Algebraic realization of the topological vertex

Jean-Emile Bourgine

Korea Institute for Advanced Study

Strings, Branes and Gauge Theories
Asia Pacific Center for Theoretical Physics
2019-07-22

Not based on JEB, Saebyeok Jeong [arXiv:1906.01625]

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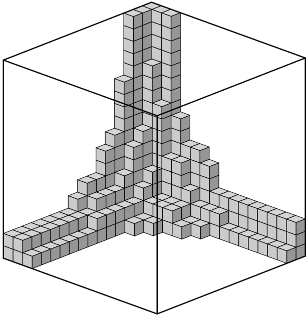
Based on **[Awata, Feigin, Shiraishi 2011]** revisited.

Topological vertex as sum over plane partitions



Sum over 3D partitions with fixed 2D asympotics λ, μ, ν:

$$C_{\lambda,\mu,\nu} = \sum_{\pi \in \mathcal{P}_{\lambda,\mu,\nu}} q^{|\pi|}$$

[Okounkov, Reshetikhin, Vafa 2003]

(In this talk, we ignore the framing factors for simplicity.)

Main applications

This simple combinatorial object is related to:

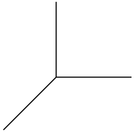
- Topological strings amplitudes
- Topological invariants of Calabi-Yau 3-folds (Gromov-Witten, Gopakumar-Vafa)
- $\mathcal{N} = (2,2)$ 2D superconformal sigma models
- $U(N)$ Chern-Simons theory on S^3 at large N
- Mirror symmetry
- 5D $\mathcal{N} = 1$ supersymmetric gauge theories (instanton partitions functions)
- (p,q)-brane webs in IIB string theory
- 5D BPS black hole
- ...
- Quantum groups and integrable systems

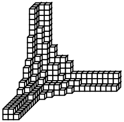
Topological strings amplitudes

Consider a toric Calabi-Yau 3-fold (i.e. a fibration of $T^2 \times \mathbb{R}$ over $\mathbb{R}^3$).

$\rightsquigarrow$ The toric diagram in $\mathbb{R}^3$ encodes the degeneration locus of the cycles.

Example I: $\mathbb{C}^3$ (= topological vertex)





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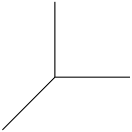
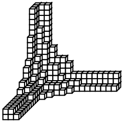
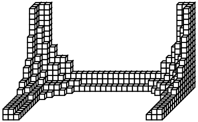




Example II: Local $\mathbb{CP}^1$ ($\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$)



Obtained by gluings $\mathcal{A} = \sum_\lambda C_{\emptyset,\emptyset,\lambda} C_{\lambda',\emptyset,\emptyset}$. ($\lambda'$ is the transposed of λ.)

Outline

Topological vertex

- The topological vertex can be written using skew Schur polynomials $s_{\lambda/\mu}(x)$:

$$C_{\lambda,\mu,\nu} = \sum_{\eta\subset\lambda',\mu} s_{\lambda'/\eta}(x) s_{\mu/\eta}(y),$$

with $x = (q^{-\nu_1+1/2}, q^{-\nu_2+3/2}, \cdots)$ and $y = (q^{-\nu'_1+1/2}, q^{-\nu'_2+3/2}, \cdots)$.

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- Skew-Schur polynomials coincide with correlators of a free boson:

$$s_{\lambda/\mu}(x)=\langle\lambda|\prod_{i>0}\Gamma_-(x_i)\left|\mu\right\rangle=\langle\mu|\prod_{i>0}\Gamma_+(x_i)\left|\lambda\right\rangle,$$

$$\text{with}\quad \Gamma_\pm(z)=e^{\sum_{k>0}\frac{z^k}{k}\alpha_{\pm k}},\quad [\alpha_k,\alpha_l]=k\delta_{k+l}.$$

(Here the state $|\lambda\rangle$ is built as $s_\lambda(X)\left|\emptyset\right\rangle$ with $\sum_i X_i^k\equiv\alpha_{-k}$.)

