Primordial black holes and secondary gravitational waves

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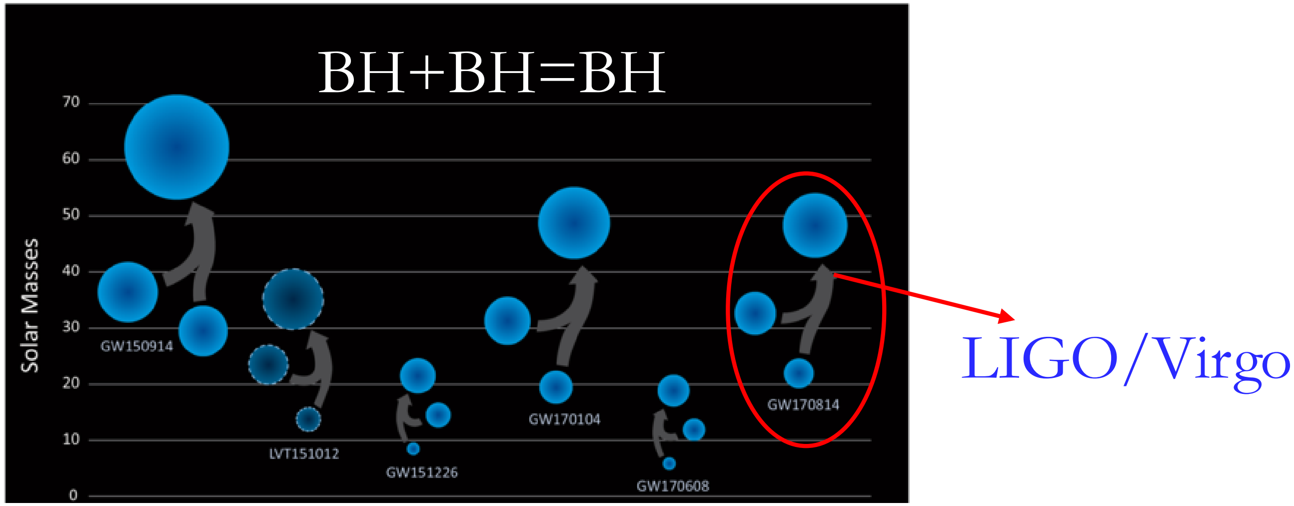
Outline

- Observational constraints on the mass of PBHs
- The generic feature of inflation for the production of PBHs
- Ultra-slow-roll inflation
- Model building
- Conclusions

Gao, Gong and Li, 1405.6451; Gong, 1707.09578;
Yi and Gong, 1712.07478

Motivation

- GWs: detection of binary black hole merger



GW150914 : PRL 116, 061102; GW151226: PRL 116, 241103; GW170104: PRL 118, 221101; GW170608: ApJ 851, L35; GW170814: PRL 119, 141101;

- PBHs: Primordial black holes as dark matter?

Bird etal., PRL 116, 201301; Sasaki etal., PRL 117, 061101;

The PBHs

- PBHs: PBH forms in the radiation era as a result of gravitational collapse of density perturbations generated during inflation
- The mass

Suppose that the mass of PBHs is of the same order of the horizon mass $M = \gamma M_H$ $\qquad 1 M_\odot \approx 2 \times 10^{33}\mathrm{g}$

$$M_H = \frac{4\pi\rho}{3H^3} = \frac{1}{2GH} \approx 2.02 \times 10^5 \left(\frac{t}{1\mathrm{s}}\right) M_\odot$$

$$M \approx 18.4 \gamma M_\odot \left(\frac{g}{10.75}\right)^{-1/6} \left(\frac{k}{10^6 \mathrm{Mpc}^{-1}}\right)^{-2}$$

$$M \approx 2 \times 10^5 \gamma M_\odot \left(\frac{g}{10.75}\right)^{-1/2} \left(\frac{T}{10^{10}\mathrm{K}}\right)^{-2}$$

The mass of PBHs

- The problem of γ

$M = \gamma M_H$ Order one quantity which depends on models

$\gamma = 3^{-3/2} \approx 0.2$ Carr, APJ 201 (75) 1

$$M = k(\delta - \delta_c)^\alpha M_H$$

$$k = 3.3,\ \alpha = 0.357,\ \delta - \delta_c = 2.37 \times 10^{-3}$$

$\gamma \approx 0.36$ Musco, Miller & Rezzolla, CQG 22 (05) 1405

$\gamma \sim 0.4$ Carr, Kuhnel & Sandstad, PRD 94 (16) 083504

$M = 0.4415 M_H$ Musco, Miller & Rezzolla, CQG 22 (05) 1405

$M^{\max} \approx 0.6 M_H$ Niemeyer & Jedamzik, PRL 80 (98) 541

The Constraints

- The mass window
  - Hawking radiation by now

    $M > 10^{15}\mathrm{g}$ Carr, APJ 201 (75) 1
  - MACHO (Massive Astrophysical Compact Halo Object)

    Exclude $10^{-7} - 10M_{\odot}$ EROS-2 Collaboration 07, AA 469, 387
  - Astrophysical constraints (femtolensing, millilensing, microlensing etc)

    Exclude $10^{17} - 10^{20}\mathrm{g}, 10^{-3} - 60M_{\odot}, 10^{6} - 10^{9}M_{\odot}$
  - μ-distortion

    Exclude $4 \times 10^{2} - 4 \times 10^{13}M_{\odot}$ Kohri etal. 14, PRD 90, 083514

Gould 92; Nemiroff etal. 01; Wilkinson etal. 01; Dalcanton etal. 94; Allsman etal. 01; Tisserand etal. 07; Carr etal. 10; Griest etal. 13

The mass window for PBHs

- Allowed mass window $1M_\odot \approx 2\times 10^{33}\mathrm{g}$

$$10^{15}-2\times10^{17}\mathrm{g}, 2\times10^{20}-4\times10^{24}\mathrm{g}, 20-100M_\odot$$

- Particular constraints
  - Microlensing constraints

    $M\sim 10^{20}\mathrm{g},\ \sigma\lesssim 0.1$ Width
  - Extended mass functions

Kuhnel etal. 17, 1705.10361; Kovetz, 1705.09182; Ali-Haimoud etal., 1709.06576

Inomata etal. PRD 96 (17) 043504

Allowed mass $M\sim 5\times10^{-16}M_\odot,\ 2\times10^{-14}M_\odot,\ 25-100M_\odot$

All observations combined: PBHs can only constitute of order 10% of the dark matter

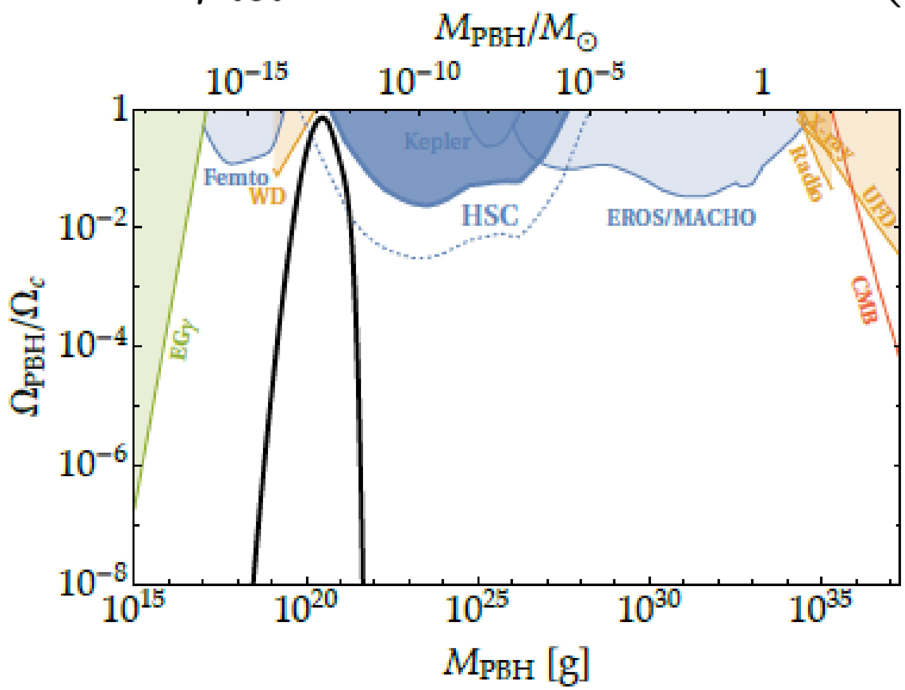
Carr etal. PRD 96 (17) 023514

Uncertainty exists!!!

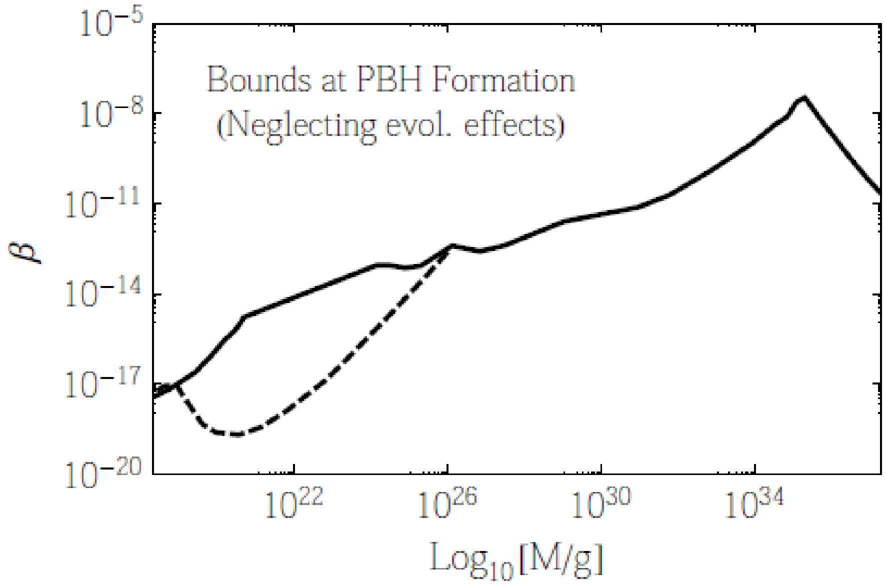
Observational constraints

The fraction of the mass of the Universe in PBHs at the formation of PBHs $\beta(M)$

$$\beta = \frac{\rho_{\rm PBH}}{\rho_{\rm tot}} = 1.76 \times 10^{-9} \gamma^{-1/2} \left(\frac{\Omega_{\rm PBH} h^2}{0.12}\right) \left(\frac{g}{10.75}\right)^{1/4} \left(\frac{M}{M_\odot}\right)^{1/2}$$



