Ghost-Free Theory with Third-Order Time Derivatives

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06/20/18@STGCOS2018, APCTP

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1711.08125, J.Phys.Soc.Jap. 87 (2018) 063401.

1804.07990, to appear in JHEP.

$c = \hbar = M_G^2 = 1/(8\pi G) = 1$

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- **Introduction**

  What is the most general scalar-tensor theory ?

- **Healthy degenerate theory for point particles**

  Review of second order (time) derivative system

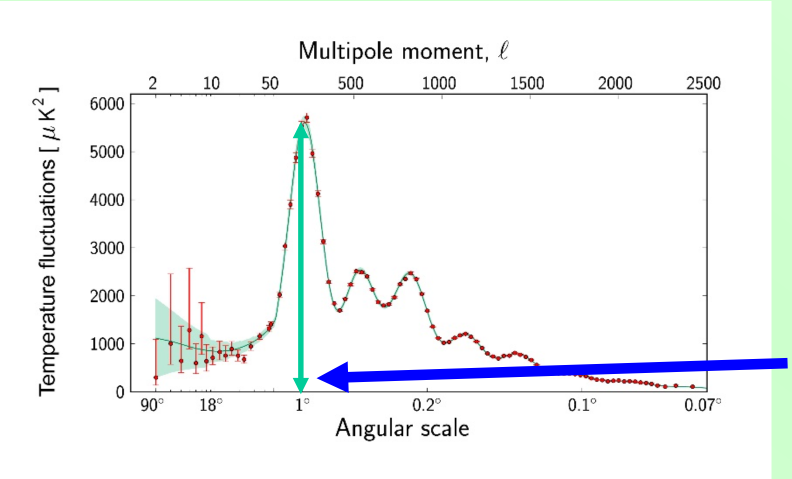
  Third order (time) derivative system

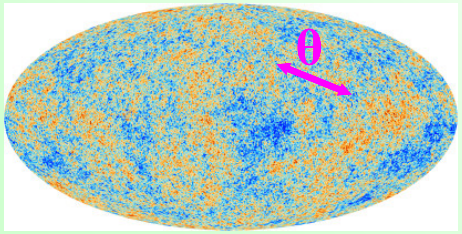
- **Discussion and conclusions**

Introduction

Inflation is strongly supported by CMB observations

Planck TT correlation :





Green line : prediction by inflation

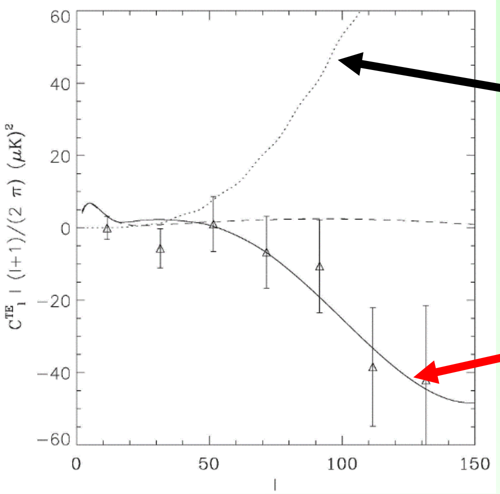
Red points : observation by PLANCK

Angle θ ~ 180° / l

Total energy density ←→ Geometry of our Universe

Our Universe is spatially flat as predicted by inflation !!

WMAP TE correlation :



Causal seed models

Superhorizon models

(adiabatic perturbations)

Unfortunately, primordial tensor perturbations have not yet been observed.

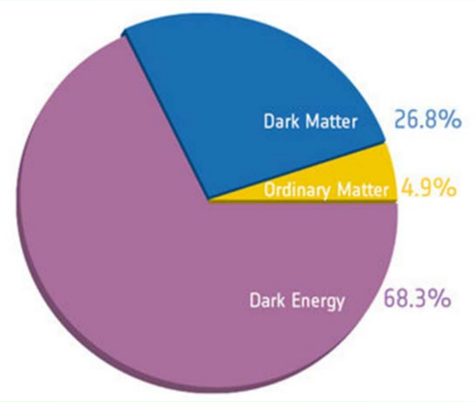
The presence of dark energy

The Universe is now accelerating !!

- **Dark Energy is introduced**

or

- **GR may be modified in the IR limit**



PLANCK

We have almost confirmed the presence of inflation and dark energy, but, unfortunately, we know neither the identification of an inflaton nor that of dark energy.

Next task is to identify the inflaton and the origin of the dark energy.

Identification methods

- **Top down approach :**

To construct the unique model from the ultimate theory like string theory.
(Recently, it may not be so actively studied.)

