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Cosmological Bell Inequality and Entangled Quantum Vacuum

based on Sugumi Kanno & J.S., arXiv:1705.06199
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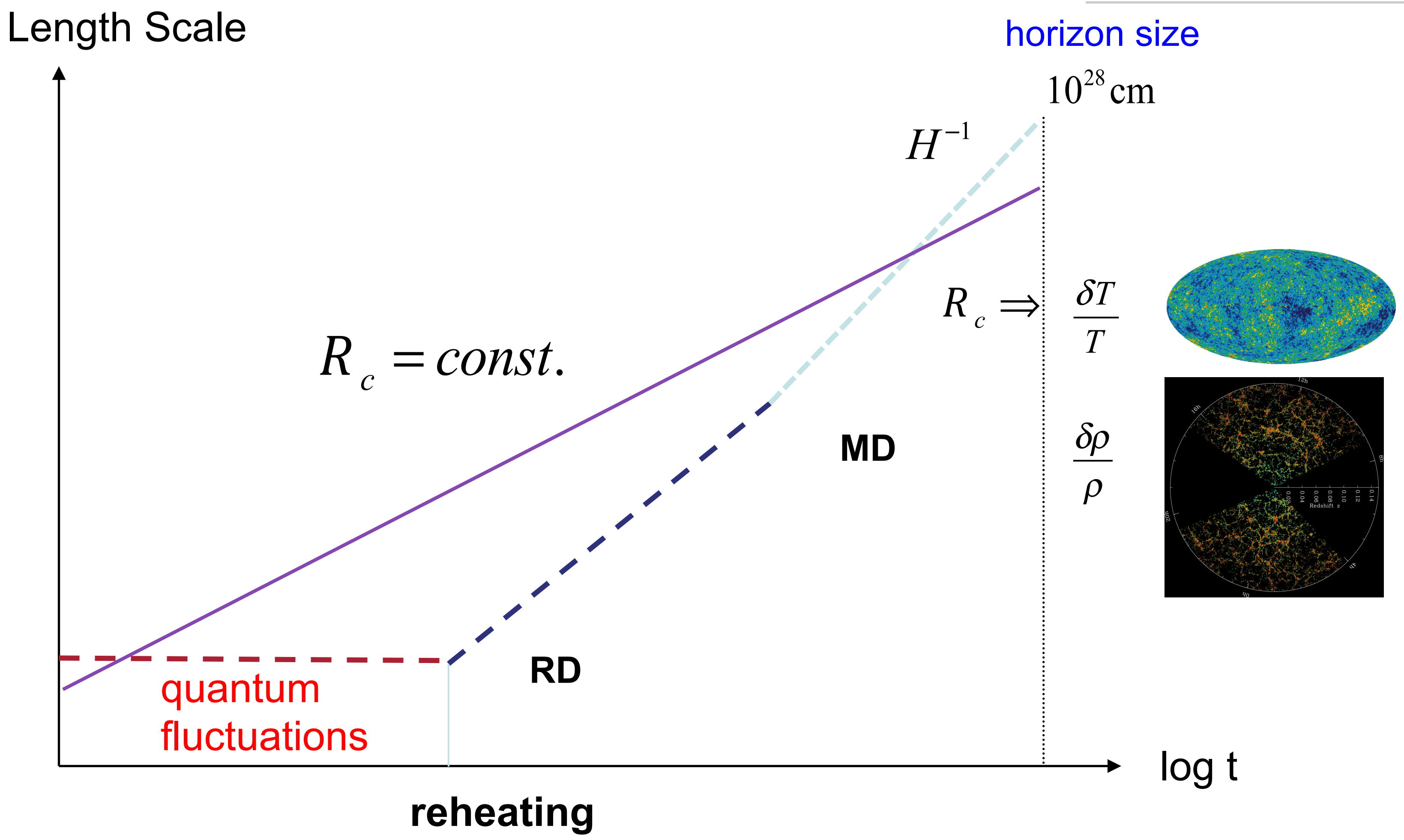
The origin of LSS is quantum fluctuations!!



It is extremely important to prove the quantumness of primordial fluctuations.

How to characterize the quantumness?

If we can observe quantumness of primordial fluctuations,
we can prove that the origin of LSS is quantum fluctuations.

In particular, detecting quantumness of PGW implies the discovery of gravitons!

To achieve the ultimate aim,
we need to characterize the quantumness of the initial quantum state.

How to find quantumness in the cosmological data?

Bell inequality

Campo & Parentani 2006

Maldacena 2016

Bell inequality can probe the entanglement of quantum states.

Hence, as a first step, we try to classify the quantumness
of the initial quantum state in terms of Bell-like inequality.

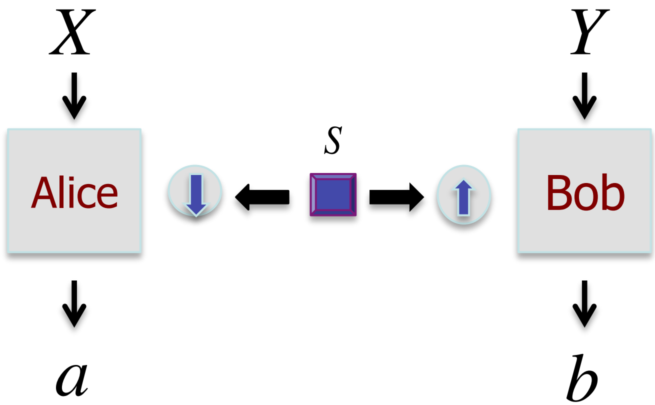
Plan of my talk

1. Bell inequality
2. Entangled quantum vacuum
3. Mermin-Klyshko inequality
4. Cosmological Bell-MK inequalities
5. Summary

Quantum non-locality vs local hidden variable theory

From the source S, two particles with opposite spins are ejected.

$$|0\rangle = |\uparrow\rangle \quad |1\rangle = |\downarrow\rangle$$



Alice and Bob are well separated
and they cannot communicate each other.

The state is a singlet and a superposition of
up-down and down-up.

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left\{|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\right\}$$

If Alice measure the spin and get up spin,
Bob should detect down spin, and vice versa.

