified into a single expression, the *Einstein Double Field Equations* (1). Note that $G_A{}^A=-DS_{(0)}$, $T_A{}^A=-DT_{(0)}$.

Restricting to the $(0,0)$ Riemannian background, the Einstein Double Field Equations reduce to

$$\begin{aligned}R_{\mu\nu}+2\nabla_\mu(\partial_\nu\phi)-\tfrac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}&=8\pi GK_{(\mu\nu)}\,,\\ \nabla^\rho\left(e^{-2\phi}H_{\rho\mu\nu}\right)&=16\pi Ge^{-2\phi}K_{[\mu\nu]}\,,\\ R+4\Box\phi-4\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}&=8\pi GT_{(0)}\,,\end{aligned}$$

which imply the conservation law, $\mathcal{D}_AT^{AB}=0$, given explicitly by

$$\nabla^\mu K_{(\mu\nu)}-2\partial^\mu\phi\,K_{(\mu\nu)}+\tfrac{1}{2}H_\nu{}^{\lambda\mu}K_{[\lambda\mu]}-\tfrac{1}{2}\partial_\nu T_{(0)}=0\,,\qquad \nabla^\mu\left(e^{-2\phi}K_{[\mu\nu]}\right)=0\,.$$

The Einstein Double Field Equations also govern the dynamics of other non-Riemannian cases, $(n,\bar{n})\neq(0,0)$, where the Riemannian metric, $g_{\mu\nu}$, cannot be defined.

Examples

– *Pure Stringy Gravity with cosmological constant:*

$$\tfrac{1}{16\pi G}e^{-2d}\left(S_{(0)}-2\Lambda_{\mathrm{DFT}}\right)\,,\qquad K_{p\bar{q}}=0\,,\qquad T_{(0)}=\tfrac{1}{4\pi G}\Lambda_{\mathrm{DFT}}\,.$$

– *RR sector, given by a* $\mathbf{Spin}(1,9)\times\mathbf{Spin}(9,1)$ *bi-spinorial potential,* $\mathcal{C}^\alpha{}_{\bar{\alpha}}$*:*

$$L_{\mathrm{RR}}=\tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}})\,,\qquad K_{p\bar{q}}=-\tfrac{1}{4}\mathrm{Tr}(\gamma_p\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}})\,,\qquad T_{(0)}=0\,,$$

where $\mathcal{F}=\mathcal{D}_+\mathcal{C}=\gamma^p\mathcal{D}_p\mathcal{C}+\gamma^{(11)}\mathcal{D}_{\bar{p}}\mathcal{C}\bar{\gamma}^{\bar{p}}$ is the RR flux set by an $\mathbf{O}(D,D)$ covariant "H-twisted" cohomology, $(\mathcal{D}_+)^2=0$, and $\bar{\mathcal{F}}=\bar{C}^{-1}\mathcal{F}^TC$ is its charge conjugate [7].

– *Spinor field:* $L_\psi=\bar{\psi}\gamma^p\mathcal{D}_p\psi+m_\psi\bar{\psi}\psi\,,\ K_{p\bar{q}}=-\tfrac{1}{4}(\bar{\psi}\gamma_p\mathcal{D}_{\bar{q}}\psi-\mathcal{D}_{\bar{q}}\bar{\psi}\gamma_p\psi)\,,\ T_{(0)}=0\,.$

– *Green-Schwarz superstring (κ-symmetric):*

$$e^{-2d}L_{\mathrm{string}}=\tfrac{1}{4\pi\alpha'}\int\mathrm{d}^2\sigma\left[-\tfrac{1}{2}\sqrt{-h}h^{ij}\Pi_i^M\Pi_j^N\mathcal{H}_{MN}-\epsilon^{ij}D_iy^M(\mathcal{A}_{jM}-i\Sigma_{jM})\right]\delta^D(x-y(\sigma))\,,$$
$$K_{p\bar{q}}(x)=\tfrac{1}{4\pi\alpha'}\int\mathrm{d}^2\sigma\sqrt{-h}h^{ij}(\Pi_i^MV_{Mp})(\Pi_j^N\bar{V}_{N\bar{q}})\,e^{2d}\delta^D(x-y(\sigma))\,,\qquad T_{(0)}=0\,,$$

where $\Sigma_i^M=\bar{\theta}\gamma^M\partial_i\theta+\bar{\theta}'\bar{\gamma}^M\partial_i\theta'$ and $\Pi_i^M=\partial_iy^M-\mathcal{A}_i^M-i\Sigma_i^M$ (doubled-yet-gauged) [9].

