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On Birkhoff's Theorem in Horava Gravity

Mu-In Park
Sogang Univ., Korea

Based on arXiv:1804.05698
Collaboration with Deniz O. Devecioglu

Plan

1. Motivations
2. Introduction to Horava gravity: [P. Horava, arXiv: 0901.3775[PRD]].
3. Birkhoff's theorem in Horava gravity
4. Future directions

1. Motivations

- We are in the era of GR test:

(1) Strong gravity test: GR waves (2015-) from BH-BH, NS-NS, etc, or from early universe perturbations;

(2) Weak gravity test: solar test (?-), binary planets [NS-NS: 1975-], or from late time cosmic acceleration [dark energy(1998-)],

- There are many candidates for beyond GR:

(1) Modifications in UV: Higher curvature gravities (f(R), Horava gravity, DFT, ...),

(2) Modification in IR: Lorentz violating gravities (Einstein aether, Horava gravity), Massive gravity, ...

Most (all) of these have additional gravity degrees of freedom ! This could be a good News or bad News, depending on its context !

- **Good News:** if it explains the missing parts in our current understanding of universe.

(1) Origin of inflaton fields (f(R) ?),

(2) Origin of dark energy (f(R), massive gravity ?)

- **Bad News:**

(1) No one has observed it yet !

(2) In solar system test, it doesn't seem to exist, or very strongly constrained: IR test !

- Cf. In UV, it's existence is still open !

- Why new gravity degrees of freedom ?
- DOF= # of variables - # of constraints.

- Higher curvature gravity (R+R^2+...):

(i) additional conjugate momenta due to higher-time derivatives,

(ii) additional constraints between new variables.

But usually increase of (i) is faster than (ii) ! : Increase of total DOF ! (>DOF=2 in GR)

- **Lorentz violating gravities (Horava gravity (2009))**
- **(i) No additional conjugate momenta due to absence of higher-time derivatives.**
- **(ii) reduced symmetries: reduce of gauge equivalent orbits (projectable N(t), non-projectable N(t,r))**

- **Total DOF "could" increase !! : The new degrees of freedom is called scalar graviton.**
- **But there have been long debates about this scalar graviton and a source of confusions !**

- Constraint analysis (for non-projectable case N(t,r)): Non-perturbative analysis !

(1) Low energy (lambda-R model):

DOF=2 !(the same as in GR)

(2) Higher energy (lambda-R+R^2)

DOF=2+1/2 !! (Bellorin et al)

(3) Extreme UV (lambda-Cotton^2)

DOF=2+1/2 !? (Li et al)

Cf. Henneaux et al: " Horava gravity is inconsistent ! "; "only N(t,r)=0 is possible !!"

- Perturbation analysis(for non-projectable case N(t,r)): IR+UV
- (1) Pert. about Minkowski background:
- No scalar graviton ! (in Horava "gauge")
- (2) Pert. about flat (k=0) FRW background with matter: No scalar graviton ! (Gao et al, Gong et al, Shin-Park).

- Cf. strong coupling problem in Minkowski background.

- These conflicts between different analyes have the source of confusions !

- Today, I am going to resolve these conflicts by considering Brikhoff’s theorem in Horava gravity.

- What is Birkhoff’s theorem and why is it important in the conflicts of Hoava gravity ?

- Birkhoff's theorem(1921,1923): No time-dependent, spherically symmetric vacuum solutions in GR.
- This implies:

(1) Uniqueness of a spherically symmetric solution as the static one : Schwarzschild solution. (cf. Newtonian gravity: Shell theorem)

(2) No gravitational radiation for pulsating or collapsing, spherically symmetric bodies !! : No spin-0 or scalar gravitons !

- Why is this important in resolving the Horava gravity conflicts ?

(i) In GR, there is no scalar graviton: no time-dependent spherically symmetric metric (Birkhoff's theorem).

(ii) Similarly, there would be no scalar graviton if there is no time-dependent spherically symmetric metric: in Birkhoff's set-up, scalar mode can be isolated, without unnecessary complications (involving usual gravitons) !