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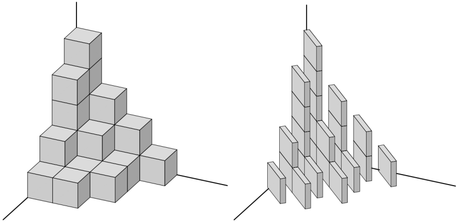
- Using this property, and $\sum_\eta |\eta\rangle\langle\eta| = 1$, we find

$$C_{\lambda,\mu,\nu} = \langle\lambda'| \prod_{i>0} \Gamma_-(x_i) \prod_{i>0} \Gamma_+(y_i) |\mu\rangle$$

Slicing interpretation

The bosonic formula is interpreted as a slicing of the 3D partition.

$$C_{\lambda,\mu,\nu} = \langle\lambda'| \prod_{i>0} \Gamma_-(x_i) \prod_{i>0} \Gamma_+(y_i) |\mu\rangle$$



(There is also a free fermion construction by fermionization of the modes α_k.)

A short computation

Let us perform this short computation: $x = (q^{-\nu_1+1/2}, q^{-\nu_2+3/2}, \cdots)$

$$\sum_i x_i = \sum_{i=1}^{\infty} q^{-\nu_i+i-1/2} = \sum_{i=1}^{\infty} q^{i-1/2} + \sum_{i=1}^{\ell(\nu)} q^{i-1/2}(q^{-\nu_i}-1)$$

$$= -\frac{1}{q^{1/2}-q^{-1/2}} + \sum_{i=1}^{\ell(\nu)} q^{i-1/2}(q^{-1}-1)\sum_{j=1}^{\nu_i} q^{-(j-1)}$$

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This gives (replacing $q \to q^k$) and denoting $\chi_{(i,j)} = q^{i-j}$:

$$\sum_i x_i^k = -\frac{1}{q^{k/2}-q^{-k/2}} - (q^{k/2}-q^{-k/2}) \sum_{(i,j)\in\nu} \chi_{(i,j)}^k,$$

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and, for the variables $y_i = q^{-\nu_i'+i-1/2}$,

$$\sum_i y_i^k = -\frac{1}{q^{k/2}-q^{-k/2}} - (q^{k/2}-q^{-k/2}) \sum_{(i,j)\in\nu} \chi_{(i,j)}^{-k}.$$

Rewriting the topological vertex

Using our previous result, we can rewrite the topological vertex:

$$C_{\lambda,\mu,\nu} = \langle\lambda'|\prod_{i>0}\Gamma_-(x_i)\prod_{i>0}\Gamma_+(y_i)|\mu\rangle = \langle\lambda'|\Phi_\nu|\mu\rangle$$

with ($\chi_\square = q^{i-j}$ for $\square = (i,j) \in \nu$):

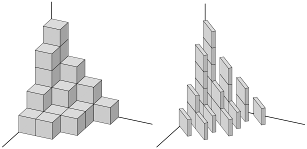
$$\Phi_\nu =: \Phi_\emptyset \prod_{\square\in\nu} \eta(\chi_\square) :,$$

$$\eta(z) =: \exp\left(-\sum_{k\neq 0}\frac{z^{-k}}{k}(q^{k/2}-q^{-k/2})\alpha_k\right) :$$

$$\Phi_\emptyset =: \exp\left(-\sum_{k\neq 0}\frac{1}{k(q^{k/2}-q^{-k/2})}\alpha_k\right) :$$

Refined topological vertex

- A refined topological vertex is obtained by tuning the weight q_a for boxes in a slice a.



$$C_{\lambda,\mu,\nu} = \sum_{\pi \in \mathcal{P}_{\lambda,\mu,\nu}} q_a^{|\pi|}$$

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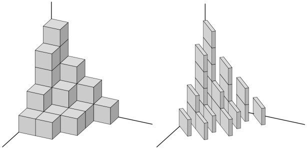

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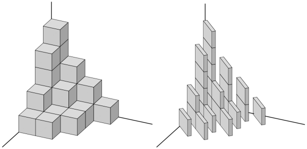

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- The weights q_a are determined in order to reproduce Nekrasov's instanton partition functions with omega-background parameters ϵ_1, ϵ_2 (unrefined case $\epsilon_1+\epsilon_2=0$).