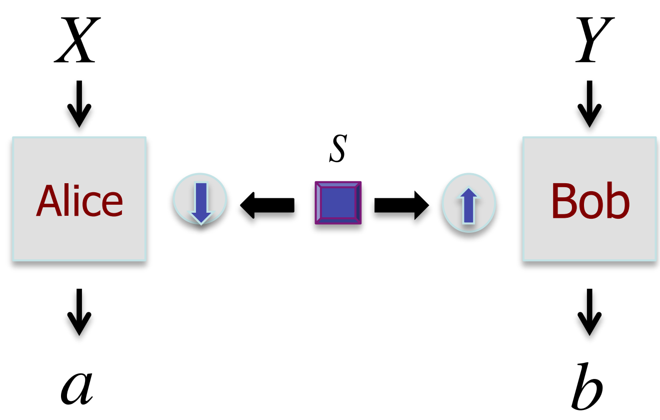
Is this a spooky action at a distance, quantum non-locality?
Is there any local hidden variable theory to explain this phenomena?

Spin system in Local hidden variable theories

From the source S, two particles with opposite spins are ejected.

Alice choose a measurement X
and get an outcome a.

Bob choose a measurement Y
and get an outcome b.



$X, Y = \{0, 1\}$ $\qquad$ $a, b = \{-1, 1\}$

After repeating the measurement many times,
we obtain a joint probability $p(ab \mid XY)$

It turned out there exists a correlation $p(ab \mid XY) \neq p(a \mid X)p(b \mid Y)$

A local hidden variable theory

$$p(ab \mid XY) = \int d\lambda \, q(\lambda) \, p(a \mid X, \lambda) p(b \mid Y, \lambda)$$

λ : a hidden variable $\qquad$ $q(\lambda)$: a probability for λ

$p(a \mid X, \lambda)$: a probability for a

Bell inequality

Bell 1964
Clauser et al (CHSH) 1969

Bell inequality

$$S = \int d\lambda q(\lambda) S_\lambda = \frac{1}{2}\left[\langle a_0 b_0\rangle + \langle a_0 b_1\rangle + \langle a_1 b_0\rangle - \langle a_1 b_1\rangle\right] \le 1$$

<proof>

$$\langle a_X b_Y\rangle = \sum_{a,b} a b p(ab \mid XY) = \int d\lambda\, q(\lambda)\ \underline{a\, p(a \mid X,\lambda)}\ \underline{b\, p(b \mid Y,\lambda)} = \int d\lambda\, q(\lambda) \underline{\langle a_X\rangle_\lambda}\, \underline{\langle b_Y\rangle_\lambda}$$

$$S_\lambda = \frac{1}{2}\left[\langle a_0\rangle_\lambda \langle b_0\rangle_\lambda + \langle a_0\rangle_\lambda \langle b_1\rangle_\lambda + \langle a_1\rangle_\lambda \langle b_0\rangle_\lambda - \langle a_1\rangle_\lambda \langle b_1\rangle_\lambda\right]$$

$$= \frac{1}{2}\langle a_0\rangle_\lambda \left\{\langle b_0\rangle_\lambda + \langle b_1\rangle_\lambda\right\} + \frac{1}{2}\langle a_1\rangle_\lambda \left\{\langle b_0\rangle_\lambda - \langle b_1\rangle_\lambda\right\}$$

$$\left|\langle a_{0,1}\rangle_\lambda\right| \le 1$$

$$S_\lambda \le \frac{1}{2}\left|\langle b_0\rangle_\lambda + \langle b_1\rangle_\lambda\right| + \frac{1}{2}\left|\langle b_0\rangle_\lambda - \langle b_1\rangle_\lambda\right|$$

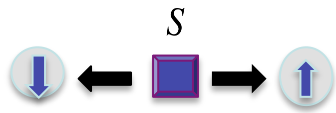
Without loosing generality, we can assume $\langle b_0\rangle_\lambda \ge \langle b_1\rangle_\lambda \ge 0$

$$S_\lambda \le \langle b_0\rangle_\lambda \le 1$$

$$\therefore S = \int d\lambda\, q(\lambda)\, S_\lambda \le 1$$

Spin system in quantum theory

$$|0\rangle = |\uparrow\rangle \quad |1\rangle = |\downarrow\rangle$$



spin operators $\quad s_x = |0\rangle\langle 1| + |1\rangle\langle 0| \qquad s_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0| \qquad s_z = |0\rangle\langle 0| - |1\rangle\langle 1|$

entangled state $\quad |\psi\rangle = \frac{1}{\sqrt{2}}\underline{\{|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\}}$

spin measurement $\quad O = \vec{n} \bullet \vec{s} = \sin\theta\, s_x + \cos\theta\, s_z$

$$(\vec{n}_1 \bullet \vec{s}) \otimes (\vec{n}_2 \bullet \vec{s})|\psi\rangle = \left[\sin\theta_1 \{\,|0\rangle\langle 1| + |1\rangle\langle 0|\} + \cos\theta_1 \{|0\rangle\langle 0| - |1\rangle\langle 1|\}\right]$$

$$\otimes\left[\sin\theta_2 \{\,|0\rangle\langle 1| + |1\rangle\langle 0|\} + \cos\theta_2 \{|0\rangle\langle 0| - |1\rangle\langle 1|\}\right]\frac{1}{\sqrt{2}}\{|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\}$$

$$= \frac{1}{\sqrt{2}}\left[\sin\theta_1 \sin\theta_2 \underline{\{-|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle\}} + \sin\theta_1 \cos\theta_2 \{-|1\rangle \otimes |1\rangle - |0\rangle \otimes |0\rangle\}\right.$$

$$\left. + \cos\theta_1 \sin\theta_2 \{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\} + \cos\theta_1 \cos\theta_2 \underline{\{-|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle\}}\right]$$

$$\therefore \langle\psi|O_1 \otimes O_2|\psi\rangle = -\cos(\theta_1 - \theta_2)$$