Gravitational effect

The regular spherical solution to the $D=4$ Einstein Double Field Equations shows that Stringy Gravity modifies GR (Schwarzschild geometry), in particular at "short" dimensionless scales, R/MG, *i.e.* distance normalized by mass times Newton constant. This might shed new light upon the dark matter/energy problems, as they arise essentially from "short distance" observations. Furthermore, it would be intriguing to view the B-field and DFT dilaton d as 'dark gravitons', since they decouple from the geodesic motion of point particles, which should be defined in string frame [10].

	Electron ($R\simeq 0$)	Proton	Hydrogen Atom	Billiard Ball	Earth	Solar System ($1\mathrm{AU}/M_\odot G$)	Milky Way (visible)	Galaxy Cluster	Universe ($M\propto R^3$)
$R/(MG)$	0^+	7.1×10^{38}	2.0×10^{43}	2.4×10^{26}	1.4×10^{9}	1.0×10^{8}	1.5×10^{6}	$\sim10^5$	0^+

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APPENDIX

Doubled-yet-Gauged Spacetime

- Let $\mathcal{F} := \{\, d, \mathcal{H}_{MN}\,, \xi^{M}\,, \partial_N d\,, \partial_L \mathcal{H}_{MN}, \cdots \}$ be the set of all the functions in DFT.
  - It contains not only the covariant physical fields, d, $\mathcal{H}_{MN}$, and local symmetry parameters, ξ^A, but also their arbitrary derivatives and products.
  - It is closed under additions, products and derivatives : if $\Phi_i, \Phi_j \in \mathcal{F}$ then

$$a\,\Phi_i + b\,\Phi_j \in \mathcal{F}\,, \qquad \Phi_i\Phi_j \in \mathcal{F}\,, \qquad \partial_A\Phi_i \in \mathcal{F}\,,$$

    where $a, b \in \mathbb{R}$.
  - The section condition,

$$\partial_M\partial^M\Phi_i = 0\,, \qquad \partial_M\Phi_i\partial^M\Phi_j = 0\,,$$

    is mathematically equivalent to certain translational invariance:

$$\Phi_i(x) = \Phi_i(x+\Delta)\,, \qquad \Delta^M = \Phi_j\partial^M\Phi_k\,,$$

    where Δ^M is said to be *derivative-index-valued.*
  - 'Physics' should be invariant under such shifts of the doubled coordinates in DFT.

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Doubled-yet-gauged spacetime

Doubled coordinates, $x^M = (\tilde{x}_\mu, x^\nu)$, are gauged through an equivalence relation,

$$x^M \sim x^M + \Delta^M(x)\,,$$

where Δ^M is derivative-index-valued.

Each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^D$.

– If we solve the section condition by letting $\tilde{\partial}^\mu \equiv 0$, and further put

$$\Delta^M = c_\mu \partial^M x^\mu \quad : \quad \mathrm{derivative\!-\!index\!-\!valued}\,,$$

we obtain explicitly,

$$(\tilde{x}_\mu\,,\,x^\nu) \;\sim\; (\tilde{x}_\mu + c_\mu\,,\,x^\nu) \quad : \quad \tilde{x}_\mu\text{'s are gauged and } x^\nu\text{'s form a section.}$$

– Then, $\mathbf{O}(D,D)$ rotates the gauged directions and the section.

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– In DFT, the usual infinitesimal one-form, $\mathrm{d}x^M$, is neither diffeomorphic covariant,

$$\delta x^M = \xi^M\,, \qquad \delta(\mathrm{d}x^M) = \mathrm{d}x^N\partial_N\xi^M \neq \mathrm{d}x^N(\partial_N\xi^M - \partial^M\xi_N)\,,$$

nor invariant under the coordinate gauge symmetry,

$$\mathrm{d}x^M \quad \longrightarrow \quad \mathrm{d}\big(x^M + \Delta^M\big) \neq \mathrm{d}x^M\,.$$

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The problems can be all cured by gauging the infinitesimal one-form explicitly,

$$Dx^M := \mathrm{d}x^M - \mathcal{A}^M \,.$$

Dx^M is a covariant vector in DFT

- The gauge potential should satisfy the same property as the coordinate gauge symmetry generator: it must be derivative-index-valued too, satisfying

$$\mathcal{A}^M \partial_M = 0 \,, \qquad \mathcal{A}_M \mathcal{A}^M = 0 \,.$$

- Essentially, half of the components are trivial, *e.g.* with $\tilde{\partial}^\mu \equiv 0$,

$$\mathcal{A}^M = A_\lambda \partial^M x^\lambda = (A_\mu \,,\, 0) \,, \qquad Dx^M = (\mathrm{d}\tilde{x}_\mu - A_\mu \,,\, \mathrm{d}x^\nu) \,.$$