$\rightsquigarrow$ q_a equals either $q=e^{\epsilon_2}$ or $t=e^{-\epsilon_1}$ depending on a and the shape of ν.

Bosonic expression for the refined vertex

- For our purpose, we only need the skew-Schur polynomials expression ($p = q/t$):

$$C_{\lambda,\mu,\nu} = \sum_{\eta\subset\lambda',\mu} p^{(|\eta|+|\mu|)/2} s_{\lambda'/\eta}(x) s_{\mu/\eta}(y),$$

with $x = (q^{-\nu_1}t^{1/2}, q^{-\nu_2}t^{3/2}, \cdots)$ and $y = (t^{-\nu_1'}q^{1/2}, t^{-\nu_2'}q^{3/2}, \cdots)$.

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- The bosonic presentation still reads

$$C_{\lambda,\mu,\nu} = \langle\lambda'| \prod_{i>0}\Gamma_-(x_i)\prod_{i>0}\Gamma_+(p^{-\frac{1}{2}}y_i)|\mu\rangle\,,$$

but now

$$\sum_i x_i^k = -\frac{1}{t^{k/2}-t^{-k/2}} - p^{k/2}(q^{k/2}-q^{-k/2})\sum_{\square\in\nu}\chi_\square^k,$$

$$\sum_i y_i^k = -\frac{1}{q^{k/2}-q^{-k/2}} - p^{-k/2}(t^{k/2}-t^{-k/2})\sum_{\square\in\nu}\chi_\square^{-k}.$$

where $\chi_\square = pt^{i-1}q^{-(j-1)}$ for $\square = (i,j)\in\nu$.

Bosonic expression for the refined vertex

Using the same trick, we end up with

$$C_{\lambda,\mu,\nu} = \langle\lambda'|\,\Phi_\nu\,|\mu\rangle\,, \quad \text{with} \quad \Phi_\nu =: \Phi_\emptyset \prod_{\square\in\nu}\eta(\chi_\square):,$$

and,

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where we have used the rescaled modes

$$\beta_k = p^{k/2}t^{-k/2}\alpha_k, \quad \beta_{-k} = \frac{1-q^k}{1-t^k}t^{k/2}p^{-k/2}\alpha_{-k} \quad \Rightarrow \quad [\beta_k,\beta_l] = k\frac{1-q^{|k|}}{1-t^{|k|}}\delta_{k+l}.$$

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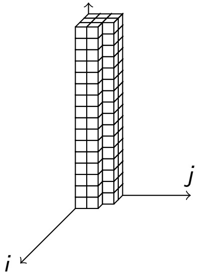
Bonus: Awata-Kanno topological vertex $\langle P_\lambda|\,\Phi_\nu\,|P_\mu\rangle$
(P_λ, P_μ Macdonald polynomials).

Interpretation in terms of 3D partitions

Noticing that (here ν_∞ is a fully filled Young diagram)

$$\Phi_\emptyset \simeq: \prod_{\square\in\nu_\infty} \eta(\chi_\square)^{-1} : \quad \Rightarrow \quad \Phi_\nu = : \prod_{\square\in\nu_\infty\setminus\nu} \eta(\chi_\square)^{-1} :$$

We can interpret $\eta(\chi_\square)^{-1}$ as a creation of a column of cubes at location $\square = (i,j)$.