Inomata etal. PRD 96 (17) 043504



Garcia-Bellido etal, JCAP 1709, 013

The production of PBHs

- The energy density (Press-Schechter)

$$\beta = \frac{\rho_{\rm PBH}}{\rho_{\rm tot}} \approx \mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2\mathcal{P}_\delta}}\right) = \mathrm{erfc}\left(\frac{9\delta_c}{4\sqrt{2\mathcal{P}_\zeta}}\right)$$

Gaussian statistics

Carr, APJ 201 (75) 1

$$\delta_c = 0.07 - 0.7$$

$$\mathcal{P}_\zeta = 0.0175,\ \delta_c = 0.12,\ \beta = 0.031$$

$$\mathcal{P}_\zeta = 0.0175,\ \delta_c = 0.45,\ \beta = 7.59 \times 10^{-16}$$

$$\beta \approx \mathrm{erfc}\left(\sqrt{\frac{1}{2} + \frac{\delta_c}{\sqrt{2\mathcal{P}_\delta}}}\right)$$

χ^2 statistics

JCAP 1709, 013

- The critical density

$\delta_c = 1/3$ Carr, APJ 201 (75) 1

$\delta_c \approx 0.7$ Niemeyer & Jedamzik, PRD 59 (99) 124013

$\delta_c = 0.3 - 0.5$ Green etal, PRD 70 (04) 041502 (R)

$\delta_c \approx 0.45$ Musco, Miller & Rezzolla, CQG 22 (05) 1405
Carr, Kuhnel & Sandstad, PRD 94 (16) 083504

The production of PBHs

- Gravitational collapse

$$\beta = \frac{\rho_{\rm PBH}}{\rho_{\rm tot}} \approx \mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2\mathcal{P}_\delta}}\right) = \mathrm{erfc}\left(\frac{9\delta_c}{4\sqrt{2\mathcal{P}_\zeta}}\right)$$

$$\mathcal{P}_\zeta = \frac{H^2}{8\pi^2\epsilon} \sim 10^{-9}$$ Larges scales

- Need an enhancement on the primordial power spectrum

$$\mathcal{P}_\zeta = \frac{H^2}{8\pi^2\epsilon} \sim 0.01$$ Small scales

The production mechanism

- Blue spectrum

$$\mathcal{P}_\zeta = \frac{H^2}{8\pi^2\epsilon} = A_s(k_*)\left(\frac{k}{k_*}\right)^{n_s-1}, \quad n_s > 1$$

- Running index

$$n_s = n_s(k_*) + \frac{1}{2}n_s'(k_*)\ln(k/k_*) + \frac{1}{6}n_s''(k_*)\ln^2(k/k_*)$$

- Designer inflation

Enhancement: Power spectrum peaks on some scales

- Parametric Resonance
- Waterfall scenario

Inflationary models

- Two-field hybrid model of inflation

  Garcia-Bellido, Linde, Wands, PRD 54 (96) 6040;

  Inomata etal. PRD 95 (17) 123510; PRD 96 (17) 043504

- Axion inflation

  Garcia-Bellido, Peloso, Unal, JCAP 1612 (16) 031

- Single field double inflation

  Yokoyama, PRD 58 (98) 083510

  Saito, Yokoyama & Nagata, JCAP 06(2008)024

  Kannike, Marzola, Raidal, Veermae, JCAP 1709 (17) 020

- Single field inflation

  Garcia-Bellido, Morales, Phys. Dark Univ. 18 (17) 47;

  Ezquiaga, Garcia-Bellido, Morales, PLB 776 (18) 345-349

  Di & Gong, 1707.09578, JCAP in press

Double inflation

- Single field double inflation: Coleman-Weinberg potential

$$V(\phi) = \frac{\lambda}{4}\phi^4\left(\ln\left|\frac{\phi}{v}\right| - \frac{1}{4}\right) + \frac{\lambda}{16}v^4$$

$$\lambda = 5.4 \times 10^{-14}, \;\; v = 0.355139 M_{\rm Pl}$$

Chaotic inflation + New inflation

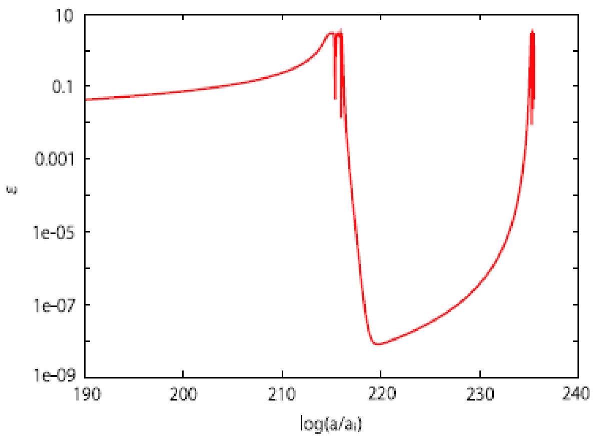
$\phi \gg M_{\rm Pl} > v$ $\qquad\qquad$ $\phi \sim 0$

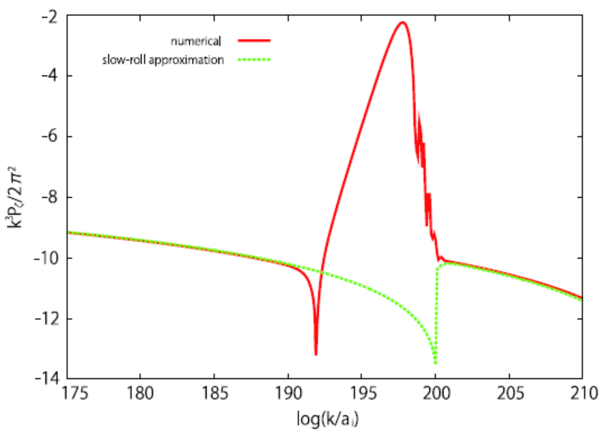
Initial condition?

$$n_s = 0.925, \;\; r = 0.4 \qquad \mathcal{P}_\zeta \sim 6.2 \times 10^{-3}$$

Yokoyama, PRD 58 (98) 083510

Saito, Yokoyama & Nagata, JCAP 06(2008)024

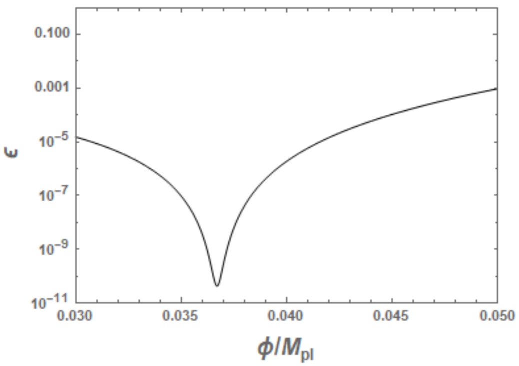


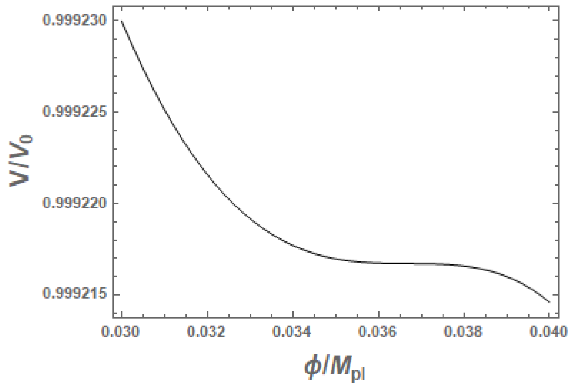


Single field inflation: Overview

- Small ϵ requires very flat potential, inflection point

$$\mathcal{P}_\zeta = \frac{H^2}{8\pi^2\epsilon}$$





- Ultra-slow-roll inflation

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi(\phi) = 0$$

$$V_\phi(\phi) = 0$$

$$\Rightarrow \quad \eta_H = -\frac{\ddot{\phi}}{H\dot{\phi}} = 3$$

Break-down of slow-roll

Tsamis, Woodard, PRD 69 (04) 084005

Ultra-slow-roll inflation

- The field rolls faster $\dot{\phi} \propto a^{-3}$ $\rho_\phi^{KE} \propto a^{-6}$

  The contribution to N during ultra-slow-roll inflation (inflection point) becomes smaller

  Dimopoulos, 1707.05644

- Mukhanov-Sasaki Equation

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0$$

$$\frac{z''}{z} = \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)$$

$$\nu = \eta_H - 3 = \frac{3}{2}$$

$$\mathcal{P}_\zeta^{1/2} = \frac{H^2}{2\pi\dot{\phi}} = \left(\frac{H^2}{8\pi^2\epsilon_H}\right)^{1/2} \to \infty$$

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi(\phi) = 0$$

Slow-roll

$$3H\dot{\phi} \approx -V_\phi(\phi) = 0$$

Kinney, PRD 72 (05) 023515

Constant-roll inflation

- Constant-roll inflation $\ddot{\phi} = nH\dot{\phi}$

$$\frac{z''}{z} = \frac{2+3n+n^2}{\tau^2} \qquad \nu = \left| n + \frac{3}{2} \right|$$

- No freezing out on Super-horizon scale

$$v_k'' - \frac{z''}{z} v_k = 0$$

$$v_k(\tau) = A_k z(\tau) + B_k z(\tau) \int \frac{d\tau}{z^2(\tau)}$$

- Power spectrum

growing mode?

$$\mathcal{P}_\zeta = 2^{2\nu-4} \left(\frac{\Gamma(\nu)}{\Gamma(3/2)} \right)^2 \left(\frac{1}{\epsilon_H} \right) \left(\frac{H}{2\pi} \right)^2 \left(\frac{k}{aH} \right)^{3-2\nu} \qquad \epsilon_H \propto e^{-6\Delta N}$$

Martin, Motohashi, Suyama PRD 87 (13) 023514
Motohashi, Starobinsky, Yokoyama, JCAP 1509 (15) 018

Constant-roll inflation[1]

- The model $\ddot{\phi} = nH\dot{\phi},\ n = -2\alpha$ $\quad \eta_H = -\frac{\ddot{\phi}}{H\dot{\phi}} = 2\alpha$

$$H(\phi) = c_1 \exp\left[\sqrt{\alpha}\,(\phi - \phi_0)\right] + c_2 \exp\left[-\sqrt{\alpha}\,(\phi - \phi_0)\right]$$

$$H(\phi) = M \exp(\pm\sqrt{\alpha}\phi), \quad c_1 c_2 = 0,$$ Power-law inflation

$$H(\phi) = M \cosh\left(\sqrt{\alpha}\phi\right), \quad c_1 c_2 > 0, \quad \dot{\epsilon}_H < 0$$ Exit problem

$$H(\phi) = M \sinh\left(\sqrt{\alpha}\phi\right), \quad c_1 c_2 < 0 \quad \dot{\epsilon}_H > 0$$