Quantum violation of Bell inequality

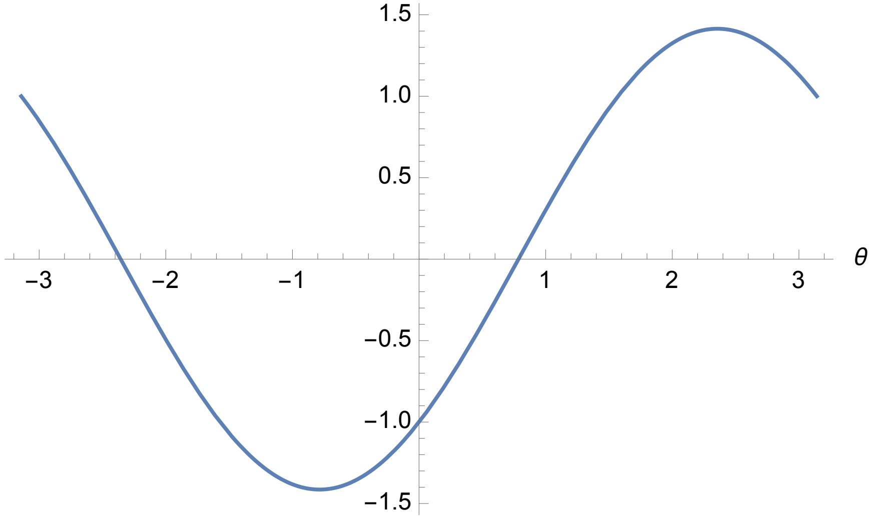
Bell operator $2M_2 = O_1 \otimes O_2 + O_1 \otimes O'_2 + O'_1 \otimes O_2 - O'_1 \otimes O'_2$

$$\langle\psi|M_2|\psi\rangle = \frac{1}{2}\left[-\cos(\theta_1-\theta_2)-\cos(\theta_1-\theta'_2)-\cos(\theta'_1-\theta_2)+\cos(\theta'_1-\theta'_2)\right]$$

What is the maximal value?

$$\theta_1 = \theta, \quad \theta'_1 = -\theta, \quad \theta_2 = 0, \quad \theta'_2 = -\frac{\pi}{2}$$

$\langle\psi|M_2|\psi\rangle$



$$\theta_1 = \frac{3\pi}{4}, \quad \theta'_1 = -\frac{3\pi}{4}, \quad \theta_2 = 0, \quad \theta'_2 = -\frac{\pi}{2}$$

$$\langle\psi|M_2|\psi\rangle = \sqrt{2} > 1$$

Thus, Bell inequality is violated in quantum theory.

Another view of Bell inequality

Cirelson 1980

It is useful to see the origin of violation of Bell inequality.

$$M_2 = \frac{1}{2}\left[O_1 O_2 + O_1 O'_2 + O'_1 O_2 - O'_1 O'_2\right] \qquad O = \vec{n}\bullet\vec{s} \qquad \vec{n}\bullet\vec{n} = 1$$

The square of Bell operator can be calculated using $s_i s_j = \delta_{ij} + i\varepsilon_{ijk} s_k$

$$(M_2)^2 = I - \frac{1}{4}[O_1, O_1'][O_2, O'_2]$$

Since $\left[s_i, s_j\right] = 2i\varepsilon_{ijk} s_k \Rightarrow \quad \left\|[O, O']\right\| \le 2$

We obtain

Quantum Bell inequality $\quad \therefore \left|M_2\right| \le \sqrt{2}$

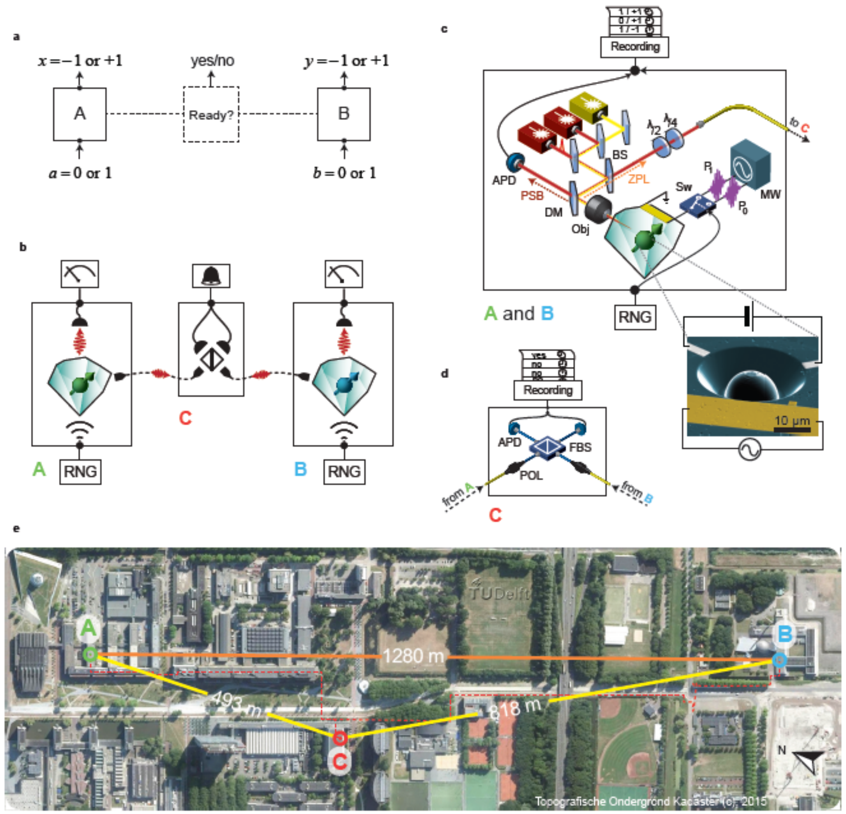
Namely, the non-commutativity is the origin of the violation of Bell inequality.
It is also important to realize that the quantum bound of the violation exists.
If this bound is violated, that means quantum theory is not enough.