Outline

Quantum toroidal $\mathfrak{gl}(1)$: definition

(or **[Ding, Iohara 1997 - Miki 2007]** algebra)

- The quantum toroidal $\mathfrak{gl}(1)$ algebra depends on the parameters

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- It is formulated in terms of a central element c and 4 Drinfeld currents

$$x^{\pm}(z) = \sum_{k\in\mathbb{Z}} z^{-k} x_k^{\pm}, \quad \psi^{\pm}(z) = \sum_{k\geq 0} z^{\mp k} \psi_{\pm k}^{\pm}.$$

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- The parameters define the structure function

$$g(z) = \prod_{\alpha=1,2,3} \frac{1 - q_\alpha z}{1 - q_\alpha^{-1} z}.$$

Quantum toroidal $\mathfrak{gl}(1)$: definition

The algebraic relations read

$$\psi^+(z)x^\pm(w) = g(q_3^{\pm c/4}z/w)^{\pm 1}x^\pm(w)\psi^\pm(z),$$
$$\psi^-(z)x^\pm(w) = g(q_3^{\mp c/4}z/w)^{\pm 1}x^\pm(w)\psi^-(z),$$
$$[\psi^\pm(z),\psi^\pm(w)] = 0, \quad \psi_0^+\psi_0^- = \psi_0^-\psi_0^+ = 1$$
$$\psi^+(z)\psi^-(w) = \frac{g(q_3^{c/2}z/w)}{g(q_3^{-c/2}z/w)}\psi^-(w)\psi^+(z),$$
$$x^\pm(z)x^\pm(w) = g(z/w)^{\pm 1}x^\pm(w)x^\pm(z),$$
$$[x^+(z),x^-(w)] = \delta(q_3^{-c/2}z/w)\psi^+(q_3^{-c/4}z) - \delta(q_3^{c/2}z/w)\psi^-(q_3^{c/4}z),$$

together with Serre relations.

(Here $\delta(z) = \sum_{k\in\mathbb{Z}} z^k$ denotes the multiplicative Dirac delta function.)

Quantum toroidal $\mathfrak{gl}(p)$: coalgebraic structure

The algebra has the structure of a Hopf algebra with the Drinfeld coproduct

$$\Delta(x^+(z)) = x^+(z)\otimes 1 + \psi^-(q_3^{c_{(1)}/4}z)\otimes x^+(q_3^{c_{(1)}/2}z),$$

$$\Delta(x^-(z)) = x^-(q_3^{c_{(2)}/2}z)\otimes \psi^+(q_3^{c_{(2)}/4}z) + 1\otimes x^-(z),$$

$$\Delta(\psi^\pm(z)) = \psi^\pm(q_3^{\pm c_{(2)}/4}z)\otimes \psi^\pm(q_3^{\mp c_{(1)}/4}z).$$

We denoted $c_{(1)} = c\otimes 1$, $c_{(2)} = 1\otimes c$, and $\Delta(c) = c_{(1)} + c_{(2)}$.

Horizontal representations

- Horizontal representations $\rho_u^{(1,n)}$ have levels $(c, \bar{c}) = (1, n)$, and weight $u \in \mathbb{C}^\times$.

(They are also called "level one", or "vertex" representations)

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- Formulated in terms of a q-deformed bosonic modes β_k,

$$[\beta_k,\beta_l]=k\frac{1-q^{|k|}}{1-t^{|k|}}\delta_{k+l}.$$

$\rightsquigarrow$ Usual vacuum $|\emptyset\rangle$ such that $\beta_{k>0}|\emptyset\rangle=0$, PBW basis acting with $\beta_{k<0}$.

(Other basis: Schur basis $|\lambda\rangle$, Macdonald basis $|P_\lambda\rangle$,...)

Horizontal representations

- Drinfeld currents are represented in terms of vertex operators,

$$\rho_u^{(1,n)}(x^\pm(z)) = u^{\pm 1} z^{\mp n} \eta^\pm(z), \quad \rho_u^{(1,n)}(\psi^\pm(z)) = \gamma^{\mp n} \varphi^\pm(z),$$

with:

$$\eta^+(z) = \exp\left(\sum_{k=1}^\infty \frac{1-t^{-k}}{k} z^k \beta_{-k}\right) \exp\left(-\sum_{k=1}^\infty \frac{1-t^k}{k} z^{-k} \beta_k\right),$$