- With the appropriate transformations of $\mathcal{A}^M$, the covariance of Dx^M is ensured:

$$\delta x^M = \Delta^M \,, \quad \delta\mathcal{A}^M = \mathrm{d}\Delta^M \qquad \Longrightarrow \qquad \delta(Dx^M) = 0 \,;$$

$$\delta x^M = \xi^M \,, \quad \delta\mathcal{A}^M = \partial^M \xi_N(\mathrm{d}x^N - \mathcal{A}^N) \qquad \Longrightarrow \qquad \delta(Dx^M) = Dx^N(\partial_N \xi^M - \partial^M \xi_N) \,.$$

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Doubled-yet-gauged spacetime

With $Dx^M = \mathrm{d}x^M - \mathcal{A}^M$, we can define **Proper Length** through a path integral,

$$\textbf{Length} := -\ln\left[\int \mathcal{D}\mathcal{A}\ \exp\left(-\int \sqrt{Dx^M Dx^N \mathcal{H}_{MN}}\right)\right]$$

which is gauged and covariant under **O**(D,D) and DFT-diffeomorphisms.

- For the Riemannian DFT-metric, we have a useful relation,

$$Dx^M Dx^N \mathcal{H}_{MN} \equiv \mathrm{d}x^\mu \mathrm{d}x^\nu g_{\mu\nu} + (\mathrm{d}\tilde{x}_\mu - A_\mu + \mathrm{d}x^\rho B_{\rho\mu})(\mathrm{d}\tilde{x}_\nu - A_\nu + \mathrm{d}x^\sigma B_{\sigma\nu})\, g^{\mu\nu}\,.$$

- Hence, after integrating out the gauge potential, A_μ, the above **O**(D,D) covariant definition of the proper length reduces to the conventional one,

$$\textbf{Length} \implies \int \sqrt{\mathrm{d}x^\mu \mathrm{d}x^\nu g_{\mu\nu}(x)}\,.$$

- Since it is independent of $\tilde{x}_\mu$, indeed it measures the distance between two gauge orbits, which is of course a desired feature.

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Doubled-yet-gauged sigma models

The definition of the proper length readily leads to 'covariant' actions:

i) Particle Action Ko-JHP-Suh 2016

$$\mathcal{S}_{\rm particle} = \int {\rm d}\tau \; e^{-1} \, D_\tau x^M D_\tau x^N \mathcal{H}_{MN}(x) - \tfrac{1}{4} m^2 e$$

ii) String Action Lee-JHP 2013, *c.f.* Hull 2006

$$\mathcal{S}_{\rm string} = \tfrac{1}{4\pi\alpha'} \int {\rm d}^2\sigma \; -\tfrac{1}{2}\sqrt{-h} h^{ij} D_i x^M D_j x^N \mathcal{H}_{MN}(x) - \epsilon^{ij} D_i x^M \mathcal{A}_{jM}$$

With the Riemannian DFT-metric plugged, after integrating out the auxiliary fields, the above actions reduce to the conventional ones:

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The scheme has been also extended to construct

iii) Doubled-yet-gauged Green-Schwarz superstring JHP 2016

$$\mathcal{S} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \; -\tfrac{1}{2}\sqrt{-h}h^{ij}\Pi_i^M\Pi_j^N\mathcal{H}_{MN} - \epsilon^{ij}D_ix^M\left(\mathcal{A}_{jM} - i\Sigma_{jM}\right) ,$$

where $\Pi_i^M := D_ix^M - i\Sigma_i^M$ and $\Sigma_i^M := \bar{\theta}\gamma^M\partial_i\theta + \bar{\theta}'\bar{\gamma}^M\partial_i\theta'$.

While this action reduces consistently to the original undoubled one,
it features the desired symmetries :

- **$\mathbf{O}(D, D)$ T-duality**
- **DFT-diffeomorphisms**
- **Worldsheet diffeomorphisms plus Weyl symmetry**
- **Coordinate gauge symmetry :** $x^M \sim x^M + \Delta^M \quad (\Delta^M\partial_M = 0)$
- **twofold Lorentz symmetry, Spin$(1,9)_L \times$ Spin$(9,1)_R$** $\Rightarrow$ **Unification of IIA & IIB**
- **Maximal 16+16 SUSY & kappa symmetry upon flat background**

All the above actions are formulated with $\mathcal{H}_{MN}$, V_{Mp}, $\bar{V}_{M\bar{p}}$ which satisfy the defining properties only, not necessarily parametrized by the Riemannian metric/vielbein.

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