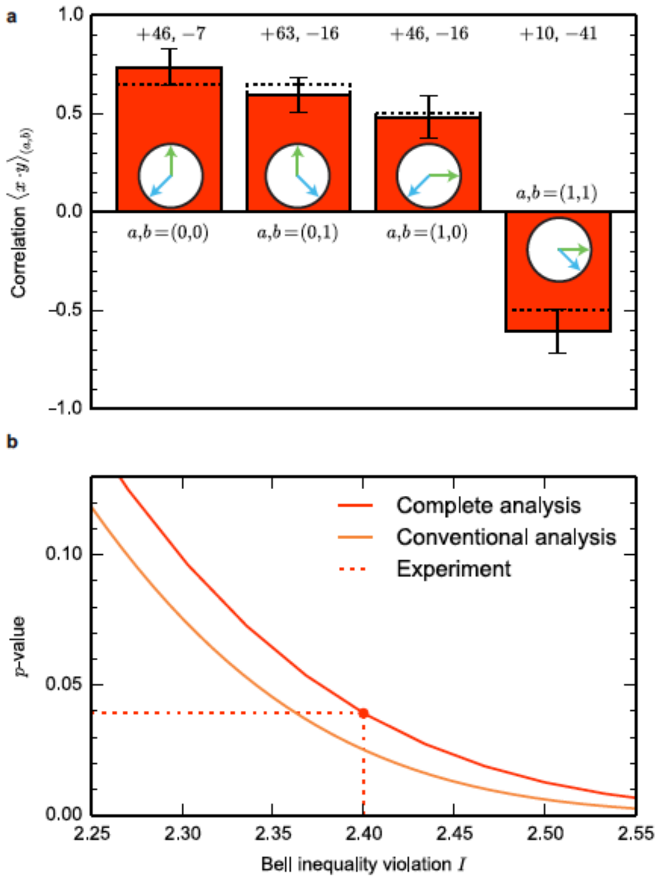
Aspect, Grangier and Roger 1981

Hensen et al. 2015; Giustina et al. 2015; Shalm et al. 2015

$$S = \langle x \cdot y \rangle_{(0,0)} + \langle x \cdot y \rangle_{(0,1)} + \langle x \cdot y \rangle_{(1,0)} - \langle x \cdot y \rangle_{(1,1)} \leq 2$$

Hensen et al. 2015





ENTANGLED QUANTUM VACUUM

Scalar field in inflationary universe

$$\left[\nabla^{\mu}\nabla_{\mu} - m^2\right]\phi = 0 \qquad ds^2 = a^2(\eta)\left[-d\eta^2 + dx^2 + dy^2 + dz^2\right]$$

de Sitter inflation $\quad a(\eta) = -\dfrac{1}{H(\eta - 2\eta_r)} \quad -\infty < \eta < \eta_r$

radiation dominant $\quad a(\eta) = \dfrac{\eta}{H\eta_r^2} \quad \eta_r < \eta$

$$a\phi_k(\eta) = u_k(\eta)a_{\mathbf{k}} + u_k^*(\eta)a_{-\mathbf{k}}^{\dagger} \qquad \left(\frac{d^2}{d\eta^2} + k^2 - \frac{a''}{a}\right)u_k(\eta) = 0$$

de Sitter inflation $\quad \left(\dfrac{d^2}{d\eta^2} + k^2 - \dfrac{2}{(\eta - 2\eta_r)^2}\right)u_k(\eta) = 0 \quad -\infty < \eta < \eta_r$

radiation dominant $\quad \left(\dfrac{d^2}{d\eta^2} + k^2\right)u_k(\eta) = 0 \quad \eta_r < \eta$

Vacuum is not unique

As usual in field theory in curved space, the vacuum is not unique.

In vacuum mode $u_k^{in}(\eta) \xrightarrow[\eta \to -\infty]{} \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k(\eta - 2\eta_r)}\right) e^{-ik(\eta - 2\eta_r)}$

Out vacuum mode $u_k^{out}(\eta) \xrightarrow[\eta_r < \eta]{} \frac{1}{\sqrt{2k}} e^{-ik\eta}$

$$a\phi_k(\eta) = u_k^{in}(\eta) a_{\mathbf{k}}^{in} + u_k^{in*}(\eta) a_{-\mathbf{k}}^{in\dagger} = u_k^{out}(\eta) a_{\mathbf{k}}^{out} + u_k^{out*}(\eta) a_{-\mathbf{k}}^{out\dagger}$$

In vacuum $a_{\mathbf{k}}^{in}|0_{in}\rangle = 0$ Out vacuum $a_{\mathbf{k}}^{out}|0_{out}\rangle = 0$

Bogoliubov transformation

$$u_k^{in}(\eta) = A_k u_k^{out}(\eta) + B_k^* u_k^{out} * (\eta) \quad \Longleftrightarrow \quad a_{\mathbf{k}}^{in} = A_k^* a_{\mathbf{k}}^{out} - B_k a_k^{out\dagger}$$

$$A_k = \langle u_k^{out}, u_k^{in} \rangle\Big|_{\eta=\eta_r} = -\frac{e^{2ik\eta_r}}{2k^2\eta_r}\left(1 - 2ik\eta_r - 2k^2\eta_r^2\right) \qquad B_k^* = -\langle u_k^{*out}, u_k^{in} \rangle\Big|_{\eta=\eta_r} = \frac{1}{2k^2\eta_r^2}$$

Two-mode squeezed vacuum

Using the relations

$$\tanh r_k = \frac{B_k}{A_k} = -e^{-2ik\eta_r} \frac{1}{1+2ik\eta_r - 2k^2\eta_r^2}$$

$$a_{\mathbf{k}}^{in} = A_k^* a_{\mathbf{k}}^{out} - B_k a_k^{out\dagger}$$

we can solve the equation $a_{\mathbf{k}}^{in}|0_{in}\rangle = 0$ as

$$|0_{in}\rangle \equiv |BD\rangle = \frac{1}{\cosh r_k} \prod_{\mathbf{k}} e^{\tanh r_k a_{\mathbf{k}}^{out\dagger} a_{-\mathbf{k}}^{out\dagger}} |0_{out}\rangle = \frac{1}{\cosh r_k} \prod_{\mathbf{k}} \sum_{n=0}^{\infty} \tanh^n r_k |n_{\mathbf{k}}^{out}\rangle \otimes |n_{-\mathbf{k}}^{out}\rangle$$

$$= \frac{1}{\cosh r_k} \prod_{\mathbf{k}} \left[|0_{\mathbf{k}}^{out}\rangle \otimes |0_{-\mathbf{k}}^{out}\rangle + \tanh r_k |1_{\mathbf{k}}^{out}\rangle \otimes |1_{-\mathbf{k}}^{out}\rangle + \cdots \right]$$

squeezing parameter

where we defined Bunch-Davies vacuum which is a standard vacuum in inflation.