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$$\varphi^+(z) = \exp\left(-\sum_{k=1}^\infty \frac{1-t^k}{k} (1-p^k) p^{-k/4} z^{-k} \beta_k\right),$$

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$$\varphi^{+}(z) = \exp\left(-\sum_{k=1}^{\infty} \frac{1-t^{k}}{k} (1-p^k) p^{-k/4} z^{-k} \beta_{k}\right),$$

$$\varphi^{-}(z) = \exp\left(\sum_{k=1}^{\infty} \frac{1-t^{-k}}{k} (1-p^k) p^{-k/4} z^{k} \beta_{-k}\right).$$

Identify $\eta^{-}(z) \equiv \eta(z)$!!!

Vertical representations

- Vertical representations $\rho_v^{(0,1)}$ have levels $(c, \bar{c}) = (0, 1)$, and weight $v \in \mathbb{C}^\times$.

(Generalize the finite dimensional representations of quantum affine algebras.)

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- Drinfeld currents act on states $|\lambda\rangle\rangle$ parameterized by a Young diagram λ:

$$\rho_v^{(0,1)}(x^+(z))\,|\lambda\rangle\rangle=\sum_{\square\in A(\lambda)}\delta(z/\chi_\square)\,\underset{z=\chi_\square}{\mathrm{Res}}\,\frac{1}{z\mathcal{Y}_\lambda(z)}\,|\lambda+\square\rangle\rangle,$$

$$\rho_v^{(0,1)}(x^-(z))\,|\lambda\rangle\rangle=q_3^{-1/2}\sum_{\square\in R(\lambda)}\delta(z/\chi_\square)\,\underset{z=\chi_\square}{\mathrm{Res}}\,z^{-1}\mathcal{Y}_\lambda(q_3^{-1}z)\,|\lambda-\square\rangle\rangle,$$

$$\rho_v^{(0,1)}(\psi^\pm(z))\,|\lambda\rangle\rangle=q_3^{-1/2}\left[\frac{\mathcal{Y}_\lambda(q_3^{-1}z)}{\mathcal{Y}_\lambda(z)}\right]_\pm|\lambda\rangle\rangle.$$

- $\chi_\square=vq_1^{i-1}q_2^{j-1}\in\mathbb{C}^\times$ for a box $\square=(i,j)\in\lambda$ ("instanton position").
- $A(\lambda)$ denote the set of boxes that can be added to λ.
- $R(\lambda)$ denote the set of boxes that can be removed from λ.
- $[f(z)]_\pm$ denotes an expansion of $f(z)$ in powers of $z^{\mp1}$.

Vertical representation

- The function $\mathcal{Y}_\lambda(z)$ is Nekrasov's $\mathcal{Y}$-observable.

⇝ It appears in recursion of formulas for the Nekrasov factor of 5D $\mathcal{N}=1$ theories,

$$\frac{N(v^{(1)},\lambda^{(1)}|v^{(2)},\lambda^{(2)}+\square)}{N(v^{(1)},\lambda^{(1)}|v^{(2)},\lambda^{(2)})}=\mathcal{Y}_{\lambda^{(1)}}(\chi_\square),$$
$$\frac{N(v^{(1)},\lambda^{(1)}+\square|v^{(2)},\lambda^{(2)})}{N(v^{(1)},\lambda^{(1)}|v^{(2)},\lambda^{(2)})}=\mathcal{Y}_{\lambda^{(2)}}(q_3^{-1}\chi_\square).$$

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- Explicitly, it writes

$$\mathcal{Y}_\lambda(z)=\frac{\prod_{\square\in A(\lambda)}1-\chi_\square/z}{\prod_{\square\in R(\lambda)}1-\chi_\square/(q_3 z)}.$$