(iii) This analysis can be fully non-linear !!

2. Horava gravity: Introduction

[P. Horava, arXiv: 0901.3775[PRD]].

The Action Construction:

- **Einstein-Hilbert action:**

$$S_{EH} = \frac{1}{16\pi G_N}\int dx^4\sqrt{-g^{(4)}}\left(R^{(4)} - 2\Lambda\right)$$

Lorentz invariant !

Lorentz scalars

$$= \frac{1}{16\pi G_N}\int d^4x\,\sqrt{g}\,N\left\{\left(K_{ij}K^{ij} - K^2\right) + R - 2\Lambda\right\}$$

$R^{(4)}$

in ADM decomposition

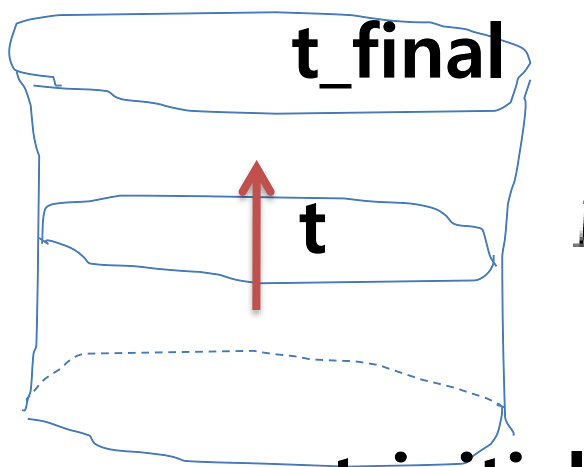
$$ds^2 = -N^2c^2dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$$

- Here, we have used the Gauss-Godacci relation (up to boundary terms)

$$R^{(4)} = (K_{ij}K^{ij} - K^2) + R$$

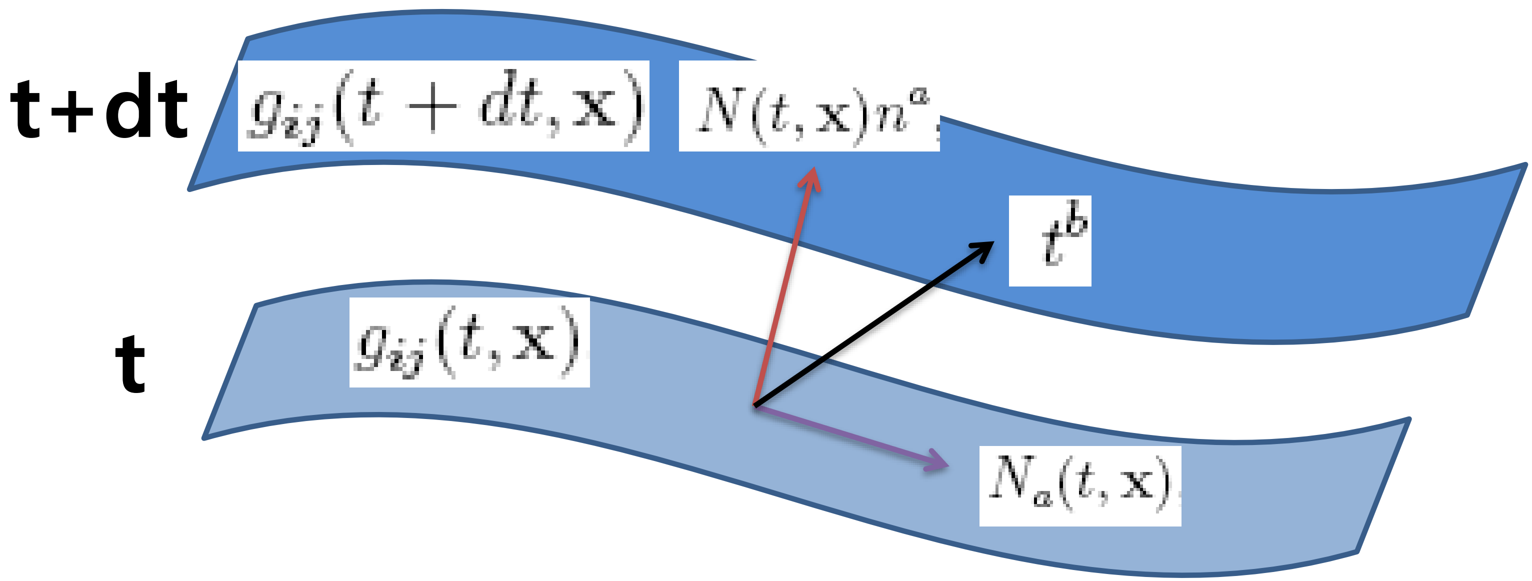
Extrinsic curvature of t=constant hypersurface