- **Bottom up approach**

To consider the most general model.
Then, we can constrain models (or to single out the true model finally) from the observational results.

In this talk, we take the latter approach

Bottom up approach

- **Effective field theory approach :** (Weinberg 2008, Cheung et al. 2008)

  The low-energy effective theory (after integrating out heavy mode with its mass M).

  A ghost seems to appear around the cut-off scale M (>> E).

$$\mathcal{L} = -\underset{\underset{\displaystyle E^2\phi^2}{\wr}}{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi} - \frac{1}{2}m^2\phi^2 - \underset{\underset{\displaystyle \frac{E^4}{M^2}\phi^2}{\wr}}{\frac{1}{2}\frac{1}{M^2}(\Box\phi)^2}$$

(E: the energy scale we pay attention to)

$$M \gg E \quad \Longleftrightarrow \quad \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \gg \frac{1}{2}\frac{1}{M^2}(\Box\phi)^2$$

- **Most general theory without ghost**

  (if we are interested in the case in which higher derivative terms play an important role in the dynamics.)

In this talk, we take the latter approach

The following question arises:

What is the most general
scalar-tensor theory without ghost ?

How widely can we extend scalar tensor theory ?

- A kinetic term of an inflaton is not necessarily canonical.

$$\mathcal{L} = X - V(\phi), \quad X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \quad \Longrightarrow \quad \mathcal{L} = K(\phi, X)$$

(k-inflation)

(Armendariz-Picon et.al. 1999)

- An inflaton is not necessarily minimally coupled to gravity.

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} M_G^2 R + \mathcal{L}_\phi \right) \quad \Longrightarrow \quad \Delta S = \int d^4x \sqrt{-g} f(\phi) R$$

(Higgs inflation)

(Cervantes-Cota & Dehnen 1995, Bezrukov & M. Shaposhnikov 2008)

- Action may include higher derivatives.

(Nicolis et.al. 2009)

$$\mathcal{L} = K(\phi, X) \quad \Longrightarrow \quad \Delta \mathcal{L} = G(\phi, X) \Box \phi$$

Theories with higher order derivatives are quite dangerous in general.

Example with higher order (time) derivatives

- $L = \frac{1}{2}\ddot{q}^2(t)$ $\xrightarrow[\text{EL eq.}]{}$ $q^{(4)} = 0$ **requires 4 initial conditions.**

$\Updownarrow$

2 (real) DOF

- $L^{(1)}_{\text{eq}} = \ddot{q}u - \frac{1}{2}u^2$ $\xrightarrow[\text{EL eq.}]{}$ $\left\{\begin{array}{rcl} \ddot{u} & = & 0, \\ \ddot{q} & = & u, \end{array}\right.$ $\Longrightarrow$ $q^{(4)} = 0$

$$x \equiv \frac{q-u}{\sqrt{2}}, \quad y \equiv \frac{q+u}{\sqrt{2}} \Longrightarrow L^{(1)}_{\text{eq}} = -\dot{q}\dot{u} - \frac{1}{2}u^2 = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\dot{y}^2 - \frac{1}{4}(x-y)^2.$$

$$\Longrightarrow H = \frac{1}{2}p_x^2 - \frac{1}{2}p_y^2 + \frac{1}{4}(x-y)^2.$$

$$(p_x \equiv \dot{x}, \quad p_y \equiv \dot{y})$$

2 (real) DOF = 1 healthy & 1 ghost

- $L^{(2)}_{\text{eq}} = \frac{1}{2}\dot{Q}^2 + \lambda(Q - \dot{q})$ $\Longrightarrow$ $p \equiv \frac{\partial L^{(2)}_{\text{eq}}}{\partial \dot{q}} = -\lambda, \quad P \equiv \frac{\partial L^{(2)}_{\text{eq}}}{\partial \dot{Q}} = \dot{Q}.$

$$\Longrightarrow H = p\dot{q} + P\dot{Q} - L^{(2)}_{\text{eq}} = \frac{1}{2}P^2 + pQ.$$

Hamiltonian is unbounded through a linear momentum !!