In the large squeezing limit, the state becomes highly entangled state.

Four-mode squeezed vacuum

Albrecht et al. 2014
Kanno 2015

Let us consider two scalar fields.

$$S=\int d\eta\sum_{\mathbf{k}}\left[a^2\left(\phi_k'\phi_k^{*\prime}-k^2\phi_k\phi_k^*\right)-a^4m_\phi^2\phi_k\phi_k^*+a^2\left(\chi_k'\chi_k^{*\prime}-k^2\chi_k\chi_k^*\right)-a^4m_\phi^2\chi_k\chi_k^*\right]$$

$$a\phi_k(\eta)=u_k^{in}(\eta)a_{\mathbf{k}}^{in}+u_k^{in*}(\eta)a_{-\mathbf{k}}^{in\dagger}\qquad a\chi_k(\eta)=v_k^{in}(\eta)b_{\mathbf{k}}^{in}+v_k^{in*}(\eta)b_{-\mathbf{k}}^{in\dagger}$$

Entangled initial state $\quad\tilde{a}_{\mathbf{k}}=\alpha_k a_{\mathbf{k}}^{in}+\beta_k^* b_{-\mathbf{k}}^{in\dagger}\qquad \tilde{b}_{\mathbf{k}}=\alpha_k b_{\mathbf{k}}^{in}+\beta_k^* a_{-\mathbf{k}}^{in\dagger}$

$$\tilde{a}_{\mathbf{k}}|\psi\rangle=0\qquad \tilde{b}_{\mathbf{k}}|\psi\rangle=0$$

$$\phi_{\mathbf{k}}\Leftrightarrow\chi_{-\mathbf{k}}\qquad \phi_{-\mathbf{k}}\Leftrightarrow\chi_{\mathbf{k}}$$

$$|\psi\rangle=N\prod_{\mathbf{k}}e^{-\frac{\beta_k}{\alpha_k}a_{\mathbf{k}}^\dagger b_{-\mathbf{k}}^\dagger}|BD\rangle$$

$$=N\prod_{\mathbf{k}}\left[|BD\rangle_\phi\otimes|BD\rangle_\chi-\frac{\beta_k}{\alpha_k}|1_{\mathbf{k}}^{in}\rangle_\phi\otimes|1_{-\mathbf{k}}^{in}\rangle_\chi-\frac{\beta_k}{\alpha_k}|1_{-\mathbf{k}}^{in}\rangle_\phi\otimes|1_{\mathbf{k}}^{in}\rangle_\chi+\cdots\right]$$

$$|\mathbf{k}\rangle_\phi\Leftrightarrow|-\mathbf{k}\rangle_\chi$$

$$\Updownarrow\qquad\Updownarrow$$

$$|-\mathbf{k}\rangle_\phi\Leftrightarrow|\mathbf{k}\rangle_\chi$$

If we consider interactions or multi fields, we could have more complicated entanglement.

MERMIN-KLYSHKO INEQUALITY

Bell-Mermin-Klyshko inequality

Mermin 1990

Klyshko 1993

We can generalize Bell inequality to multi-partite systems.

Recursion relation
$$M_n = \frac{M_{n-1}}{2} \otimes (O_n + O'_n) + \frac{M'_{n-1}}{2} \otimes (O_n - O'_n)$$

$$M_1 = O_1 \qquad M'_1 = O'_1$$

Ex:
$$2M_2 = O_1 \otimes O_2 + O_1 \otimes O'_2 + O'_1 \otimes O_2 - O'_1 \otimes O'_2$$

$$O_n = \vec{a}_n \cdot \vec{s} \qquad O_n{}' = \vec{a}'_n \cdot \vec{s}$$

$$2M_3 = M_2 \otimes (O_3 + O'_3) + M'_2 \otimes (O_3 - O'_3)$$

Since these operators dichotomic quantities with outcome +1 and -1, we have

BMK inequality $\left| < M_n > \right| \le 1$

Property of BMK operators

Alsina et al. 2016

We can deduce

$$M_n^2 = I + \sum_{s=1}^{\left[\frac{n}{2}\right]} \frac{(-1)^s}{2^{2s}} \sum_{i_j} \prod_{j=1}^{2s} \left[O_{i_j}, O'_{i_j} \right]$$

Ex: $(M_3)^2 = I - \frac{1}{4}[O_1,O_1'][O_2,O'_2] - \frac{1}{4}[O_1,O_1'][O_3,O'_3] - \frac{1}{4}[O_2,O_2'][O_3,O'_3]$

$$\begin{aligned}(M_4)^2 = I &- \frac{1}{4}[O_1,O_1'][O_2,O'_2] - \frac{1}{4}[O_1,O_1'][O_3,O'_3] - \frac{1}{4}[O_2,O_2'][O_3,O'_3] \\ &- \frac{1}{4}[O_1,O_1'][O_4,O'_4] - \frac{1}{4}[O_2,O_2'][O_4,O'_4] - \frac{1}{4}[O_3,O_3'][O_4,O'_4] \\ &- \frac{1}{16}[O_1,O_1'][O_2,O'_2][O_3,O_3'][O_4,O'_4]\end{aligned}$$

From the above formula, we can get the maximal value

$$\left\langle M_n^2 \right\rangle_{\max} = 1 + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{2\left[\frac{n}{2}\right]} = 2^{n-1}$$