Vertical representation

Remark: Note that we can also write the Cartan currents $\psi^{\pm}(z)$ as

$$\psi^{\pm}(z) = \psi_0^{\pm} \exp\left(\pm \sum_{k>0} z^{\mp k} a_{\pm k} \right)$$

with

$$\rho_v^{(0,1)}(a_{k>0})\,|\lambda\rangle\rangle = \frac{1}{k}(p^{-1/2}v)^k \frac{t^{k/2} - t^{-k/2}}{p^{k/2} - p^{-k/2}} \sum_i x_i^k \; |\lambda\rangle\rangle,$$

$$\rho_v^{(0,1)}(a_{k<0})\,|\lambda\rangle\rangle = \frac{1}{k}(p^{-1/2}v)^k \frac{q^{k/2} - q^{-k/2}}{p^{k/2} - p^{-k/2}} \sum_i y_i^{-k} \; |\lambda\rangle\rangle,$$

where $x = (q^{-\lambda_1} t^{1/2}, q^{-\lambda_2} t^{3/2}, \cdots)$ and $y = (t^{-\lambda'_1} q^{1/2}, t^{-\lambda'_2} q^{3/2}, \cdots)$.

Intertwining operators

- Intertwining operators introduced as a generalization of Virasoro vertex operators. ($\rightsquigarrow$quantum Knizhnik-Zamolodchikov equations) **[Frenkel, Reshetikhin 1992]**

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$$\rho_{u'}^{(1,n+1)}(e)\Phi^+ = \Phi^+ \left(\rho_v^{(0,1)} \otimes \rho_u^{(1,n)}\ \Delta(e)\right),$$
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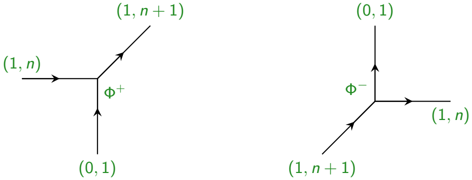
- The solution is decomposed on the vertical basis,

$$\Phi^+ = \sum_\lambda \Phi_\lambda^+ \langle\langle\lambda| , \quad \Phi^- = \sum_\lambda \Phi_\lambda^- |\lambda\rangle\rangle, \quad \text{with} \quad \Phi_\lambda^\pm =: \Phi_\emptyset^\pm \prod_{\square\in\lambda} \eta^\pm(\chi_\square) :,$$

where $\Phi_\lambda^\pm$ are vertex operators acting on horizontal modules.

Identification of the topological vertex



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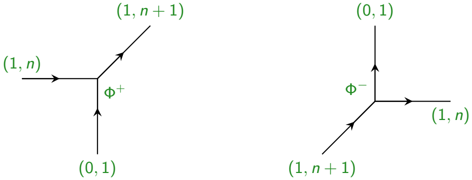

The intertwiner Φ^-_λ identifies with the refined topological vertex Φ_λ!

[Awata, Feigin, Shiraishi, 2011]

Recent results

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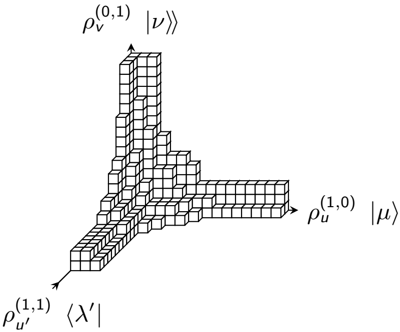
$\rightsquigarrow$ Determine the algebra from the Nekrasov factor, construct the topological vertex.

Other interests of the construction: (q,t)-Knizhnik-Zamolodchikov equations, Zamolodchikov-Faddev algebra, q-AGT correspondence,...

Outline

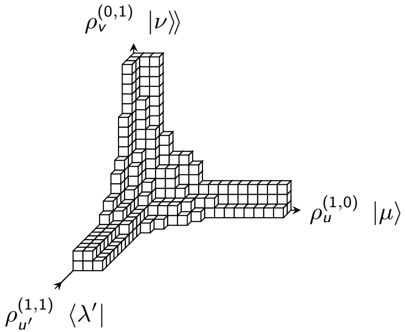
Correspondence algebra - topological strings



Each direction of the 3D partition is assigned to a representation:

- Levels $(c, \bar{c})$ label the cycle of the T^2-fibration that degenerates.
- Weights u, v, u' give the Kähler moduli of the Calabi-Yau.