Ostrogradski's theorem

(Ostrogradsky 1850)

Assume that $L = L(\ddot{q}, \dot{q}, q)$ **and** $\frac{\partial L}{\partial \ddot{q}}$ **depends on** $\ddot{q}$ **:**

(Non-degeneracy)

⇒ $$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}^2 t}\left(\frac{\partial L}{\partial \ddot{q}}\right) = 0, \quad \Longrightarrow \quad q^{(4)} = q^{(4)}\left(q^{(3)}, \ddot{q}, \dot{q}, q\right).$$

Canonical variables :
$$\begin{cases} q, & p := \dfrac{\partial L}{\partial \dot{q}} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \ddot{q}} \quad \left(= \dfrac{\partial L_{\mathrm{eq}}}{\partial \dot{q}}\right), \\ Q := \dot{q}, & P := \dfrac{\partial L}{\partial \ddot{q}} \quad \left(= \dfrac{\partial L_{\mathrm{eq}}}{\partial \dot{Q}}\right). \end{cases}$$

$$L_{\mathrm{eq}} = L(\dot{Q}, \dot{q}, q) + \lambda(Q - \dot{q})$$

Non-degeneracy ⇔ $\ddot{q} = \ddot{q}\left(q, \dot{q}, \frac{\partial L}{\partial \ddot{q}}\right)$ ⇔ $\dot{Q} = \ddot{q} = \ddot{q}\,(q, Q, P)$

Hamiltonian:
$$\begin{aligned} H(q, Q, p, P) &:= p\dot{q} + P\dot{Q} - L \\ &= pQ + P\,\ddot{q}(q, Q, P) - L(q, Q, \ddot{q}(q, Q, P)). \end{aligned}$$

⇒ **p depends linearly on H so that no system of this form can be stable !!**

N.B. $$\frac{\partial L}{\partial \phi} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu \phi)}\right) + \partial_\mu\partial_\nu\left(\frac{\partial L}{\partial(\partial_\mu\partial_\nu\phi)}\right) = 0. \quad \Longrightarrow \quad \frac{i}{(p^2+m_1^2)(p^2+m_2^2)} = \frac{1}{m_2^2 - m_1^2}\left(\frac{i}{p^2+m_1^2} - \frac{i}{p^2+m_2^2}\right).$$

(propagators)

How to circumvent Ostrogradsky’s arguments to obtain healthy higher order derivative theories ?

Loophole of Ostrogradski's theorem

We can break the non-degeneracy condition which requires that $\frac{\partial L}{\partial \ddot{q}}$ depends on ddot{q}.

(NB: another interesting possibility is infinite derivative theory)

In case Lagrangian depends on only a position q and its velocity dot{q}, degeneracy implies that EOM is first order, which represents not the dynamics but the constraint.

⇕

In case Lagrangian depends on q, dot{q}, ddot{q}, degeneracy implies that EOM can be (more than) second order, which can represent the dynamics.

Generalized Galileon = Horndeski

Deffayet et al. 2009, 2011

equivalence ↑

Horndeski 1974

Kobayashi, MY, Yokoyama 2011

$$\left\{\begin{aligned}
\mathcal{L}_2 &= \boxed{K(\phi, X)}\\
\mathcal{L}_3 &= -\boxed{G_3(\phi, X)}\Box\phi,\\
\mathcal{L}_4 &= \boxed{G_4(\phi, X)}R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right],\\
\mathcal{L}_5 &= \boxed{G_5(\phi, X)}G_{\mu\nu}\nabla^\mu\nabla^\nu\phi\\
&\quad -\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right].
\end{aligned}\right.$$

$$X = -\frac{1}{2}(\nabla\phi)^2, \qquad G_{iX} \equiv \partial G_i/\partial X.$$

This is the most general scalar tensor theory whose Euler-Lagrange EOMs are up to second order though the action includes second derivatives.
Many of inflation and dark energy models can be understood in a unified manner.

NB : **● $G_4 = M_G^2 / 2$ yields the Einstein-Hilbert action**

● $G_4 = f(\varphi)$ yields a non-minimal coupling of the form $f(\varphi)R$

● The new Higgs inflation with $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ comes from $G_5 \propto \varphi$ after integration by parts.

Horndeski theory

Horndeski 1974

In 1974, Horndeski presented the most general action (in four dimensions) constructed from the metric g, the scalar field φ, and their derivatives, $\partial g_{\mu\nu}, \partial^2 g_{\mu\nu}, \partial^3 g_{\mu\nu}, \cdots, \partial\phi, \partial^2\phi, \partial^3\phi, \cdots$ still having second-order equations.

$$\begin{aligned}\mathcal{L}_H \;=\;& \delta^{\alpha\beta\gamma}_{\mu\nu\sigma}\left[\kappa_1 \nabla^\mu\nabla_\alpha\phi R_{\beta\gamma}{}^{\nu\sigma} + \frac{2}{3}\kappa_{1X}\nabla^\mu\nabla_\alpha\phi\nabla^\nu\nabla_\beta\phi\nabla^\sigma\nabla_\gamma\phi + \kappa_3\nabla_\alpha\phi\nabla^\mu\phi R_{\beta\gamma}{}^{\nu\sigma} + 2\kappa_{3X}\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi\nabla^\sigma\nabla_\gamma\phi\right] \\ &+\delta^{\alpha\beta}_{\mu\nu}\left[(F+2W)R_{\alpha\beta}{}^{\mu\nu} + 2F_X\nabla^\mu\nabla_\alpha\phi\nabla^\nu\nabla_\beta\phi + 2\kappa_8\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi\right] - 6\left(F_\phi + 2W_\phi - X\kappa_8\right)\Box\phi + \kappa_9.\end{aligned}$$