Quantum BMK inequality $\left| < M_n > \right| \le 2^{\frac{n-1}{2}}$

Pseudo-spin operators

Chen et al. 2002

In order to extend Bell inequality to field theory, we define pseudo-spin operators

$$s_z = \sum_{n=0}^{\infty} \left\{ |2n+1\rangle\langle 2n+1| - |2n\rangle\langle 2n| \right\}$$

$$s_- = \sum_{n=0}^{\infty} |2n\rangle\langle 2n+1| = \left(s_+\right)^\dagger$$

which satisfies the same commutation relations as the spin operators

$$\left[s_z,\ s_\pm\right] = \pm 2\, s_\pm \qquad\qquad \left[s_+,\ s_-\right] = s_z$$

With a unit vector $\vec{a} = \left(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\right)$

We can define

$$O = \vec{a}\cdot\vec{s} = \cos\theta\, s_z + \sin\theta\left(e^{i\varphi} s_- + e^{-i\varphi} s_+\right) \quad \Longrightarrow \quad O^2 = 1$$

COSMOLOGICAL BELL-MK INEQUALITIES

Bell inequality in Bunch-Davies vacuum

The standard vacuum in de Sitter space is the Bunch-Davies vacuum

$$\left|BD\right\rangle = \prod_{\mathbf{k}} \frac{1}{\cosh r_k} \sum_{n=0}^{\infty} \tanh^n r_k \left|n_{\mathbf{k}}^{out}\right\rangle \otimes \left|n_{-\mathbf{k}}^{out}\right\rangle$$

The density matrix is given by $\rho = \left|BD\right\rangle\left\langle BD\right|$

For two-partite system, we can take $O = \cos\theta\, s_z + \sin\theta\left(s_- + s_+\right)$

Using the relation

$$\sum_{n=0}^{\infty} \frac{\tanh^n r_k}{\cosh r_k}\left|n_{\mathbf{k}}^{out}\right\rangle \otimes \left|n_{-\mathbf{k}}^{out}\right\rangle = \sum_{n=0}^{\infty} \frac{\tanh^{2n} r_k}{\cosh r_k}\left|(2n)_{\mathbf{k}}^{out}\right\rangle \otimes \left|(2n)_{-\mathbf{k}}^{out}\right\rangle + \sum_{n=0}^{\infty} \frac{\tanh^{2n+1} r_k}{\cosh r_k}\left|(2n+1)_{\mathbf{k}}^{out}\right\rangle \otimes \left|(2n+1)_{-\mathbf{k}}^{out}\right\rangle$$

we can calculate

$$\begin{aligned} E\left(a_1, a_2\right) &= \mathrm{Tr}\, O_1 \otimes O_2\, \rho \\ &= \cos\theta_1 \cos\theta_2 + \tanh 2r_k \sin\theta_1 \sin\theta_2 \end{aligned}$$

Bell inequality in Bunch-Davies vacuum

Taking $\theta_1' = -\theta_1, \quad \theta_2 = 0, \quad \theta_2' = \dfrac{\pi}{2}$

$$\langle BD | M_2 | BD \rangle = \frac{1}{2}\left[E(a_1, a_2) + E(a_1, a_2') + E(a_1', a_2) - E(a_1', a_2') \right]$$

$$= \cos\theta_1 + \tanh 2r_k \sin\theta_1$$

maximum at $\tan\theta_1 = \tanh 2r_k$

$$\langle BD | M_2 | BD \rangle = \sqrt{1 + \tanh^2 2r_k} > 1$$

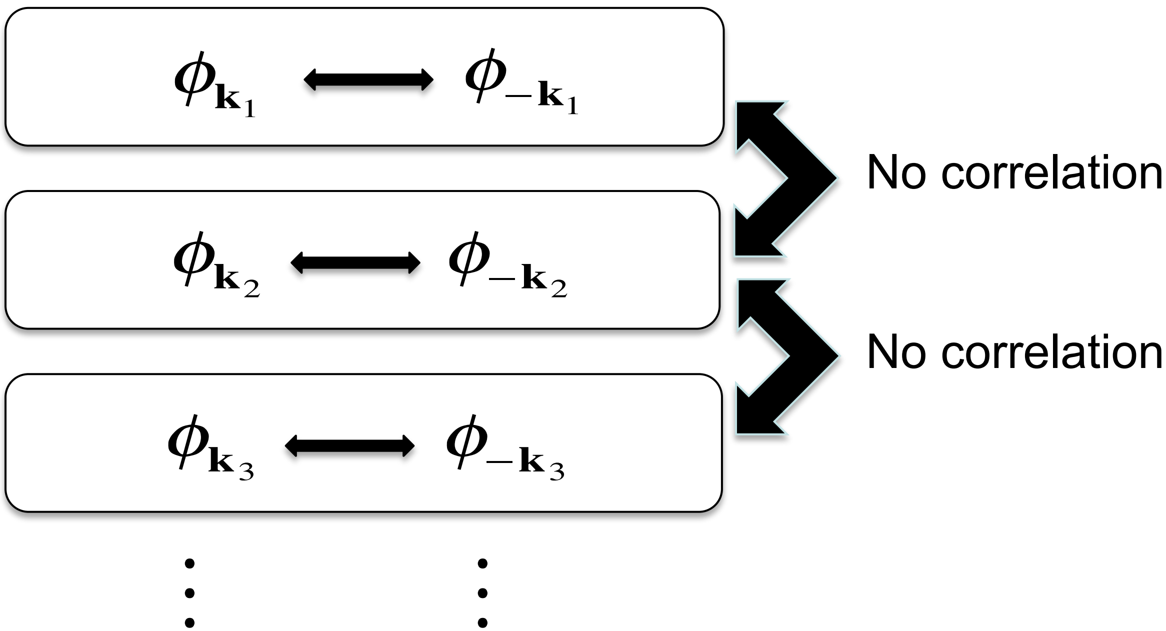
The Bell inequality is always violated in the Bunch-Davies vacuum.

In the large squeezed limit, we obtain

$$\therefore \langle BD | M_2 | BD \rangle \xrightarrow[r\to\infty]{} \sqrt{2}$$

Increasing the number of modes

It should be stressed that there are infinitely many modes in field theory.



We focus on the 2n partite system.

$$H_{2n} = \underbrace{H_2 \otimes H_2 \otimes \cdots \otimes H_2}_{2n \text{ partite sysyem}}$$

Bell is sufficient for BD vacuum

Kanno & Soda 2017

What if we use BMK inequalities?