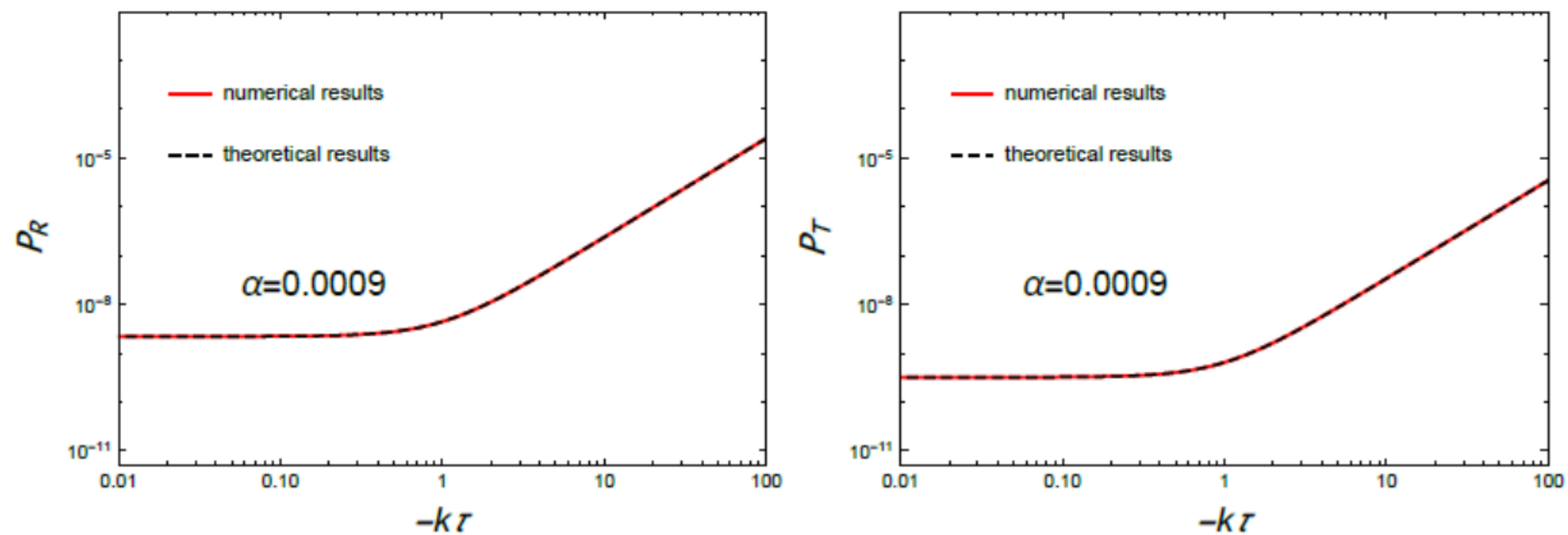
$$H(\phi) = c_1 \sin\left(\sqrt{-\alpha}\phi\right) + c_2 \cos\left(\sqrt{-\alpha}\phi\right)$$

$$H(\phi) = M \sin\left[\sqrt{-\alpha}(\phi - \phi_1)\right]$$

$$V(\phi) = V_0 \sinh^2\left(\sqrt{\alpha}\phi\right) - V_1 \cosh^2\left(\sqrt{\alpha}\phi\right)$$

$$a = a_0 \sin^{1/(2\alpha)}[2\alpha M(t_0 - t)] \quad \pi/4 < \alpha M\,(t_0 - t) < \pi/2$$

- Validity of analytical approximation



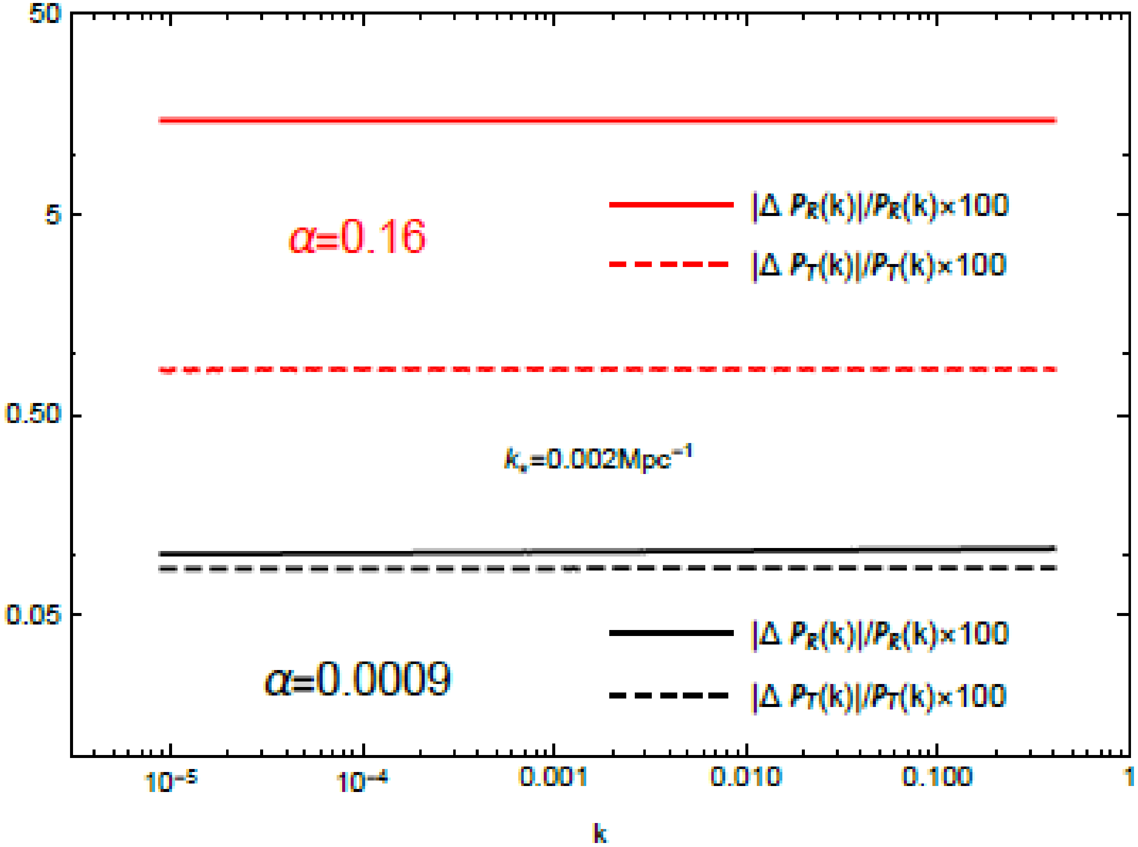
$$H(\phi) = M \sinh\left(\sqrt{\alpha}\phi\right)$$

Slow-roll inflation

Yi and Gong, JCAP 1803 (18) 052

Constant-roll inflation3

- Validity of analytical approximation



$$\frac{1}{aH} \approx (-1 + \epsilon_H)\tau$$

$$\frac{1}{aH} \approx \left(-1 + \frac{\epsilon_H}{1 + 2\eta_H}\right)\tau$$

- Needs to do numerical calculation

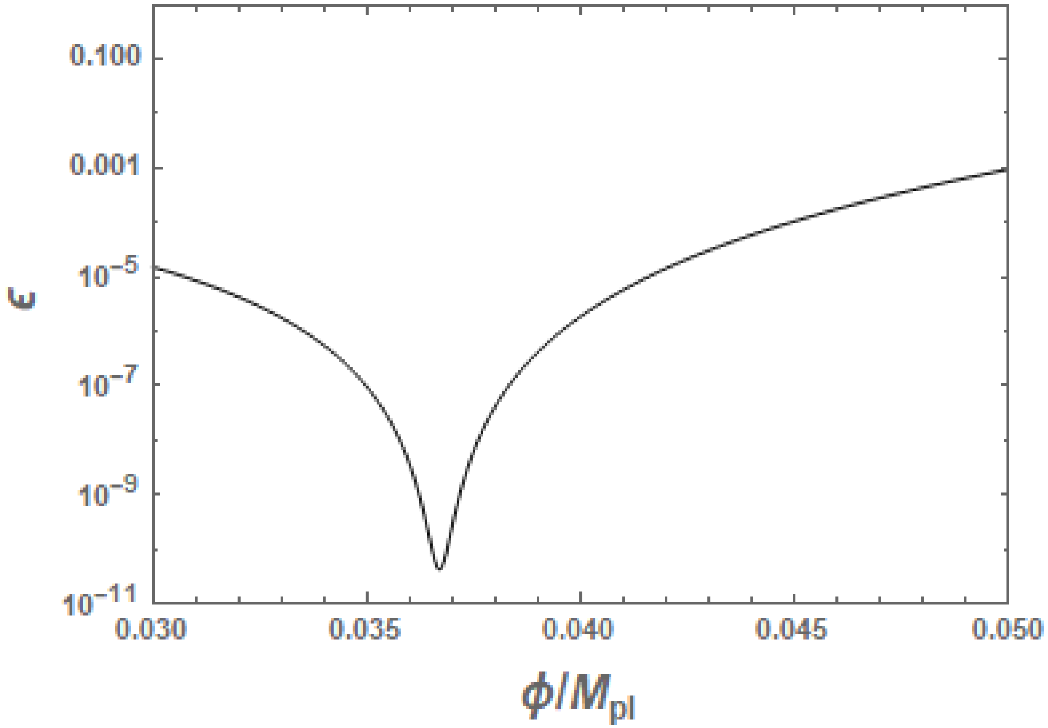
Yi and Gong, JCAP 1803 (18) 052

USR inflation: generic feature

- Very small ϵ at small scales

$$\mathcal{P}_\zeta = \frac{H^2}{8\pi^2\epsilon}$$

Slow-roll inflation



$$\epsilon = \frac{1}{2}\left(\frac{V_\phi}{V}\right)^2 \sim 0$$

Motivation and Goal: Decrease by 7 orders of magnitude

Slow-roll parameter ϵ cannot increase monotonically

The challenges

- Breakdown of slow-roll inflation

$$\frac{d\ln\epsilon_H}{dN} = 2(\eta_H - \epsilon_H) \approx 2(\eta - 2\epsilon)$$