κ1, κ3, κ8, κ9, F : functions of φ & X with

W = W(φ)

$$\delta^{\alpha_1\alpha_2\ldots\alpha_n}_{\mu_1\mu_2\ldots\mu_n} = n!\,\delta^{[\alpha_1}_{\mu_1}\delta^{\alpha_2}_{\mu_2}\ldots\delta^{\alpha_n]}_{\mu_n}.$$

$$F_X = 2(\kappa_3 + 2X\kappa_{3X} - \kappa_{1\phi}).$$

What is the relation between Generalized Galileon and Horndeski's models ?

⇒ **Both models are completely equivalent :** **Kobayashi, MY, Yokoyama 2011**

$$\left\{\begin{aligned} K &= \kappa_9 + 4X\int^X dX'\left(\kappa_{8\phi} - 2\kappa_{3\phi\phi}\right), \\ G_3 &= 6F_\phi - 2X\kappa_8 - 8X\kappa_{3\phi} + 2\int^X dX'(\kappa_8 - 2\kappa_{3\phi}), \\ G_4 &= 2F - 4X\kappa_3, \\ G_5 &= -4\kappa_1, \end{aligned}\right.$$

$$\left\{\begin{aligned} \mathcal{L}_2 &= K(\phi, X), \\ \mathcal{L}_3 &= -G_3(\phi, X)\Box\phi, \\ \mathcal{L}_4 &= G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\ \mathcal{L}_5 &= G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\ &\quad -\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]. \end{aligned}\right.$$

Beyond Horndeski theory

- Gleyzes, Langlois, Piazza, and Vernizzi (GLPV) pointed out that there is extended theory with the number of propagating degrees of freedom unchanged, even though apparent EOMs of the theory are higher (third) order.

(Gleyzes et al. 2014, Zumalacarregui & Garcia-Bellido 2014)

- Noui and Langlois pointed out the importance of degeneracy of kinetic matrix of terms with different order derivatives and proposed degenerate higher order scalar-tensor (DHOST) theory.

(Noui & Langlois 2016, etc)

- Another direction is to consider infinitely many higher derivatives (nonlocal theory). (Barnaby & Kamran 2008, References therein, also Leonardo’s talk, Yun Soo’s talk etc)

e.g. $$e^{-\Box/\Lambda^2}(\Box - m^2)\phi = 0. \Longrightarrow \frac{i}{e^{p^2/\Lambda^2}(p^2+m^2)}$$

Only a pole of propagator appears with $p^2 = m^2$.

(Biswas, A. Mazumdar, W. Siegel 2006)

We are interested in an issue whether further extension is possible.

As far as I know, all of the (finite) higher order derivative theories without ghosts include up to second time derivatives.

(Xian Gao proposed arbitrary higher spatial derivative theory in 2014.)

What happens if we consider “third” order time derivative theories ?

(Is it straightforward extension of second order time derivative theories ???)

Healthy degenerate theories with “second” order derivatives for point particles

(as a first step with keeping in mind future extension to scalar-tensor theory)

Lagrangian up to second order derivatives

(Motohashi, Noui, Suyama, MY, Langlois. 2016)

$$L(\ddot{\phi}^a, \dot{\phi}^a, \phi^a; \dot{q}^i, q^i) \ (a = 1, \cdots, n; \ i = 1, \cdots, m)$$

Variables with second order derivatives: (φ^a)
Variables with first order derivatives: (q^i)

φ^a and q^i generically obey fourth order and second order equations of motion, respectively ➔ Ostrogradsky instability

Let us derive the conditions to escape such an instability by Hamiltonian analysis, starting from the equivalent Lagrangian:

$$L_{eq}^{(1)}(\dot{Q}^a, Q^a; \dot{\phi}^a, \phi^a; \dot{q}^i, q^i, \lambda_a) \equiv L(\underset{\overset{\uparrow}{\ddot{\phi}^a}}{\dot{Q}^a}, \underset{\overset{\uparrow}{\dot{\phi}^a}}{Q^a}, \phi^a; \dot{q}^i, q^i) + \lambda_a(\dot{\phi}^a - Q^a).$$