By induction, we can prove the relation

$$M_{2n} = \frac{M_{2n-2}}{2} \otimes \left(M_2 + M'_2\right) + \frac{M'_{2n-2}}{2} \otimes \left(M_2 - M'_2\right)$$

Assume that 2n-2 and 2 have no correlation

$$\langle M_{2n} \rangle = \frac{1}{2} \langle M_{2n-2} \rangle \left(\langle M_2 \rangle + \langle M'_2 \rangle \right) + \frac{1}{2} \langle M'_{2n-2} \rangle \left(\langle M_2 \rangle - \langle M'_2 \rangle \right)$$

$$\langle M'_{2n} \rangle = \frac{1}{2} \langle M'_{2n-2} \rangle \left(\langle M'_2 \rangle + \langle M_2 \rangle \right) + \frac{1}{2} \langle M_{2n-2} \rangle \left(\langle M'_2 \rangle - \langle M_2 \rangle \right)$$

From these, we get

$$B_{2n} \equiv \langle M_{2n} \rangle^2 + \langle M'_{2n} \rangle^2 = \frac{1}{2} \left(\langle M_{2n-2} \rangle^2 + \langle M'_{2n-2} \rangle^2 \right) \left(\langle M_2 \rangle^2 + \langle M'_2 \rangle^2 \right)$$

Thus, we can deduce

$$B_{2n} \equiv \frac{1}{2} B_{2n-2} \, B_2 = \left(\frac{1}{2} \right)^m B_{2n-2m} \, B_2^m = \left(\frac{1}{2} \right)^{n-1} B_2^n = 2^{(\log_2 q_2 - 1)n + 1} \qquad q_2 \equiv \langle \psi | M_2 | \psi \rangle^2$$

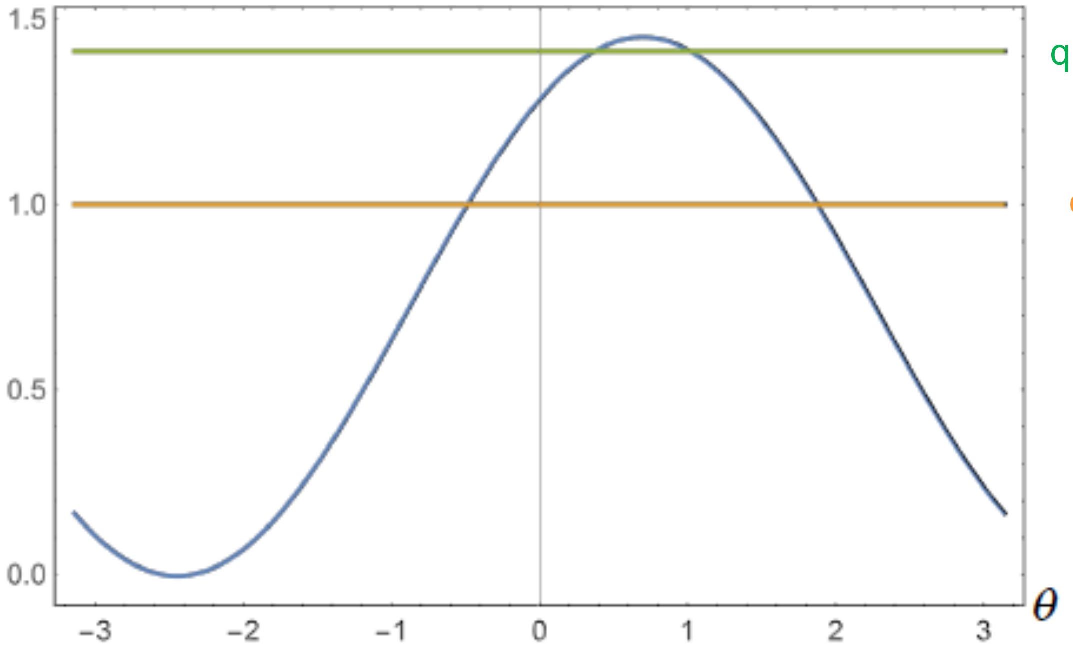
Since $q_2 \leq 2$, Bell operator is most effective test of quantumness.

BMK inequality in non-BD vacuum

A BMK operator for four-partite system reads

$$
\begin{aligned}
4M_4 = {} & O'_1\otimes O_2\otimes O_3\otimes O_4 - O'_1\otimes O'_2\otimes O'_3\otimes O_4 + O_1\otimes O'_2\otimes O_3\otimes O_4 + O_1\otimes O_2\otimes O'_3\otimes O_4 \\
& + O'_1\otimes O_2\otimes O_3\otimes O'_4 - O'_1\otimes O'_2\otimes O'_3\otimes O'_4 + O_1\otimes O'_2\otimes O_3\otimes O'_4 + O_1\otimes O_2\otimes O'_3\otimes O'_4 \\
& + O_1\otimes O'_2\otimes O'_3\otimes O_4 - O_1\otimes O_2\otimes O_3\otimes O_4 + O'_1\otimes O_2\otimes O'_3\otimes O_4 + O'_1\otimes O'_2\otimes O_3\otimes O_4 \\
& + O_1\otimes O'_2\otimes O'_3\otimes O'_4 + O_1\otimes O_2\otimes O_3\otimes O'_4 + O'_1\otimes O_2\otimes O'_3\otimes O'_4 + O'_1\otimes O'_2\otimes O_3\otimes O'_4
\end{aligned}
$$