Need to apply numerical calculation

- Large contribution to N

$$N = \int_{\phi}^{\phi_e} \frac{d\phi}{\sqrt{2\epsilon(\phi)}}$$

Breakdown of Lyth bound
Helps to reduce N

N can be much larger than 60

- Ruin of the featureless and smooth $\mathcal{P}_{\mathcal{R}}(k)$ over the observational scales, leads to big μ distortion $0.008\mathrm{Mpc}^{-1} \lesssim k \lesssim 0.1\mathrm{Mpc}^{-1}$

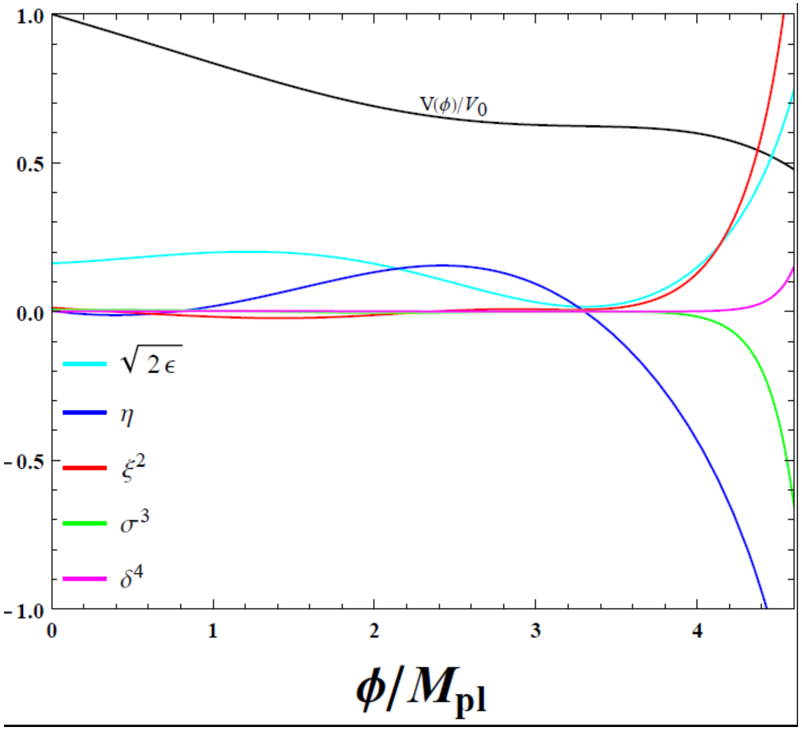
Planck15, AA 594 (16) A20

The Challenges

- Slow-roll inflation

  ϵ increases monotonically till $\epsilon \sim 1$

- Very small slow-roll parameter



$$V(\phi) = V_0 \left[1 + \sum_{m=1} \lambda_m (\phi - \phi_*)^m \right]$$

$$\lambda_1 = -0.162,\ \lambda_2 = 0.00161,$$

$$\lambda_3 = -0.0132,$$

$$\lambda_4 = 0.01,\ \lambda_5 = -0.0014576$$

Gao, Gong and Li, PRD 91 (15) 063509

Modified Lyth bound

- Lyth bound

$$N(\phi) = \frac{1}{M_{pl}} \int_{\phi_f}^{\phi} \frac{1}{\sqrt{2\epsilon_H}} d\phi$$

$$\Delta\phi > N(\phi_*)\sqrt{2\epsilon(\phi_*)}$$

$$\frac{\Delta\phi}{M_{pl}} > N(\phi_*)\sqrt{r/8}$$

- Smaller field excursion: breakdown of Lyth bound

Slow-roll Results

$$n_s = 0.9595,\ r = 0.21,\ n_s' = -0.0292,\ N = 60,\ \phi_e = 4.43$$

$$\Delta\phi = 4.43 < N\sqrt{2\epsilon} = 9.72$$

Numerical Results

$$n_s = 0.9584,\ n_s' = -0.039,\ r = 0.2,\ N = 63.1$$

$$\Delta\phi = 5.08 M_{\rm Pl}$$

Examples: Inflection point inflation

- Toy model

$$V(x) = \frac{\lambda v^4}{12} \frac{x^2(6 - 4ax + 3x^2)}{(1 + bx^2)^2},$$

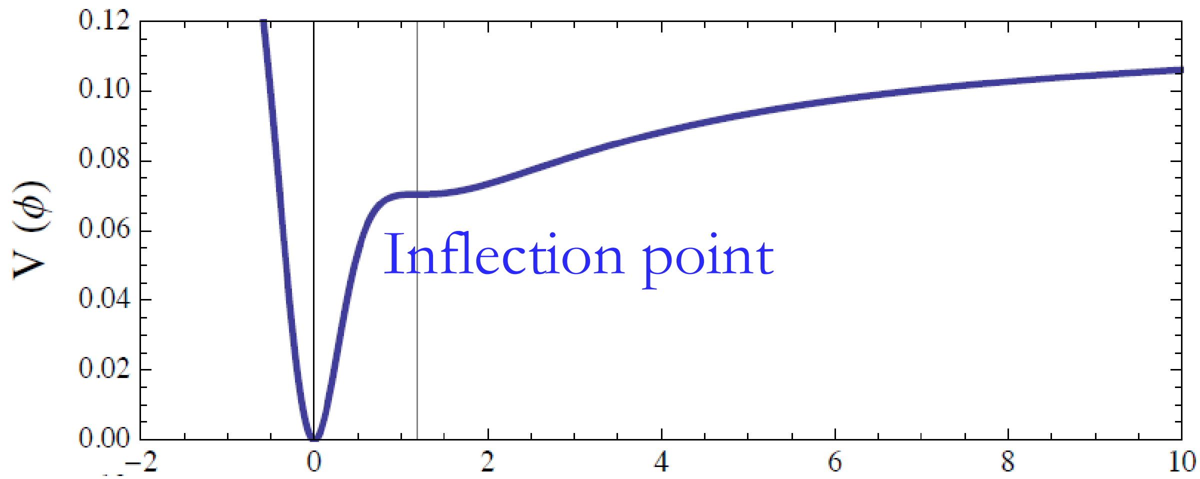
$$x = \frac{\phi}{v}, \; m^2 = \lambda v^2, \; a = 1, \; b = 1.435$$

$$\epsilon = \left(\frac{1 - \eta_H/3}{1 - \epsilon_H/3}\right)^2 \epsilon_H \approx 0$$

$$V_{,\phi} \approx 0$$

$$\eta = \frac{\epsilon_H + \eta_H - \eta_H^2/3 - \xi_H/3}{1 - \epsilon_H/3} \approx 0$$

$$V_{,\phi\phi} \approx 0$$



$$\eta_H \approx 3$$

$$\epsilon_H \ll 1, \; \xi_H \ll 1$$

Garcia-Bellido, Morales, Phys. Dark Univ. 18 (17) 47

The toy model

- The change of ϵ

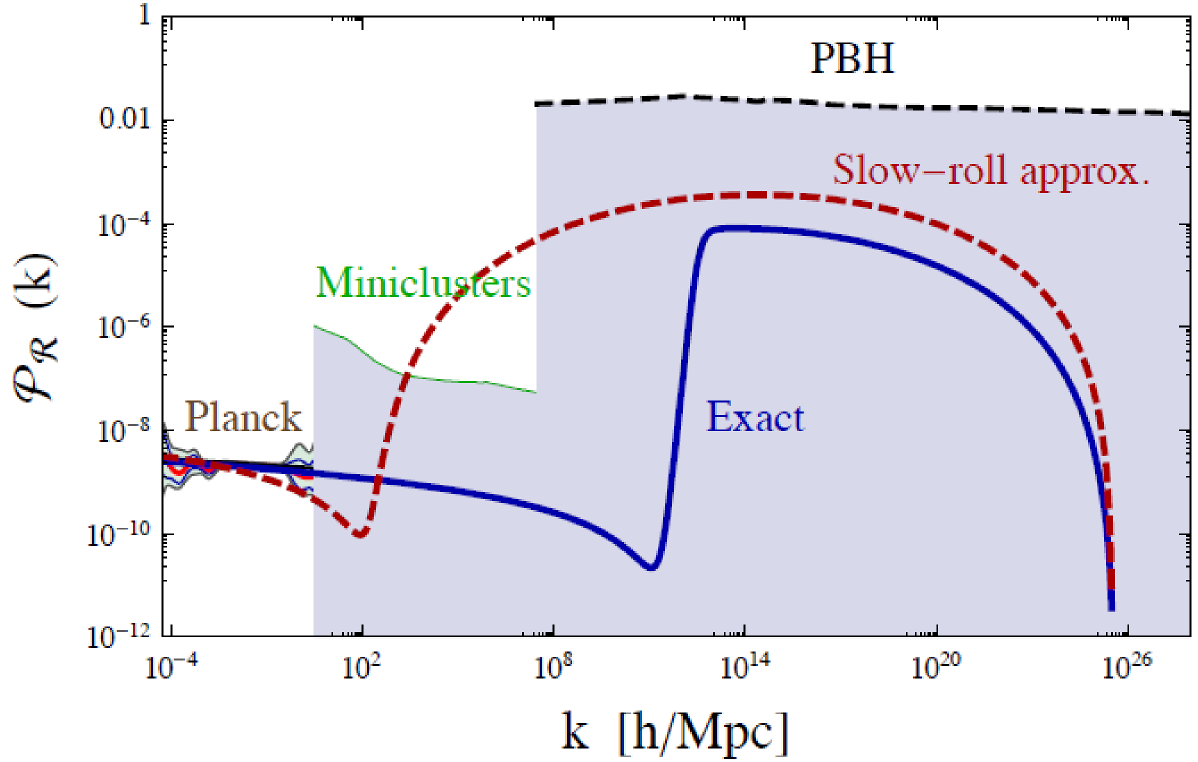
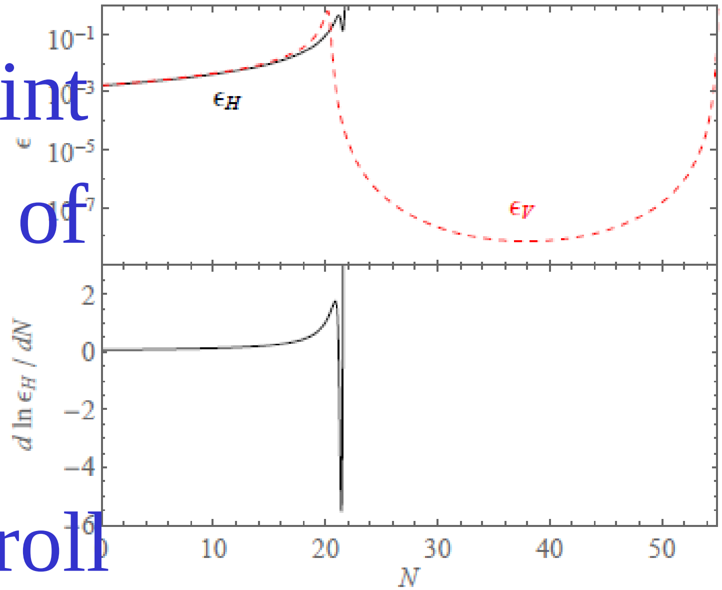


Figure 4. The exact matter power spectrum (blue line), for model parameters: $a = 1$, $b = b_c(1) - \beta$, $\kappa^2 v^2 = 0.108$ and $\beta = 1 \times 10^{-4}$. We have also plotted the SRA (red-dashed line), and the range of values allowed by Planck (2015), by compact minihalos (green line) and by PBH (black dashed line), at 95% c.l. (figure adapted from ref. [45]).