Hamiltonian analysis

$$L^{(1)}_{eq}(\dot{Q}^a, Q^a; \dot{\phi}^a, \phi^a; \dot{q}^i, q^i, \lambda_a) \equiv L(\dot{Q}^a, Q^a, \phi^a; \dot{q}^i, q^i) + \lambda_a(\dot{\phi}^a - Q^a).$$

$\dot{\phi}^a$

(3n+m) canonical variables : Q^a, φ^a, q^i, ~~χ~~^{a}

(3n+m) canonical momenta : Pa, πa, pi, ~~ρa~~

(4n+2m)
phase DOF
n ghosts!!

$$P_a = \frac{\partial L}{\partial \dot{Q}^a} \equiv L_{\dot{Q}} \;, \quad \pi_a = \frac{\partial L^{(1)}_{eq}}{\partial \dot{\phi}^a} = \lambda_a \;, \quad p_i = \frac{\partial L}{\partial \dot{q}^i} \equiv L_{\dot{q}} \;, \quad \rho_a = \frac{\partial L^{(1)}_{eq}}{\partial \dot{\lambda}^a} = 0 \,.$$

Two sets of n primary constraints:

$$\Phi_a = \pi_a - \lambda_a \approx 0 \;, \quad \Psi_a = \rho_a \approx 0 \,,$$

$$H = H_0 + \pi_a Q^a \quad \text{with} \quad H_0 = P_a \dot{Q}^a + p_i \dot{q}^i - L(\dot{Q}^a, Q^a, \phi^a; \dot{q}^i, q^i)$$

π^a appears linearly in Hamiltonian and hence it is unbounded if the system is nondegenerate without further primary constraints.

Degenerate Lagrangian

For healthy theories, we have to eliminate n DOF from constraints

$$\begin{pmatrix} \delta P_a \\ \delta p_i \end{pmatrix} = K \begin{pmatrix} \delta \dot{Q}^b \\ \delta \dot{q}^j \end{pmatrix}, \quad K \equiv \begin{pmatrix} L_{\dot{Q}^a \dot{Q}^b} & L_{\dot{Q}^a \dot{q}^j} \\ L_{\dot{q}^i \dot{Q}^b} & L_{\dot{q}^i \dot{q}^j} \end{pmatrix} = \begin{pmatrix} L_{ab} & L_{aj} \\ L_{ib} & L_{ij} \end{pmatrix}$$

Kinetic matrix must be degenerate !!

⟹ $$L_{ab} - L_{ai} L^{ij} L_{jb} = 0$$ **(First DC)**

Assume det Lij ≠ 0 (q^i is a normal variable)

$$\because \quad \left(\det K = \det(L_{ab} - \underset{\uparrow}{L_{ai}} L^{ij} L_{jb}) \det \underset{\uparrow}{L_{ij}} \right)$$

(all n eigenvalues are zero) **(m non-zero eigenvalues)**

⟺ $$\Xi_a \equiv P_a - F_a(p_i, Q^b, \phi^b, q^i) \approx 0.$$

Assume det Lij ≠ 0

(Additional primary constraints)

Additional primary constraints

(Motohashi & Suyama 2014)

$$\Xi_a \equiv P_a - F_a(p_i, Q^b, \phi^b, q^i) \approx 0.$$

$$\Longrightarrow \quad H_T = H(P_a, p_i, \pi_a, Q^a, q^i, \phi^a) + \mu^a \Phi_a + \nu^a \Psi_a + \xi^a \Xi_a$$

$$(\Phi_a = \pi_a - \lambda_a \approx 0, \ \ \Psi_a = \rho_a \approx 0)$$

$$\Longrightarrow \quad \begin{cases} \dot{\Phi}_a = \{\Phi_a, H\} - \nu^a + \xi^b \partial F_b / \partial \phi^a \approx 0, \\ \dot{\Psi}_a = \mu^a \approx 0, \qquad (\{\Phi_a, \Xi_b\} = \partial F_b / \partial \phi^a) \\ \dot{\Xi}_a = \{\Xi_a, H\} + \xi^b \underset{\overset{|||}{M_{ab}}}{\{\Xi_a, \Xi_b\}} \approx 0. \end{cases} \quad \Longrightarrow \ \textbf{Fix } \mu^a \ \& \ \nu^a$$

If det Mab ≠ 0, all ξ^a are fixed and no secondary constraints.