Correspondence algebra - topological strings



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- Levels $(c, \bar{c})$ label the cycle of the T^2-fibration that degenerates.
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In the preferred direction, $x^{\pm}$ describe the variation of the Young diagram.

(This diagram encodes the instanton configurations of the 5D gauge theory.)

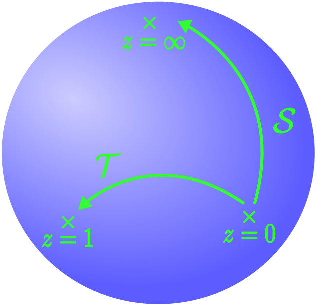
How does the algebra acts in other directions?

Model B perspective

The topological vertex describes the Calabi-Yau $\mathbb{C}^3$ in model A.

$\rightsquigarrow$ Becomes the Calabi-Yau $uv - H(x,p) = 0$ with $H(x,p) = e^x + e^p + 1$ in model B.

The degenerate locus of the fibration $H(x,p) = 0$ is a sphere with three punctures.



The modular group $PSL(2,\mathbb{Z})$ acts on the Riemann sphere.

Generators $\mathcal{S} : z \to -1/z$ and $\mathcal{T} : z \to z + 1$ map the different punctures.

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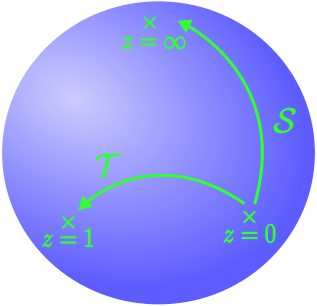

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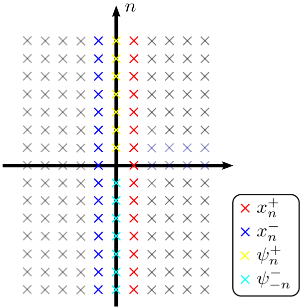
Generators $\mathcal{S} : z \to -1/z$ and $\mathcal{T} : z \to z + 1$ map the different punctures.

$\rightsquigarrow$ Identify the three marked points with the axis of the toric diagram!

Automorphisms of the algebra

The quantum toroidal $\mathfrak{gl}(1)$ algebra has a group of $SL(2,\mathbb{Z})$ automorphisms,



- $\mathcal{S}$ rotates the generators by 90°. **[Miki 2007]**
- $\mathcal{T}$ twist the generators (move up modes on the left, down on the right).

⚠ $\mathcal{S}$ is now of order four! (extra orientation)

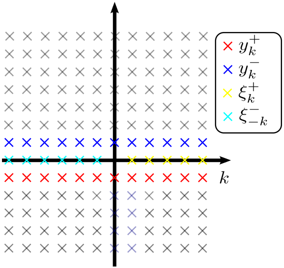
Automorphisms of the algebra

- Explicitly:

$$\mathcal{T}(c,\bar{c}) = (c, \bar{c}+c), \quad \mathcal{T}(x_k^\pm) = x_{k\mp 1}^\pm, \quad \mathcal{T}(\psi_{\pm k}^\pm) = p^{\mp c/2}\psi_{\pm k}^\pm,$$
$$\mathcal{S}(c,\bar{c}) = (-\bar{c}, c), \quad \mathcal{S}(x_k^\pm) = y_k^\pm, \quad \mathcal{S}(\psi_{\pm k}^\pm) = \xi_{\pm k}^\pm$$

with

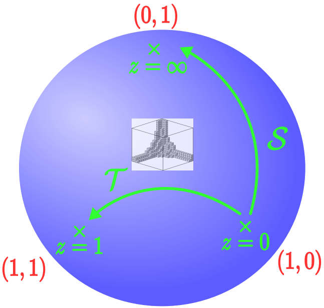
$$y_k^\pm \propto \left(\mathrm{ad}_{x_0^+}\right)^{k-1} x_{\mp 1}^+, \quad y_{-k}^\pm \propto \left(\mathrm{ad}_{x_0^-}\right)^{k-1} x_{\mp 1}^-,$$
$$\xi_{\pm k}^\pm \propto \mathrm{ad}_{x_{\mp 1}^\pm}\left(\mathrm{ad}_{x_0^\pm}\right)^{k-2} x_{\pm 1}^\pm, \quad \xi_{\pm 1}^\pm \propto x_0^\pm.$$