Intrinsic curvature : 3 curvature



$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i)$$

- **Here,** $N(t,\mathbf{x}), N_i(t,\mathbf{x})$ **determine the time foliations completely (with** $g_{ij}(t,\mathbf{x})$ **) :**



- Under the full Diff.

$$\delta x^i = -\zeta^i(t,\mathbf{x}),\ \delta t = -\zeta^t(t,\mathbf{x}),$$

the foliation is not preserved generally (no absolute time).

- But, one may consider foliation preserving Diff.

$$\begin{aligned}
\delta x^i &= -\zeta^i(t,\mathbf{x}),\ \delta t = -f(t),\\
\delta g_{ij} &= \partial_i\zeta^k g_{jk} + \partial_j\zeta^k g_{ik} + \zeta^k\partial_k g_{ij} + f\dot{g}_{ij},\\
\delta N_i &= \partial_i\zeta^j N_j + \zeta^j\partial_j N_i + \dot{\zeta}^j g_{ij} + f\dot{N}_i + \dot{f}N_i\\
\delta N &= \zeta^j\partial_j N + f\dot{N} + \dot{f}N.
\end{aligned}$$

↑ Exact symmetry for N(t,x) !

- In the anisotropic scaling (mom.) dimensions,

$$[\mathbf{x}] = -1, \qquad [t] = -z,$$

we do not need to keep the Lorentz invariant combinations only. (Planck unit)

- For example, we may consider

$$\left(K_{ij}K^{ij} - \lambda K^2\right) + \beta R$$

, in which the Lorentz symmetry is explicitly broken for

$$\lambda \neq 1, \beta \neq 1$$

but there is still Foliation Preserving diffeomorphisms (FPDiff).

- However, in order not to introduce higher-time derivatives to avoid the ``possible" ghost problems, we do not consider "simply" the following terms

$$(K_{ij}K^{ij})^2, K^4 \cdots$$

but only consider

$$R^2,\ R_{ij}R^{ij},\ \nabla_k R_{ij}\nabla^k R^{ij},\ \ \cdots$$

So, the action can be written as

$$S_{\text{Horava}} = \frac{2}{\kappa^2}\int dt\, d^D\mathbf{x}\, \sqrt{g}N\left(K_{ij}K^{ij} - \lambda K^2\right)$$

Kinetic term

$$+ \int dt\, d^D\mathbf{x}\, \sqrt{g}N\, V[g_{ij}]$$

Potential term

- Horava gravity (z=D) is (power-counting) renormalizable, higher-derivative gravity theory, without the ghost problem in the usual covariant higher-curvature gravities, by considering higher-derivatives in space but not in time !

- Cf. Proof of renormalizability ? : done for 3D projectable case. We need to extend to 4D, non-projectable case !

3. Birkhoff's in Horava Gravity

- In addition to the standard Horava action, we consider the modification of potential $V[gij]$, with $a_i = \partial_i lnN$

$$\delta\mathcal{V}[g_{ij}, a_i] = -\frac{\sigma}{2} a_i a^i,$$

[Blas et al,; Extended Horava gravity]

This could be another source of IR Lorentz violation generally.