⟹ **Not sufficient number of constraints to eliminate all ghosts.**

⟹ **Simplest case for healthy theory : Mab = 0 (second DC)**

$$\Longrightarrow \quad \Theta_a \equiv \{\Xi_a, H\} = -\pi_a + \{\Xi_a, H_0\} - \partial F_a / \partial \phi_b Q^b \approx 0$$

(New constraint, which fixes all π_a in terms of the other phase space variables)

Summary of second order system

$$L(\ddot{\phi}^a, \dot{\phi}^a, \phi^a; \dot{q}^i, q^i)$$

$$\Longleftrightarrow \quad L_{eq}^{(1)}(\dot{Q}^a, Q^a; \dot{\phi}^a, \phi^a; \dot{q}^i, q^i, \lambda_a) \equiv L(\dot{Q}^a, Q^a, \phi^a; \dot{q}^i, q^i) + \lambda_a(\dot{\phi}^a - Q^a).$$

(3n+m) canonical variables : $\overset{\dot{\phi}^a \swarrow}{Q^a}$, **φ^a, q^i,** ~~**λ^a**~~

(3n+m) canonical momenta : ~~**Pa**~~**,** ~~**πa**~~**, pi,** ~~**ρa**~~

⇒ **(2n+2m)**
phase DOF
no ghosts!!

- **First DC :** $L_{ab} - L_{ai}L^{ij}L_{jb} = 0$

$$\left[\begin{pmatrix}\delta P_a \\ \delta p_i\end{pmatrix} = K\begin{pmatrix}\delta \dot{Q}^b \\ \delta \dot{q}^j\end{pmatrix}, \quad K \equiv \begin{pmatrix} L_{\dot{Q}^a\dot{Q}^b} & L_{\dot{Q}^a\dot{q}^j} \\ L_{\dot{q}^i\dot{Q}^b} & L_{\dot{q}^i\dot{q}^j}\end{pmatrix} = \begin{pmatrix} L_{ab} & L_{aj} \\ L_{ib} & L_{ij}\end{pmatrix}\right]$$

Assume det Lij ≠ 0

$$\Rightarrow \quad \Xi_a \equiv P_a - F_a(p_i, Q^b, \phi^b, q^i) \approx 0.$$

(primary constraints)

- **Second DC :** $M_{ab} = \{\Xi_a, \Xi_b\} = -\frac{\partial F_a}{\partial Q^b} + \frac{\partial F_b}{\partial Q^a} + \frac{\partial F_a}{\partial q_i}\frac{\partial F_b}{\partial p_i} - \frac{\partial F_a}{\partial p_i}\frac{\partial F_b}{\partial q_i} = 0.$

$$\Rightarrow \quad \Theta_a \equiv \{\Xi_a, H\} = -\pi_a + \{\Xi_a, H_0\} - \partial F_a/\partial\phi_b\, Q^b \approx 0$$

(secondary CS)

$$H = H_0 + \pi_a Q^a \quad \text{with} \quad H_0 = P_a\dot{Q}^a + p_a\dot{q}^a - L(\dot{Q}^a, Q^a, \phi^a; \dot{q}^i, q^i)$$

EOMs : $(L_{ab} - L_{ai}L^{ij}L_{jb})\phi^{b\,(4)} + M_{ab}\phi^{b\,(3)} + \cdots = 0$ ⇒ **second order**

Is the extension to third order (time) derivative system straightforward ???

No essential difference ???

⇨ There is big difference !!

Quadratic model with third order derivatives

(Motohashi, Suyama, MY 2018)

$$L = \frac{a_{nm}}{2}\dddot{\psi}^n\dddot{\psi}^m + \frac{b_{nm}}{2}\ddot{\psi}^n\ddot{\psi}^m + \frac{c_{nm}}{2}\dot{\psi}^n\dot{\psi}^m + \frac{d_{nm}}{2}\psi^n\psi^m + e_{nm}\dddot{\psi}^n\ddot{\psi}^m + f_{nm}\ddot{\psi}^n\dot{\psi}^m + \frac{A_{ij}}{2}\dot{q}^i\dot{q}^j + \frac{B_{ij}}{2}q^iq^j + C_{ij}\dot{q}^iq^j + \alpha_{ni}\dddot{\psi}^n\dot{q}^i\,.$$

$(n = 1, \cdots, N)$

$(i = 1, \cdots, I)$

$$\left(\psi^n = \psi^n(t),\ q^i = q^i(t), \quad a_{nm}, b_{nm}, \cdots : \mathrm{const}, \quad \mathrm{det}A_{ij} \neq 0\right)$$

⟺ $$L_{\mathrm{eq}} = L(\underset{\overset{\uparrow}{\dddot{\psi}^n}}{\dot{Q}^n}, \underset{\overset{\uparrow}{\ddot{\psi}^n}}{Q^n}, \underset{\overset{\uparrow}{\dot{\psi}^n}}{R^n}, \psi^n, \dot{q}^i, q^i) + \xi_n(\dot{\psi}^n - R^n) + \lambda_n(\dot{R}^n - Q^n).$$