$\langle\psi|M_4|\psi\rangle$



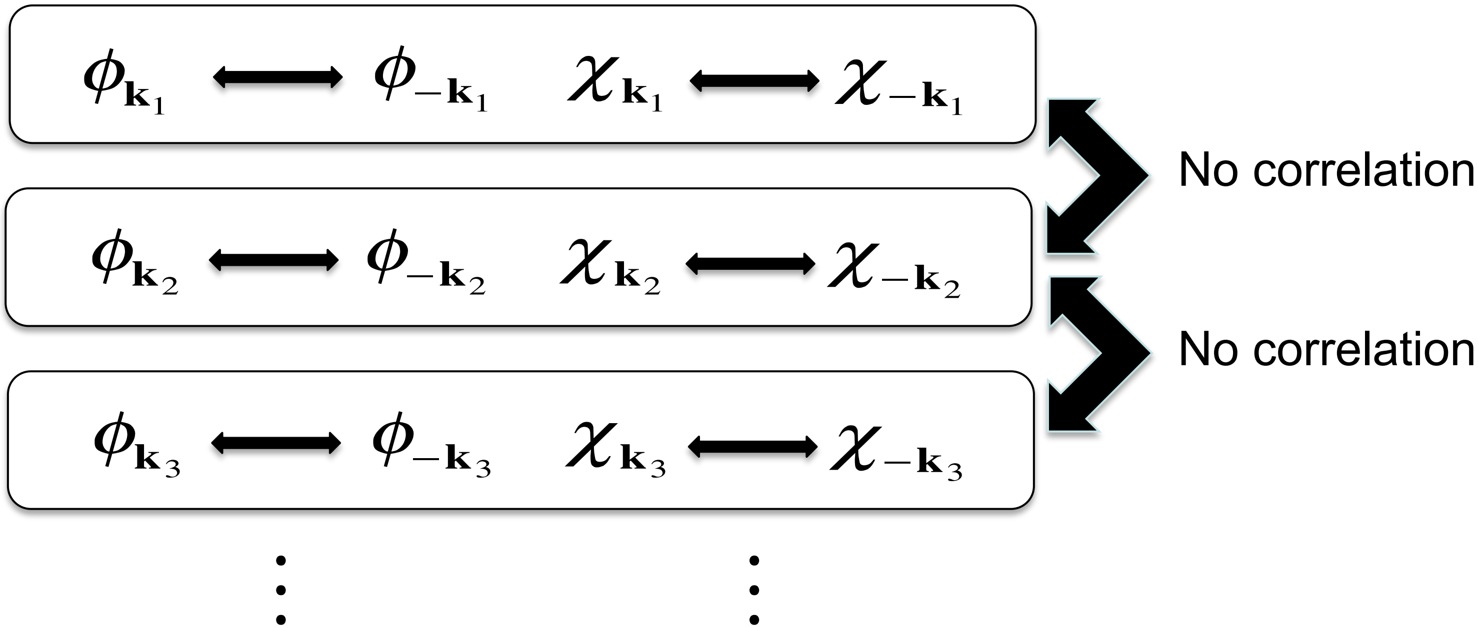
quantum bound
for a two-partite system

classical bound

The maximal vale is about 1.45.

Increasing the number of modes

Again, it should be stressed that there are infinitely many modes in field theory.



We focus on the 4n partite system.

$$H_{4n} = \underbrace{H_4 \otimes H_4 \otimes \cdots \otimes H_4}_{4n \text{ partite sysyem}}$$

Infinite violation

Kanno & Soda 2017

$$M_{4n} = \frac{M_{4n-4}}{2} \otimes \left(M_4 + M'_4 \right) + \frac{M'_{4n-4}}{2} \otimes \left(M_4 - M'_4 \right)$$

Assume that 4n-4 and 4 have no correlation

$$\langle M_{4n} \rangle = \frac{1}{2} \langle M_{4n-4} \rangle \left(\langle M_4 \rangle + \langle M'_4 \rangle \right) + \frac{1}{2} \langle M'_{4n-4} \rangle \left(\langle M_4 \rangle - \langle M'_4 \rangle \right)$$

$$\langle M'_{4n} \rangle = \frac{1}{2} \langle M'_{4n-4} \rangle \left(\langle M'_4 \rangle + \langle M_4 \rangle \right) + \frac{1}{2} \langle M_{4n-4} \rangle \left(\langle M'_4 \rangle - \langle M_4 \rangle \right)$$

From these, we get

$$B_{4n} \equiv \langle M_{4n} \rangle^2 + \langle M'_{4n} \rangle^2 = \frac{1}{2} \left(\langle M_{4n-4} \rangle^2 + \langle M'_{4n-4} \rangle^2 \right) \left(\langle M_4 \rangle^2 + \langle M'_4 \rangle^2 \right)$$

Thus, we can deduce

$$B_{4n} \equiv \frac{1}{2} B_{4n-4} \; B_4 = \left(\frac{1}{2} \right)^m B_{4n-4m} \; B_4^m = \left(\frac{1}{2} \right)^{n-1} B_4^n = 2^{(\log_2 q_4 - 1)n + 1} \qquad q_4 \equiv \langle \psi | M_4 | \psi \rangle^2$$

For q>2, the expectation value of BMK exponentially large.

For the present case, we have $q = 1.45^2 = 2.103$

Hence, we can see infinite violation of BMK inequality.

More general state

$$M_{mn} = \frac{M_{mn-m}}{2} \otimes \left(M_m + M'_m\right) + \frac{M'_{mn-m}}{2} \otimes \left(M_m - M'_m\right)$$

Assume that mn-m and m have no correlation

Thus, we can deduce

$$B_{mn} \equiv \frac{1}{2} B_{mn-m} \quad B_m = \left(\frac{1}{2}\right)^k B_{mn-km} \quad B_m^k = \left(\frac{1}{2}\right)^{n-1} B_m^n = 2^{\left(\log_2 q_m - 1\right)n+1} \qquad q_m \equiv \langle \psi | M_m | \psi \rangle^2$$

In the maximal case, we have $q_m = 2^{m-1}$ Cf. $\left|< M_n >\right|^2 \leq 2^{n-1}$

Hence, we can see infinite violation of BMK inequality.

$$B_{mn} = 2^{(m-2)n+1}$$

Thus, we can classify entangled vacuum by BMK inequality.

Summary

- We have formulated BMK inequality in inflation

- It is shown that Bell inequality is maximally violated in Bunch-Davies vacuum

- The violation of BMK inequality gets exponentially larger for non-Bunch-Davies vacuum

- We have shown that we can characterize the initial quantum state in terms of BMK inequalities.

- The huge violation indicates the detectability of quantumness.

- We need to invent a concrete method for detecting the quantumness.