- **In order to study Birkhoff's theorem, let's consider spherically symmetric, time-dependent ansatz**

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

The EQM are

$$\begin{aligned}
\mathcal{H} &= -\frac{2(1-\lambda)}{\kappa^2}\dot{\beta}^2 e^{-2\alpha} + \frac{\kappa^2\mu^2}{8(1-3\lambda)}\left[\frac{2(\Lambda_W-\omega)e^{-2\beta}}{r^2}\left(2r\beta' + e^{2\beta} - 1\right) - 3\Lambda_W^2\right] \\
&\quad + \frac{\kappa^2\mu^2 e^{-4\beta}}{8(1-3\lambda)r^4}\left\{-2(1-\lambda)r^2\beta'^2 - 4\lambda r\beta'(e^{2\beta}-1) - (1-2\lambda)(e^{2\beta}-1)^2\right\} \\
&\quad - \sigma e^{-2\beta}\left(\frac{1}{2}\alpha'^2 + \alpha'' + \frac{2}{r}\alpha' - \alpha'\beta'\right) = 0\,, \\
\mathcal{H}^r &= \frac{e^{-(\alpha+2\beta)}}{r}\left[(2 - r(1-\lambda)\alpha')\,\dot{\beta} + r(1-\lambda)\dot{\beta}'\right] = 0\,,
\end{aligned}$$

$$\begin{aligned}
E_\beta &= \frac{2(1-\lambda)}{\kappa^2}(\ddot{\beta} - \dot{\alpha}\dot{\beta})e^{2(\beta-\alpha)} + \frac{\kappa^2\mu^2(\Lambda_W-\omega)}{4(1-3\lambda)r}(\alpha'+\beta') - \frac{\sigma}{2}\left(\alpha'' + \frac{2}{r}\alpha' - \alpha'\beta'\right) \\
&\quad + \frac{\kappa^2\mu^2 e^{-2\beta}}{4(1-3\lambda)r^4}\left\{(1-\lambda)\left[r^2(\beta'^2 - \beta'' - \alpha'\beta') + (e^{2\beta}-1)\right] - \lambda r(\alpha'+\beta')(e^{2\beta}-1)\right\} = 0
\end{aligned}$$

- **GR** case ($\lambda = 1, \sigma = 0$): $\{\cdots\}$ **terms in** $\mathcal{H}$ **,** E_β **are absent: One can obtain the unique sol**

$$\dot{\beta} = 0 \text{ from } \mathcal{H}_i = 0$$

$$\dot{\alpha}' = 0 \text{ from } E_\beta = 0$$

$$\boxed{\alpha(t,r) = a(t) + b(r)}$$

a(t) can be removed by redefining time t !

: time-independence of metric (Birkhoff's theorem !)

General Cases: $\lambda \neq 1$ or $\sigma \neq 0$

$$\dot{\mathcal{H}} = -\sigma e^{-2\beta}\left[\left(\alpha' - \beta' + \frac{2}{r}\right)\dot{\alpha}' + \dot{\alpha}''\right] = 0\,, \quad (8)$$

$$\begin{aligned}\dot{E}_\beta &= \frac{\kappa^2\mu^2 e^{-2\beta}}{4(1-3\lambda)r^4}\left[-\lambda r(e^{2\beta}-1) - (1-\lambda)r^2\beta' + (\Lambda_W - \omega)r^3 e^{2\beta}\right]\dot{\alpha}' - \frac{\sigma}{2}\left[\left(-\beta' + \frac{2}{r}\right)\dot{\alpha}' + \dot{\alpha}''\right] \\ &= \left\{\frac{\sigma}{2}\alpha' + \frac{\kappa^2\mu^2}{4(1-3\lambda)r^4}\left[-\lambda r(1-e^{-2\beta}) - (1-\lambda)r^2 e^{-2\beta}\beta' + (\Lambda_W - \omega)r^3\right]\right\}\dot{\alpha}' = 0\,, \quad (9)\end{aligned}$$

- **A. Case $\dot{\beta} = 0$:**
- **Either $\lambda = 1$ or $\lambda \neq 1$, we can generally prove $\dot{\alpha}' = 0$ (Birkhoff-like), except the undetermined lapse case for $\sigma = 0$:**

$$\beta(t,r) = -ln\sqrt{1 + (\omega - \Lambda_W)r^2 + Cr^{\frac{2\lambda}{\lambda-1}}} \qquad C = 0,\ \omega(\omega - 2\Lambda_W) = 0$$