Canonical momenta for $(Q^n, R^n, \psi^n, q^i, \lambda_n, \xi_n)$ **:**

$$P_{Q^n} = a_{nm}\dot{Q}^m + \alpha_{ni}\dot{q}^i + e_{nm}Q^m, \qquad p_i = \alpha_{ni}\dot{Q}^n + A_{ij}\dot{q}^j + C_{ij}q^j,$$

$$\boxed{P_{R^n} = \lambda_n, \quad \pi_{\psi^n} = \xi_n, \quad \rho_{\lambda_n} = 0, \quad \rho_{\xi_n} = 0.}$$

Primary constraints

Linear dependence of momenta on Hamiltonian

Canonical variables : $(\underset{\nwarrow\ddot{\psi}^n}{Q^n}, \underset{\nwarrow\dot{\psi}^n}{R^n}, \psi^n, q^i, \cancel{\chi_n}, \cancel{\xi_n})$ primary constraints

Canonical momenta : $(P_{Q^n}, P_{R^n}, \pi_{\psi^n}, p^i, \cancel{\rho_{\chi_n}}, \cancel{\rho_{\xi_n}})$

$$\Rightarrow \quad H = H_0 + P_{R^n} Q^n + \pi_{\psi^n} R^n .$$

$$\left(H_0 = \frac{1}{2}A^{ij}\tilde{p}_i\tilde{p}_j - \frac{1}{2}b_{nm}Q^nQ^m - \frac{1}{2}c_{nm}R^nR^m - \frac{1}{2}d_{nm}\psi^n\psi^m - f_{nm}Q^nR^m - \frac{1}{2}B_{ij}q^iq^j \;:\; \text{no dependece on } \; P_{R^n}, \pi_{\psi^n} \right)$$

**Due to the linear dependence of momenta,
Hamiltonian is unbounded !!**

According to Ostrogradsky arguments and the lesson from second order derivative system, we expect that we have only to remove this linear momentum dependence for a healthy theory.

Conditions for a healthy theory

$$H = H_0 + P_{R^n} Q^n + \pi_{\psi^n} R^n.$$

- **First DC :** $a_{nm} - \alpha_{ni} A^{ij} \alpha_{mj} = 0.$ **(kinetic matrix is degenerate)**

  ➡ $\Psi_n \equiv P_{Q^n} - e_{nm} Q^m - A^{ij} \alpha_{nj} \tilde{p}_i \approx 0.$ **(primary constraint)**

  Assume det Aij ≠ 0

- **Second DC :** $\{\Psi_n, \Psi_m\} = -2[e_{nm} + \alpha_{ni}(A^{-1} C A^{-1})^{ij} \alpha_{mj}] = 0.$

  ➡ $\Upsilon_n \equiv -\{\Psi_n, H\} = P_{R^n} - b_{nm} Q^m + \cdots \approx 0,$ **(secondary CS)**

- **Third DC :** $\{\Upsilon_n, \Psi_m\} = -b_{nm} - \alpha_{ni}[(4\bar{C}^2 + \bar{B}) A^{-1}]^{ij} \alpha_{mj} = 0.$

  ➡ $\Lambda_n \equiv -\{\Upsilon_n, H\} = \pi_{\psi^n} + 2 f_{nm} Q^m + \cdots \approx 0,$ **(tertiary CS)**

Now, the linear dependences are completely eliminated !!

No ghost (no instability) ???

The ghosts still lurk in the Hamiltonian

After erasing $P_{R^n}, b_{nm}, \pi_{\psi^n}$ using DCs and constraints, the Hamiltonian H reduces to

$$
\begin{aligned}
H = {} & \frac{1}{2}A^{ij}\bar{p}_i\bar{p}_j - \frac{1}{2}B_{ij}\bar{q}^i\bar{q}^j + \frac{1}{2}c_{nm}R^nR^m - \frac{1}{2}d_{nm}\psi^n\psi^m \\
& + \alpha_{ni}[(4\bar{C}^2 + \bar{B})A^{-1}]^{ij}\bar{p}_jR^n - 2\alpha_{ni}(\bar{C}\bar{B})^i{}_j\bar{q}^jR^n \\
& - 2[f_{nm} + 4\alpha_{ni}(\bar{C}^3A^{-1})^{ij}\alpha_{mj}]Q^mR^n,
\end{aligned}
$$

$$
(\bar{p}_i \equiv p_i - C_{ik}q^k + 2\alpha_{nk}A^{kl}C_{li}Q^n, \quad \bar{q}^i \equiv q^i + \alpha_{nk}A^{ki}Q^n)
$$

Q^n (= $\ddot{\psi}^n$) appears only linearly in H, making the Hamiltonian H unbounded !!