The criticism

- Breakdown of slow-roll approximation
- Inflation ends before reaching the inflection point
- Not enough enhancement of the amplitude
- Needs to apply the formalism for ultra-slow-roll inflation

Overshot from standard inflation, large suppression of the amplitude



Germani, Phys. Dark Univ. 18 (17) 6

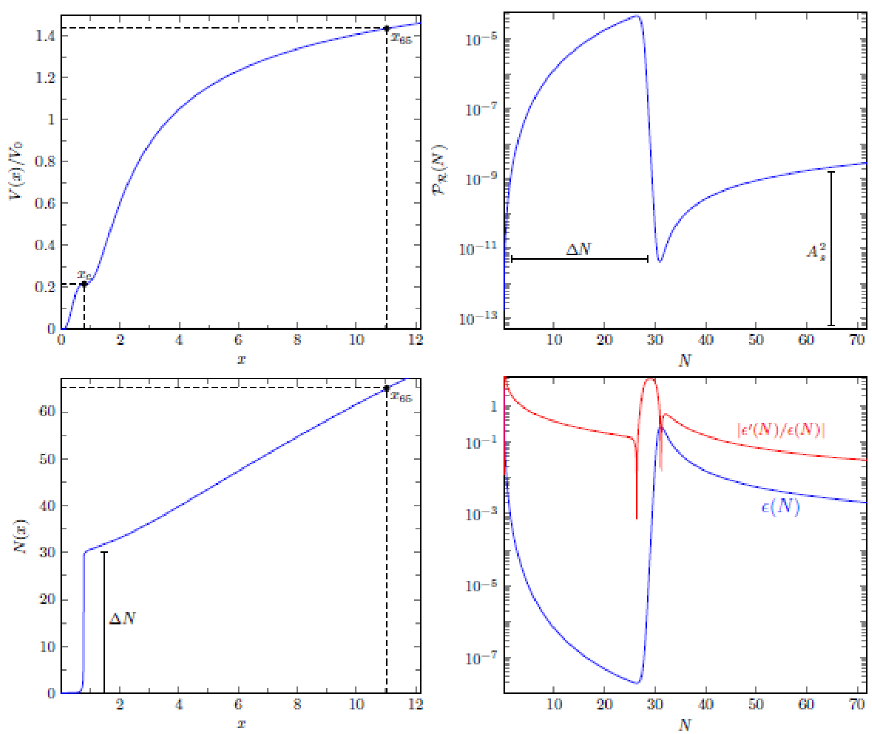
Motohashi, Hu, PRD 96 (17) 063503

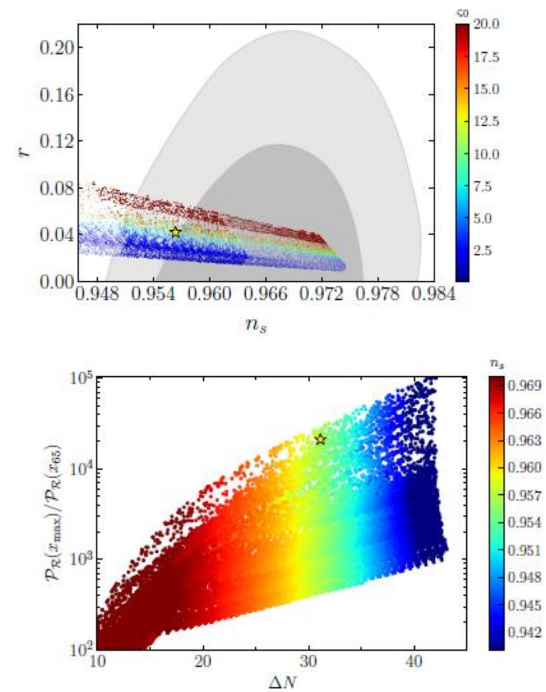
Critical Higgs Inflation

- The Higgs inflation (nonminimal coupling)

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{2}(1+\xi\phi^2)R-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi-\frac{1}{4}\lambda(\phi)\phi^4\right]$$

$$\lambda(\phi)=\lambda_0+b_\lambda\ln^2(\phi/\mu) \qquad \xi(\phi)=\xi_0+b_\xi\ln(\phi/\mu)$$



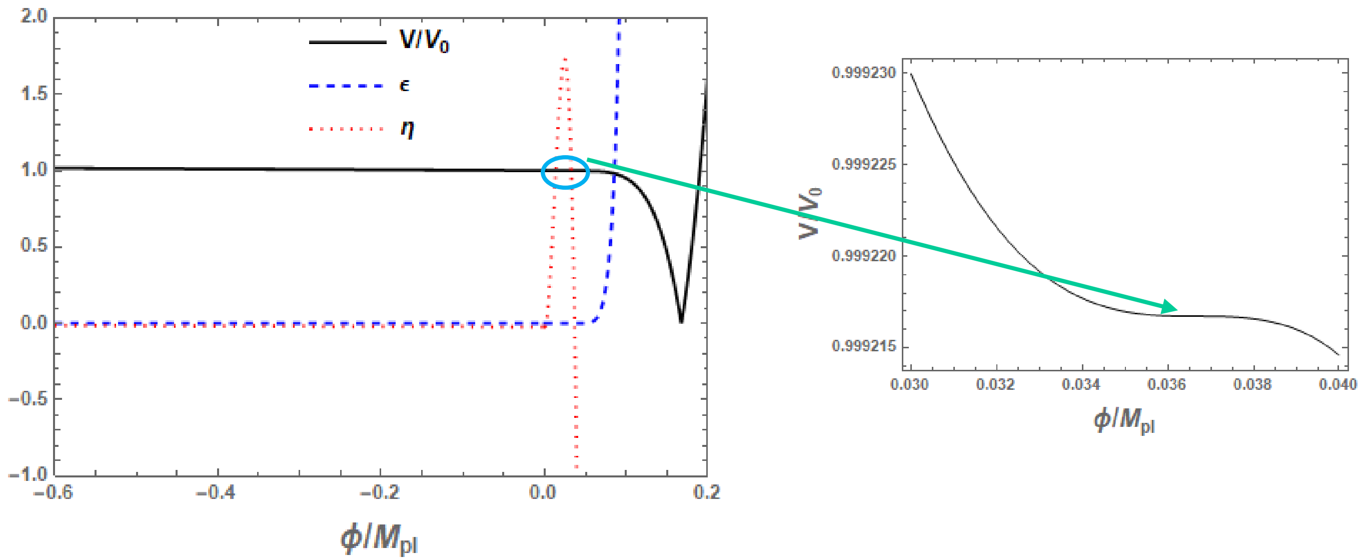


Ezquiaga, Garcia-Bellido, Morales, PLB 776 (18) 345

Our model

- Polynomial potential

$$V(\phi) = \begin{cases} V_0 \left| 1 + \sum_{m=1}^{m=5} \lambda_m \left(\frac{\phi}{M_{\rm Pl}} \right)^m \right|, & \phi \geq 0, \\ V_0 \left[1 + \sum_{m=1}^{m=3} \lambda_m \left(\frac{\phi}{M_{\rm Pl}} \right)^m \right], & \phi < 0, \end{cases}$$



The model parameters

- The parameters: determined by observational data

Planck15: AA 594 (16) A20

$$k_* = 0.05\text{Mpc}^{-1},\ n_s = 0.9674,\ r = 0.005,\ n'_s = -0.0008$$

$$\mathcal{P}_\zeta = 2.2 \times 10^{-9}$$

- Specify $\phi_*,\ N$

$$\phi_* = -0.5,\ \phi_{\text{infl}} = 0.0376$$

$$\lambda_1 = -0.0353553,\ \lambda_2 = -0.0115783,\ \lambda_3 = -0.00235702,$$

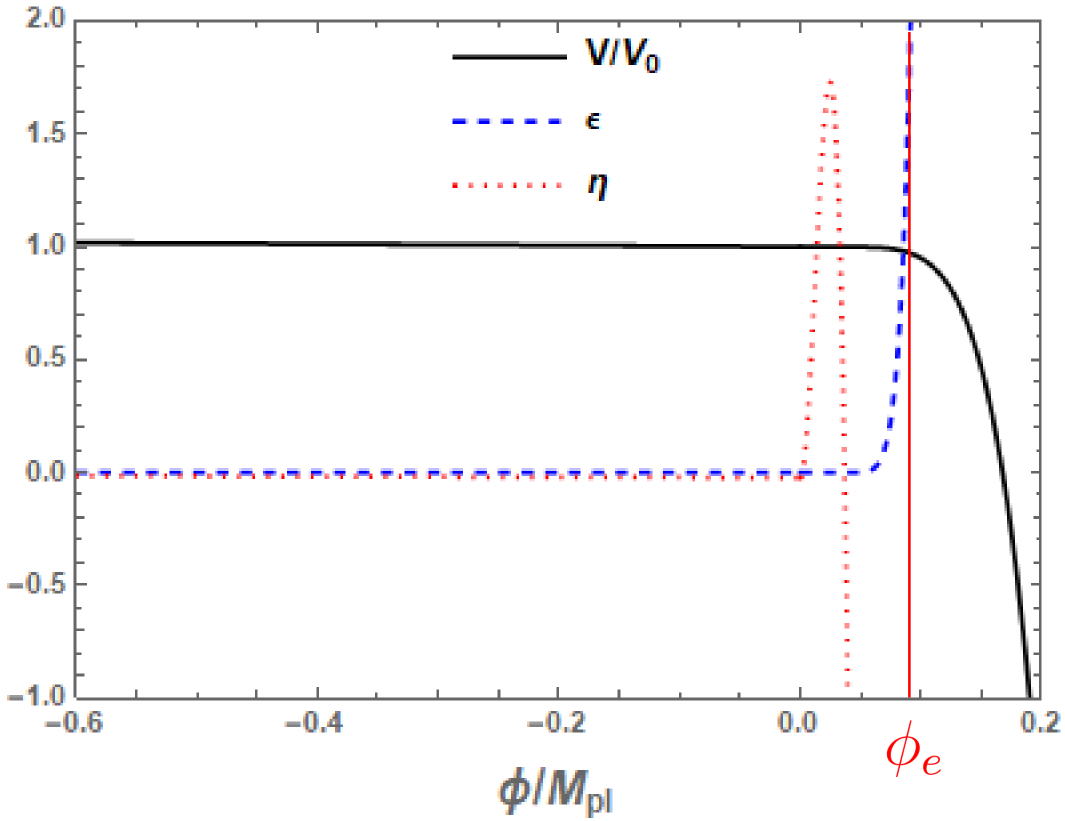
$$\lambda_4 = 728.239, \lambda_5 = -11882.9$$