But these do not have GR limit !!:

$$\lambda \to 1,\ \mu \to 0,\ \omega \to \infty,\ \Lambda_W \to \infty \text{ with ` } \mu^2\omega,\ \mu^2\Lambda_W \sim \text{fixed}$$

- B: Case $\dot{\beta} \neq 0$:
- For $\lambda \neq 1$, the case $\dot{\beta} \neq 0$, other than the usual sol. $\dot{\beta} = 0$, is possible: From momentum constraint one can get

$$\alpha'(t,r) = \frac{2}{r(1-\lambda)} + \frac{\dot{\beta}'}{\dot{\beta}}$$

and can be integrated as

$$\alpha(t,r) = ln\left(\dot{\beta}(t,r)r^{\frac{2}{1-\lambda}}\right) + a(t),$$

with undetermined, time-dependent function a(t).

- **Then, the Hamiltonian constraint reduces to (for $\sigma = 0$)**

$$\frac{16(1-\lambda)(1-3\lambda)}{\kappa^4\mu^2} r^{\frac{4}{\lambda-1}} e^{-2a(t)} = \frac{2(\Lambda_W - \omega)}{r^2}(-rf' + 1 - f) - 3\Lambda_W^2 + \frac{1}{r^4}\left\{-\frac{1}{2}(1-\lambda)r^2 f'^2 + 2\lambda r f'(1-f) - (1-2\lambda)(1-f)^2\right\}$$

where $f(t,r) \equiv e^{-2\beta(t,r)}$ **.**

1. With UV term $\{\cdots\}$ only in RHS, we can solve "exactly":

$$f(t,r) = 1 \pm \frac{4|\lambda - 1| r^{\frac{2\lambda}{\lambda-1}}}{\epsilon\sqrt{-\kappa^4\mu^2}} e^{-a(t)}, \quad \epsilon \equiv \mathrm{sign}(3\lambda - 1)$$

From $\dot{\beta} \propto \dot{a} \neq 0$, this manifestly violates Birkhoff's theorem in UV.

$$\alpha(t,r) = ln\left[\frac{2|\lambda-1|r^2\dot{a}(t)}{\pm\epsilon\sqrt{-\kappa^4\mu^2} + 4|\lambda-1|r^{\frac{2\lambda}{\lambda-1}}e^{-a(t)}}\right]$$

- Solution exists only for "dS" branch

$$\Lambda \propto -\mu^2 > 0$$

, like our accelerating Universe !

2. Only with IR terms (first two terms in RHS), one can prove that no sol exists:

No time-dependent sol in IR. (Birkhoff's)

- 3. With extension terms and IR terms, one can find one exact, time-dependent (but particular) sol.

$$\alpha(t,r) = ln\left(\frac{8\dot{C}_1(t)}{r\sqrt{-\kappa^4\mu^2\omega}}\right), \quad \beta(t,r) = \frac{C_1(t)}{r^2} + b(r),$$

with $\lambda = -1, \Lambda_W = 0, \sigma = -\mu^2\kappa^2\omega/8, a(t) = -ln\sqrt{-\kappa^4\mu^2\omega/64}$

two arbitrary functions $C_1(t)$ and $b(r)$.

- For other values of parameters, one can find the solution numerically (Mathematica).

• Numerical sols:

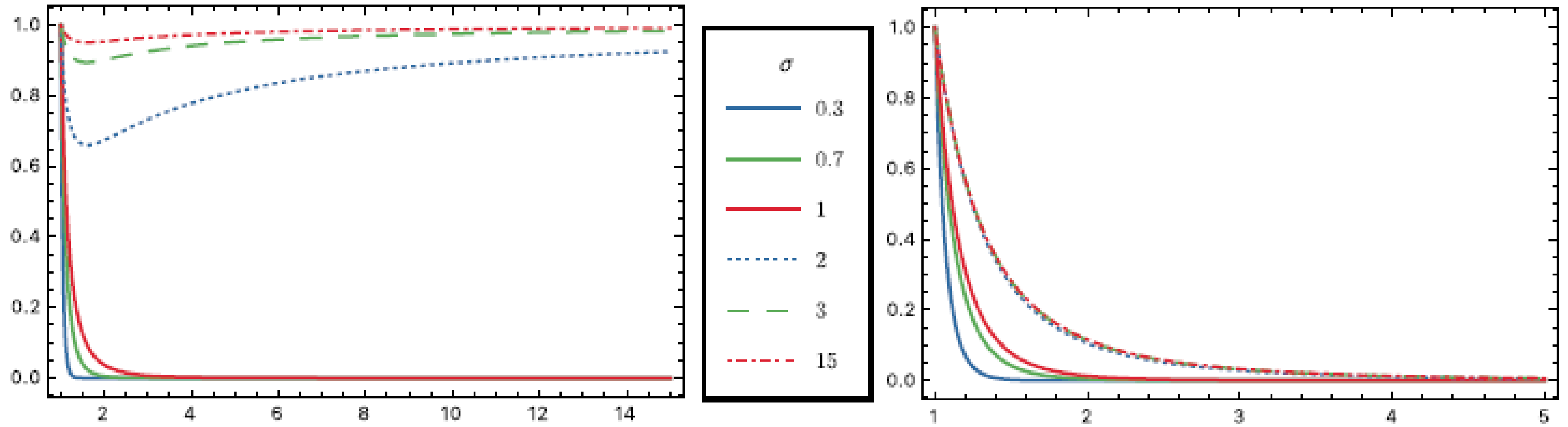


FIG. 1: Plots of numerical solutions for $f(t,r)^{-1} = e^{2\beta(t,r)}$ (left), $\dot{\beta}(t,r)$ (right) vs. r for varying σ, at $t = t_0$. Here, we have considered $\lambda = 0.35,\ \Lambda_W = 0,\ \omega = 0.225,\ \mu = 3,\ \kappa = 1,\ a(t_0) = e^3, \dot{a}(t_0) = 0$. These show two different branches of solutions with different asymptotes, $f = 1$ for $\sigma > 1$ (upper curves) or $f = \infty$ for $\sigma \leq 1$ (lower curves).

Remarks:

- Time-dependent sol in UV (without extensions) is genuine non-linear effect:

$$f^{-1} = e^{2\beta} = (1+\zeta^{-1})(1+a(t)\zeta^{-1}+\cdots),$$
$$N^2 = e^{2\alpha} = \frac{1}{4}r^{\frac{4}{1-\lambda}}\dot{a}^2(t)\zeta^{-2}[1+2a(t)\zeta^{-1}+\cdots]$$

$$\zeta = 1 \pm \frac{\epsilon\sqrt{-\kappa^4\mu^2}}{4|\lambda-1|}r^{\frac{-2\lambda}{\lambda-1}}$$

- For small a(t) and zeta(r)^-1 near Minkowski, lambda>1, small r, time-dependence appear only sub-leading (non-linear) order:
- $\dot{a}^2(t)$ factor in $N^2 = e^{2\alpha}$ can be removed by $dt \rightarrow dt' = dt/\dot{a}(t)$

- This is consistent with Constraint analysis [Bellorin, et al.] (cf. Li, et al.)
- This implies the scalar graviton, which is represented by a(t), is excited in UV as non-linear effect but “decoupled” in IR.
- So this implies that GR is recovered in IR (even) when fully non-linear effect is considered. (cf. Vainshtein mechanism in massive gravity)

(cf. strong coupling problem in linear analysis (Blas et al.))

4. Future directions

- The role of UV scalar graviton in early Universe based on Horava gravity: Role of inflaton field ? We need non-linear cosmological perturbations !!

- Can Horava gravity provide a consistent framework for Big Bang cosmology without artificial primordial (inflaton) scalar field and inflation scenario but with non-linearly excited UV scalar gravitons ?

- Gravitational radiation (power) of spherically collapsing stars: Test in future GRW detectors ??