Eliminating the linear momentum terms is always necessary to kill the Ostrogradsky ghosts, but is not sufficient for higher-than-second-order derivative system.

⇒ **Needs to fix Q^n as well.**

Healthy theory

Canonical variables : $(\underset{\nwarrow\,\ddot{\psi}^n}{\cancel{Q}^n}, \underset{\nwarrow\,\dot{\psi}^n}{R^n}, \psi^n, q^i, \cancel{\chi}_n, \cancel{\xi}_n)$ **primary constraints**

Canonical momenta : $(P_{\cancel{Q}^n}, P_{\cancel{R}^n}, \pi_{\cancel{\psi}^n}, p^i, \rho_{\cancel{\chi}_n}, \rho_{\cancel{\xi}_n})$

$$H = H_0 + P_{R^n}Q^n + \pi_{\psi^n}R^n.$$

secondary & tertiary CS

- **Fourth DC :** $\{\Lambda_n, \Psi_m\} = 2(f_{nm} - \alpha_{ni}M^{ij}\alpha_{mj}) = 0.$

⇒ $\Omega_n \equiv -\{\Lambda_n, H\} = c_{nm}Q^m - d_{nm}\psi^m + \cdots \approx 0,$ **(quaternary CS)**

Q^n is now fixed (and expressed in terms of other variables).

⇒ **Only healthy DOFs** $(\psi^n, R^n(= \dot{\psi}^n), q^i, p^i)$ **remain.**

Hamiltonian is bounded and no ghosts.

⇒ **EOMs can be reduced to the second order system**

Dirac matrix

$$D = \begin{array}{c|cccccc} & \multicolumn{3}{c}{\overbrace{\hphantom{\Phi_\beta \quad \bar{\Phi}_\beta \quad \Psi_m}}^{\text{C1}}} & \text{C4} & \text{C2} & \text{C3} \\ & \Phi_\beta & \bar{\Phi}_\beta & \Psi_m & \Omega_m & \Upsilon_m & \Lambda_m \\ \hline \Phi_\alpha & 0 & -\mathbf{1} & * & * & * & * \\ \bar{\Phi}_\alpha & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \Psi_n & * & 0 & 0 & -Z_{mn} & 0 & 0 \\ \Omega_n & * & 0 & Z_{nm} & * & * & * \\ \Upsilon_n & * & 0 & 0 & * & 0 & Z_{mn} \\ \Lambda_n & * & 0 & 0 & * & -Z_{nm} & * \end{array}$$

Condition to complete Dirac procedure :

$$\det Z_{nm} = \det\{\Omega_n, \Psi_m\} \neq 0$$

⇒ **det D $\neq 0$** ⇒ **All constraints are second class**

⇒ **Healthy (N+I) DOFs.**

Reduction

(Motohashi, Suyama, MY 2018)

$L(q^i, \dot{q}^i, \ddot{q}^i, \dddot{q}^i, \cdots)$ → **Healthy theory**

↑

A chain of constraints : $\Phi_A \approx 0 \; (A = 1, 2, \cdots)$

If all of them are second class, according to Maskawa & Nakajima (1976), a canonical transformation exists such that new variables are classified to

$$(Q^a, P_a), \quad (\tilde{Q}^m, \tilde{P}_m) \quad \text{with} \quad \tilde{Q}^m \approx 0 \quad \& \quad \tilde{P}_m \approx 0$$

↖

unconstrained and governed by H(Q^a,Pa), that is, $\dot{Q}^a = \frac{\partial H}{\partial P_a}, \quad \dot{P}_a = -\frac{\partial H}{\partial Q^a}$

↓

$$P_a = P_a(Q^b, \dot{Q}^b)$$

$$\Rightarrow \quad L'(Q^a, \dot{Q}^a) = P_a \dot{Q}^a - H(Q^a, P_a)$$

This Lagrangian is nondegenerate and contains up to first-order time derivatives of Qa.

(If a theory has first-class constraints, only gauge fixing is necessary.)

Summary

- We have investigated how to obtain healthy degenerate theory with higher-than-second derivative.
- Eliminating linear momentum terms in the Hamiltonian is necessary and sufficient to kill the ghosts for second order derivative system.
- On the other hand, this is necessary but not sufficient for the Lagrangian with higher than second-order derivatives. The (lurked) linear dependence of canonical variables corresponding to second (even higher) derivatives must be removed as well.