Action of the algebra on plane partitions

- Use these automorphisms to define the action in non-preferred direction!

$$\rho^{(1,0)} = \rho^{(0,1)} \circ \mathcal{S}^{-1}, \quad \rho^{(1,1)} = \rho^{(1,0)} \circ \mathcal{T}.$$



⚠ We have to choose the correct basis in the horizontal module: $|P_\lambda\rangle$.

⇝ $y^{\pm}(z)$ add/remove boxes to the states $|P_\lambda\rangle$.

($\mathcal{S}$ maps vertical to horizontal, sending the Cartan modes a_k to the oscillators β_k.)

Unrefined limit

- When $q = t$, the quantum toroidal $\mathfrak{gl}(1)$ algebra reduces to quantum $W_{1+\infty}$:

(more precisely $a_k \to X_{0,k}$, $x_k^{\pm} \to X_{\pm 1,k}$)

$$[X_{m,n}, X_{m',n'}] = (q^{mn'} - q^{m'n})X_{m+m',n+n'} + \delta_{n+n'}\delta_{m+m'}(m\bar{c} + nc)q^{-mn}.$$

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- This algebra has a fermionic representation of levels $(1,0)$.
($\{\psi_r, \psi_s^\dagger\} = \delta_{r+s}$, $\{\psi_r, \psi_s\} = \{\psi_r^\dagger, \psi_s^\dagger\} = 0$)

$$\rho^{(1,0)}(X_{m,n}) = \sum_{r\in\mathbb{Z}+1/2} q^{m(r-1/2)} :\psi_{r+n}\psi_{-r}^\dagger:$$

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- Accordingly the free boson of the $(1,0)$ representation is fermionized, $\psi(z) = e^{\phi(z)}$.

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- Under Miki's automorphism $\mathcal{S}$, the generators transform as

$$X_{m,n} = \oint :\psi(z)z^n q^{mz\partial_z}\psi^\dagger(z): \to X_{m,n}^{\mathcal{S}} = \oint :\psi(z)q^{nz\partial_z}z^{-m}\psi^\dagger(z):$$

$\rightsquigarrow$ Classically it reduces to a Fourier transform $(z,p) \to (-p,z)$.

[Sasa, Watanabe, Matsuo 2019]

Unrefined limit

This fermionic description is well-known for the topological vertex!

- To each patch of the three-punctured sphere correspond a free fermion.
- Classically, the fermions in different patches are related by a Fourier transform.
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This leads to several interesting questions:

- Interpretation of $\eta(z) = \psi(qz)\psi^\dagger(z)$ as brane-antibrane bound state? (distance $\sim g_s$)
- What is the action of the full quantum toroidal algebra from the B-model perspective?
- What is the connection with (q,t)-deformed integrable hierarchies? with topological recursion?

Perspectives

New formalism to describe the action of (strings) S-duality!

It can be employed to address the several important problems:

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$\rightsquigarrow$ Some of these ideas are discussed in **[Bourgine 2018]**

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Why should you care?

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- Intertwiners were defined as a generalization of 2D CFT vertex operators.

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- Quantum affine algebras were introduced to describe the mathematical structure of quantum integrable systems.
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Isolating these mathematical structures offers the possibility to generalize the method to solve new problems.

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- Geometric intuition: quantum integrable system $\Leftrightarrow$ topological strings on $X_4 \times X_6$.

$\rightsquigarrow$ X_4 determines the symmetry algebra $\mathcal{A}$.

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Thank you !!!