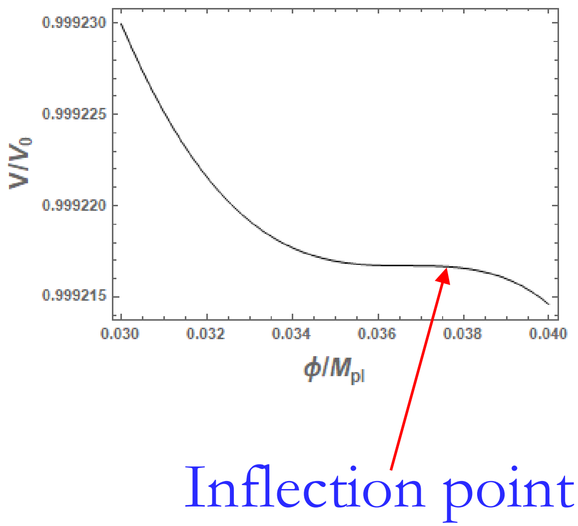
$$\phi_e = 0.14,\ N = 60.99$$

Inflection point

- Slow-roll parameters

$$\epsilon = \frac{1}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \qquad \eta = \frac{V_{,\phi\phi}}{V}$$





Ultra-slow-roll Inflation

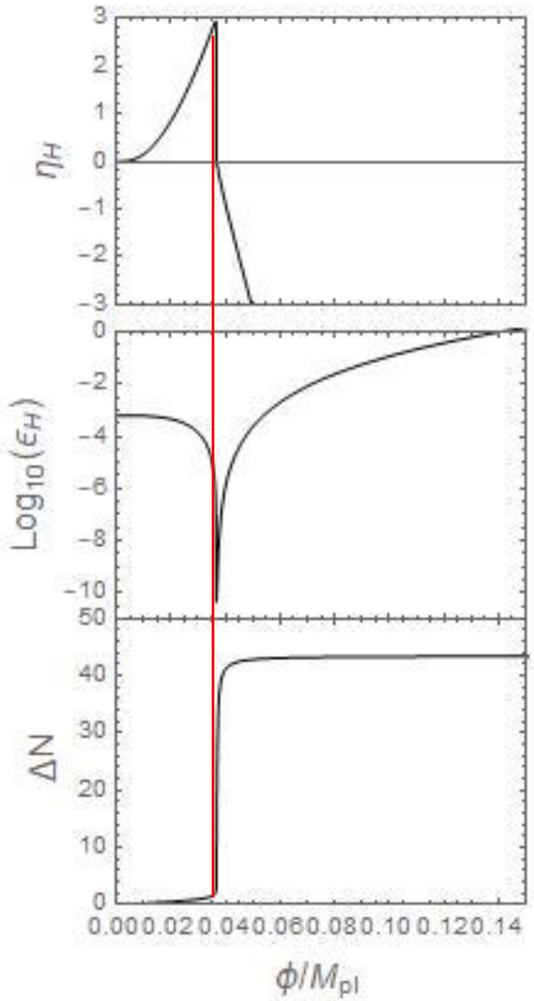
- Inflation

$$\eta_H = -\frac{\ddot{\phi}}{H\dot{\phi}}$$

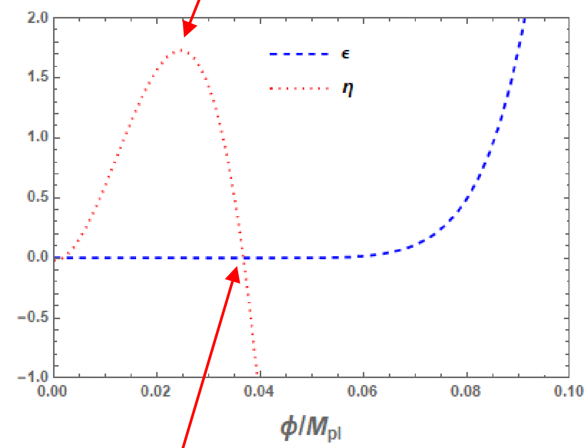
$$\epsilon_H = -\frac{\dot{H}}{H^2} \leq 1$$

$$\phi_{\rm infl} = 0.0376$$

$$\phi_e = 0.14$$



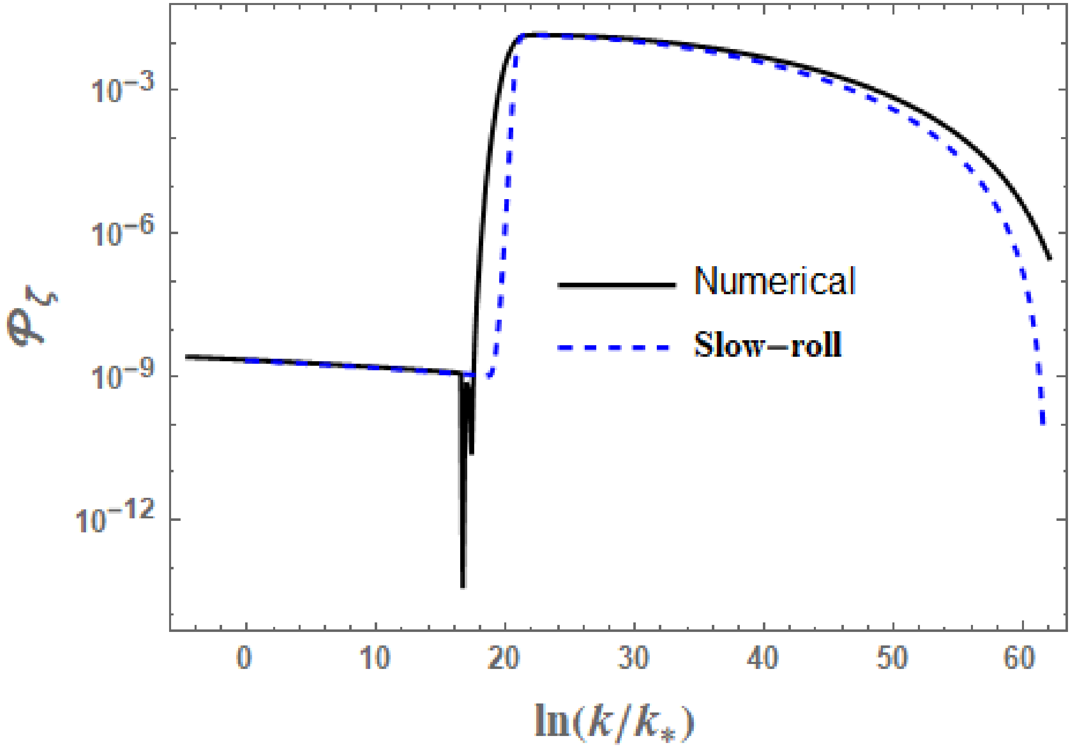
Ensure the reach of inflection point



Inflection point

The power spectrum

- Numerical solution $$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0$$



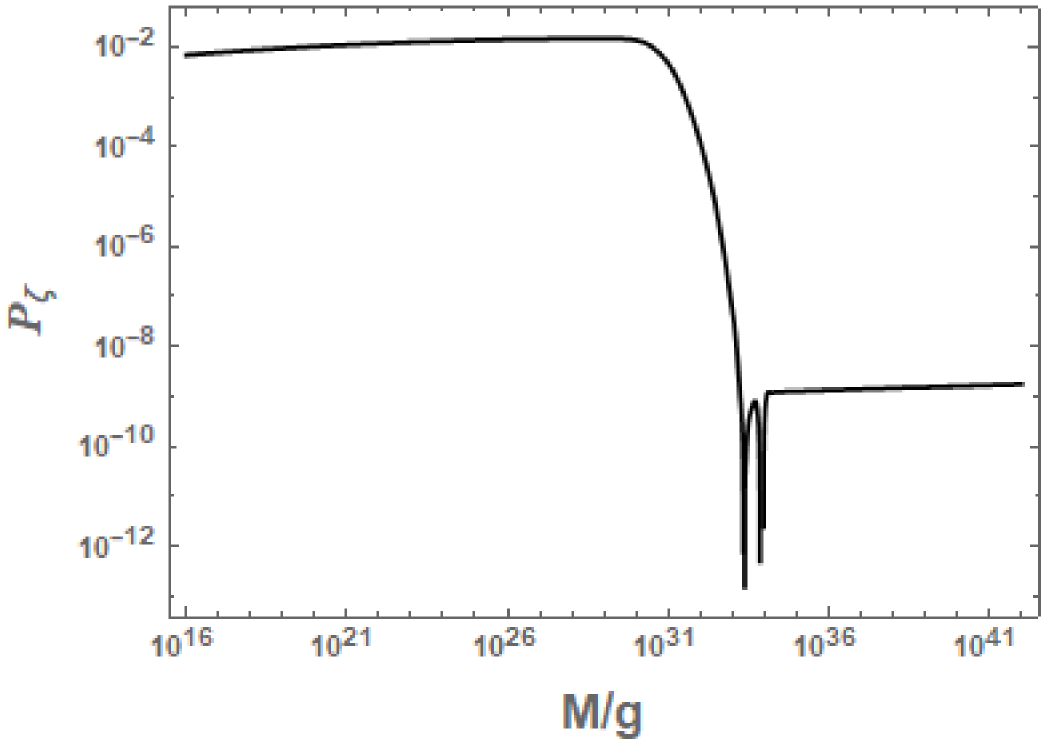
Gong, 1707.09578

$$k_* = 0.05\mathrm{Mpc}^{-1},\ n_s = 0.9674,\ r = 0.005,\ n_s' = -0.0008$$

Peak value $$\mathcal{P}_\zeta = 0.0175$$

The power spectrum

- Mass scale

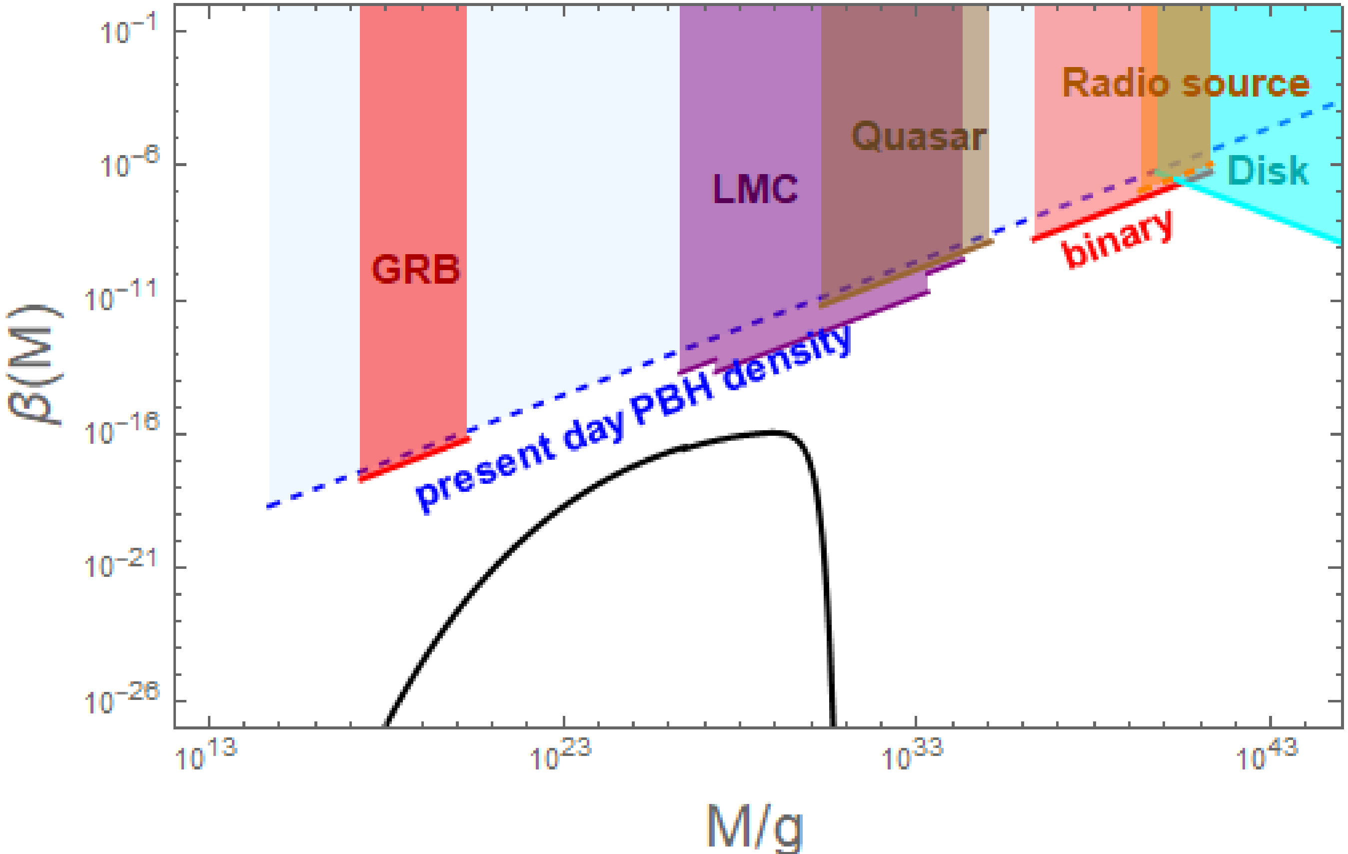


$$g = \begin{cases} 106.75, & T > 300\text{GeV} \\ 3.36, & T \lesssim 0.5\text{MeV} \\ 10.75, & \text{otherwise} \end{cases}$$

$$M \approx 18.4\gamma M_\odot \left(\frac{g}{10.75}\right)^{-1/6} \left(\frac{k}{10^6\text{Mpc}^{-1}}\right)^{-2} \qquad \gamma = 3^{-3/2}$$

The PBH

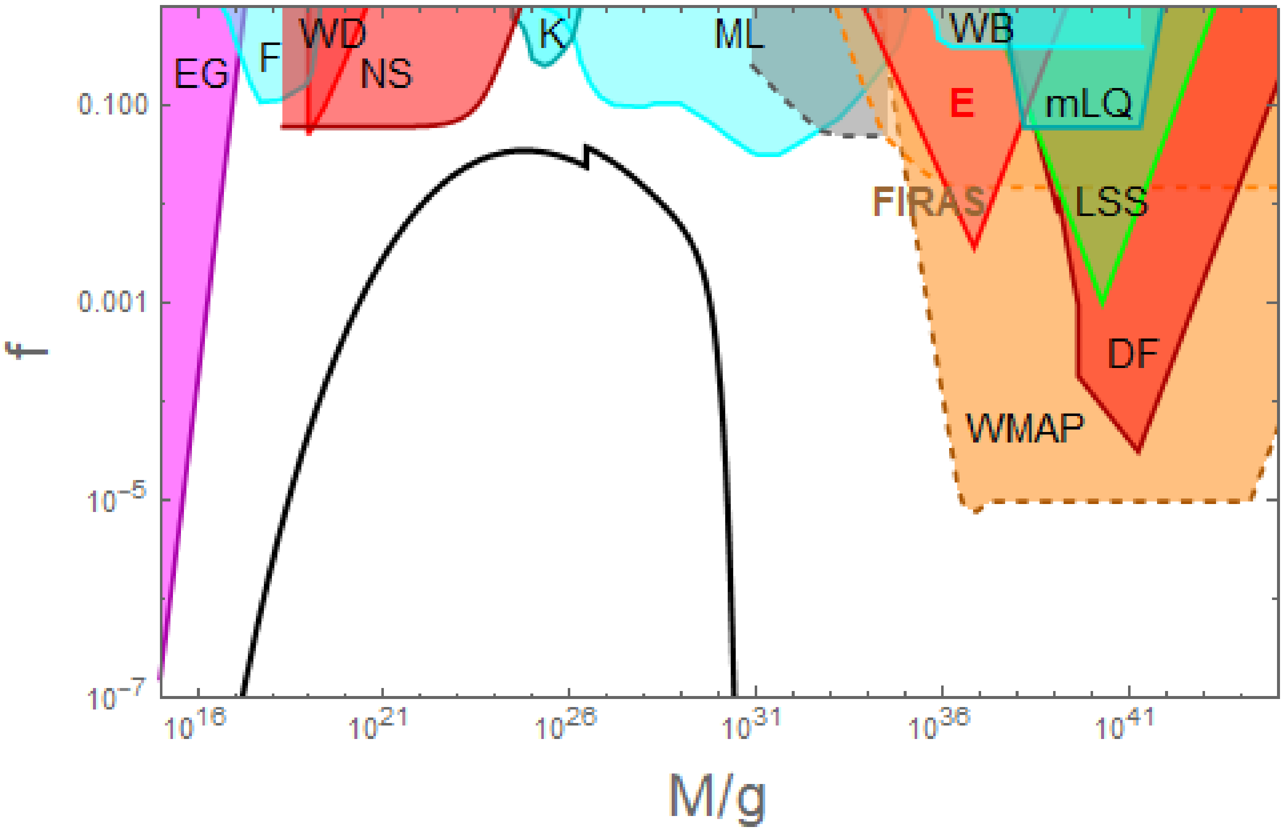
- Mass fraction



$$\beta(M) \approx \mathrm{erfc}\left(\frac{9\delta_c}{4\sqrt{2\mathcal{P}_\zeta(M)}}\right) \qquad \delta_c = 0.45$$

PBH DM

- The constraint



$$\delta_c = 0.41$$

$$\gamma = 3^{-3/2}$$

$$f = 5.68 \times 10^8 \beta(M) \frac{0.12}{\Omega_c h^2} \left(\frac{g}{10.75}\right)^{-1/4} \left(\frac{\gamma M_\odot}{M}\right)^{1/2}$$

The other constraint

- The μ distortion

Inomata etal. PRD 95 (17) 123510;
Nakama etal., PRD 95 (17) 121302

$$\mu_{\rm ac} \approx \int_{k_{\rm min}}^{\infty} \frac{dk}{k} \mathcal{P}_\zeta(k) W_\mu(k),$$

$$W_\mu(k) = 2.8A^2 \left[\exp\left(-\frac{[\hat{k}/1360]^2}{1 + [\hat{k}/260]^{0.3} + \hat{k}/340} \right) - \exp\left(-\left[\frac{\hat{k}}{32}\right]^2 \right) \right],$$

$$k_{\rm min} \approx 1{\rm Mpc}^{-1}, \quad A \approx 0.9, \ \hat{k} = k/[1\ {\rm Mpc}^{-1}]$$

$$\mu_{\rm ac} = 1.95 \times 10^{-8}$$

The induced GWs

- The generation of secondary GWs from large density perturbation

Matarrese, Mollerach, Bruni, PRD 58 (98) 043504
Ananda, Clarkson, Wands, PRD 75 (07) 123518
Baumann, Steinhardt, Takahashi, PRD 76 (07) 084019
Saito, Yokoyama, PRL 102 (09) 161101; 107 (11) 069901, Prog. Theor. Phys. 123 (10) 867
Alabidi, Kohri, Sasaki Sendouda, JCAP 1209 (12) 017
Bugaev, Klimai, PRD 81 (10) 023517; 83 (11) 083521
Inomata, Kawasaki, Mukaida, Tada, Yanagida, PRD 95 (17) 123510
Orlofsky, Peirce, Wells, PRD 95 (17) 063518
Nakama, Silk, Kamionkowski, PRD 95 (17) 043511
Garcia-Bellido, Peloso, Unal, JCAP 1709 (17) 013

Secondary GWs

- RD

$$\frac{d^2 h(\vec{k},\eta)}{d\eta^2} + \frac{2}{\eta}\frac{dh(\vec{k},\eta)}{d\eta} + k^2 h(\vec{k},\eta) = S(\vec{k},\eta),$$

$$S(\vec{k},\eta) = \int \frac{d^3\tilde{k}}{(2\pi)^{3/2}}\tilde{k}^2 \left[1 - \left(\frac{\vec{k}\cdot\vec{\tilde{k}}}{k\tilde{k}}\right)^2\right] \left\{12\Phi(\vec{k}-\vec{\tilde{k}},\eta)\Phi(\vec{\tilde{k}},\eta) + 8\left[\eta\Phi(\vec{k}-\vec{\tilde{k}},\eta) + \frac{\eta^2}{2}\frac{d\Phi(\vec{k}-\vec{\tilde{k}},\eta)}{d\eta}\right]\frac{d\Phi(\vec{\tilde{k}},\eta)}{d\eta}\right\},$$

$$\frac{d^2\Phi}{d\eta^2} + \frac{4}{\eta}\frac{d\Phi}{d\eta} + \frac{1}{3}k^2\Phi = 0.$$

Bardeen potential

- Bardeen equation

$$w = c_s^2$$

$$\Phi'' + \frac{6(1+w)}{(1+3w)\eta}\Phi' + wk^2\Phi = 0$$

- Radiation Dominated (inside horizon)

$$\Phi = 3\Phi_p \left[\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3})\cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3}\right]$$

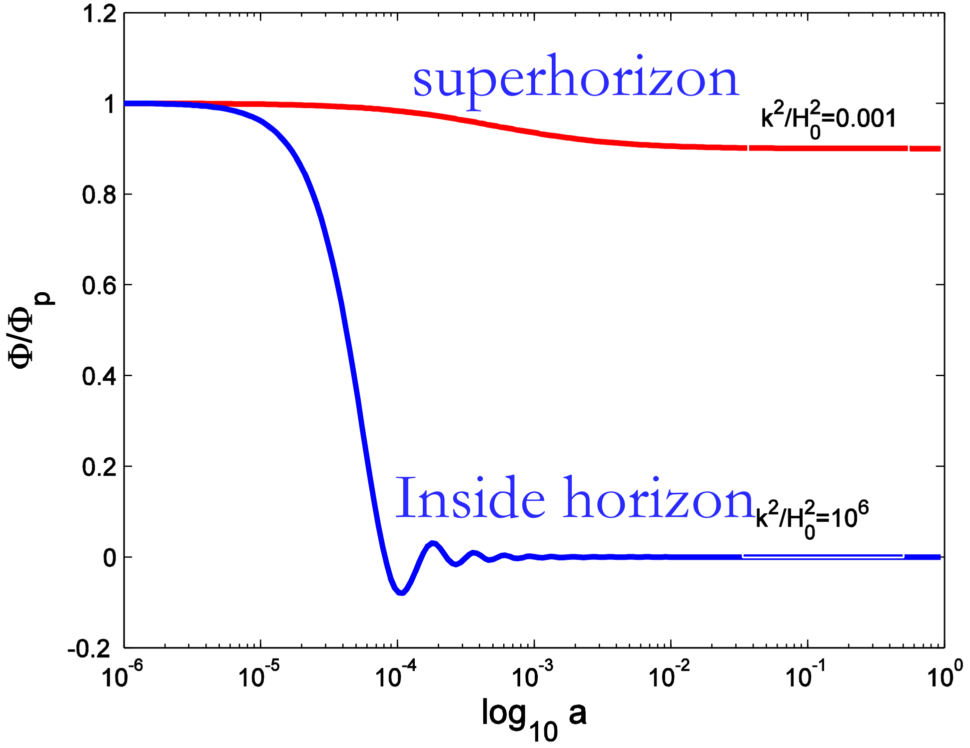
- Matter Dominated (inside horizon)

$$\Phi = A + \frac{B}{(k\eta)^5}$$

Density Perturbation

- The density perturbation $\Phi = \frac{3(1+w)}{5+3w}\zeta$

$$\delta = -\frac{4}{\sqrt{3}} j_1[k/(\sqrt{3}aH)]\left(\frac{k}{aH}\right)\zeta$$



Green's function

- Secondary GWs

$$h''(\vec{k},\eta) + 2\mathcal{H}h'(\vec{k},\eta) + k^2 h(\vec{k},\eta) = S(\vec{k},\eta)$$

- Green's function $g = ah$

$$g'' + \left(k^2 - \frac{a''}{a}\right) g = \delta(\eta - \tilde{\eta})$$

$$g(\eta,\tilde{\eta}) = \frac{\sin[k(\eta-\tilde{\eta})]}{k}, \ \mathrm{RD}$$

$$g(\eta,\tilde{\eta}) = -k\eta\tilde{\eta}\left[j_1(k\eta)y_1(k\tilde{\eta}) - j_1(k\tilde{\eta})y_1(k\eta)\right], \ \mathrm{MD}$$

- Solution

$$h(\vec{k},\eta) = \frac{1}{a(\eta)}\int d\tilde{\eta}\, g(\eta,\tilde{\eta})a(\tilde{\eta})S(\vec{k},\tilde{\eta})$$

The power spectrum

- The second-order tensor power spectrum

$$P_h(k,\eta) = \int_0^\infty d\tilde{k} \int_{-1}^{1} d\mu P_\Phi(|\vec{k} - \vec{\tilde{k}}|) P_\Phi(\tilde{k}) \mathcal{F}(k, \tilde{k}, \mu, \eta),$$

$$\mathcal{F}(k, \tilde{k}, \mu, \eta) = \frac{(1-\mu^2)^2}{a^2(\eta)} \frac{k^3 \tilde{k}^3}{|\vec{k} - \vec{\tilde{k}}|^3} \int_0^\eta d\eta_1 a(\eta_1) g(\eta, \eta_1) f(\vec{k}, \vec{\tilde{k}}, \eta_1)$$

$$\times \int_0^\eta d\eta_2 a(\eta_2) g(\eta, \eta_2) [f(\vec{k}, \vec{\tilde{k}}, \eta_2) + f(\vec{k}, \vec{k} - \vec{\tilde{k}}, \eta_2)]$$

$$f(\vec{k}, \vec{\tilde{k}}, \eta) = 12\Phi(\vec{k} - \vec{\tilde{k}}, \eta)\Phi(\vec{\tilde{k}}, \eta) + 8\eta\Phi(\vec{k} - \vec{\tilde{k}}, \eta)\Phi'(\vec{\tilde{k}}, \eta)$$

$$+4\eta^2\Phi'(\vec{k} - \vec{\tilde{k}}, \eta)\Phi'(\vec{\tilde{k}}, \eta)$$

RD

The power spectrum

- The induced GWs

$$\Omega_{GW} = \frac{1}{12}\left(\frac{k}{aH}\right)^2 P_h$$

$$\Omega_{GW} = A_{GW} P_\zeta^2 \begin{cases} \frac{a(\eta)}{a_{eq}}\frac{k}{k_{eq}}, & k < k_{eq} \\ \frac{a(\eta)}{a_{eq}}\left(\frac{k}{k_{eq}}\right)^{2-2\gamma}, & k_{eq} < k < k_c(\eta) \\ \frac{a_{eq}}{a(\eta)}, & k > k_c(\eta) \end{cases}$$

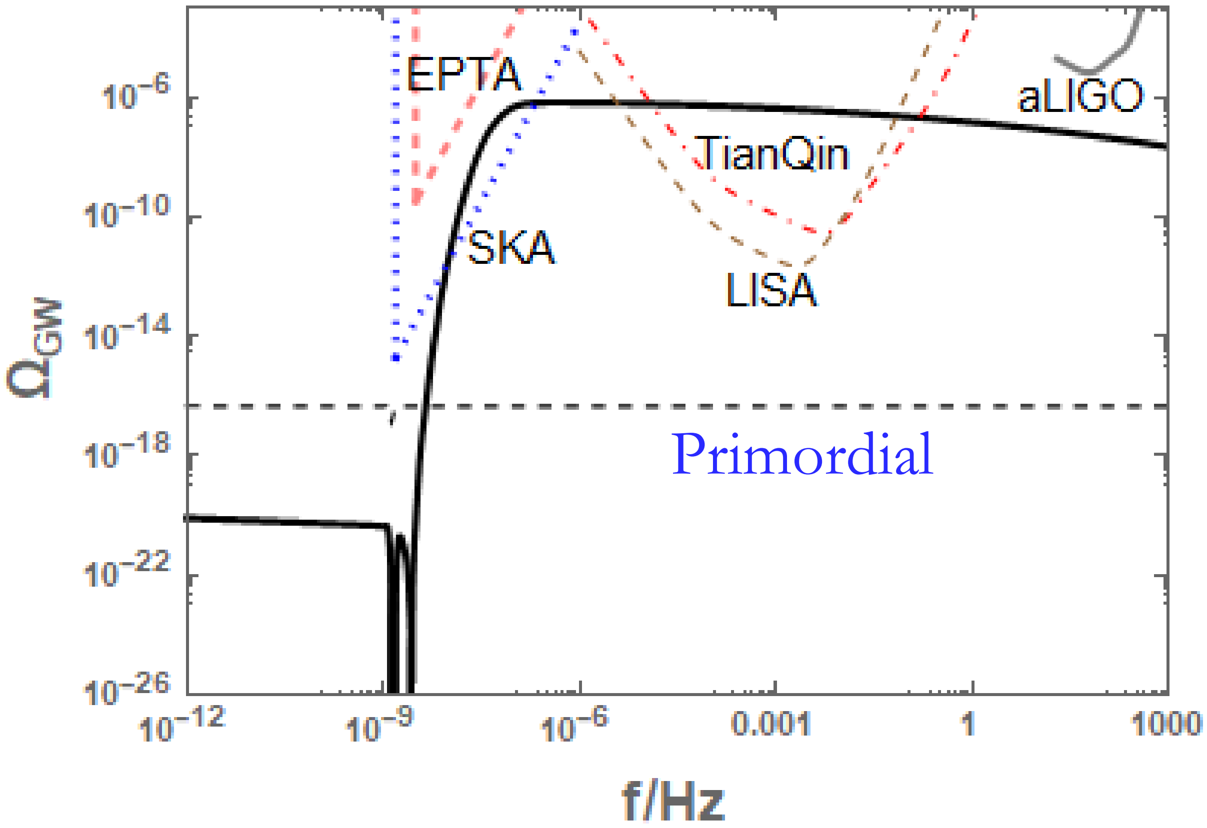
$$k_c(\eta) = \left(\frac{a(\eta)}{a_{eq}}\right)^{1/(\gamma-1)} k_{eq}, \; A_{GW} \approx 10, \; \gamma \approx 3$$

Baumann, Steinhardt, Takahashi, PRD 76 (07) 084019

Possible Detection

- The secondary GWs

$$\Omega_{GW}(k,\eta_0) \approx 10 P_\zeta^2 a_{eq}, \;\; a_0 = 1$$



Conclusions

- The large enhancement on the power spectrum at small scales is possible from ultra-slow-roll inflation
- To reach the ultra-slow-roll inflation, it is required that $|\eta| > 1$ for a brief time before the scalar field reaches the inflection point
- The scalar field is not trapped in the ultra-slow-roll inflation because of the inflection point
- The enhancement is large enough to produce PBHs and generate observable secondary GWs

Thank You