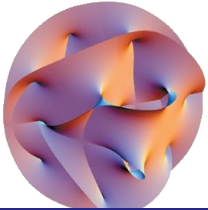
K3 Metrics from Little String Theory

Max Zimet

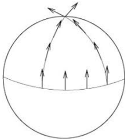
Stanford University

Pohang, South Korea – 7/22/19



Introduction

- Calabi-Yau (CY) compactification has played a central role in string theory. Reduced holonomy $\Rightarrow$ low-energy SUSY
- Type II compactifications preserve 4d $\mathcal{N}=2$ and are the setting of mirror symmetry
- Heterotic and orientifold compactifications preserve 4d $\mathcal{N}=1$ and provide semi-realistic starting points for string phenomenology
- Setting in which much of our non-perturbative understanding of string theory has been developed



K3

- K3 has played a particularly important role
- $SU(2) = Sp(1)$, so in 4d Calabi-Yau = hyper-Kähler. Only compact examples are K3 and T^4
- A concrete way to think about K3 is as T^4/Z_2 orbifold.

Introduction (continued...)

- Since K3 is hyper-Kähler, preserves even more SUSY (e.g. K3$\times T^2$ has 4d $\mathcal{N} = 4$)
- Heterotic (on T^4) - type IIA (on K3) duality plays an essential role in our understanding of how the various perturbative superstring theories are related. Can fiber this duality over a $\mathbb{P}^1$ base to find dual 4d $\mathcal{N} = 2$ theories
- Earliest example of black hole microstate counting in string theory

Introduction (continued...)

- Remarkably, all of this was achieved without an explicit form of the metric! Indeed, no smooth (compact, non-toroidal) Ricci-flat Calabi-Yau metric is known!
- Why might this matter to a string theorist? Supposedly, (tree-level) string vacuum from CFT, such as non-linear sigma model with action

$$\frac{i}{8\pi\alpha'}\int (g_{ij}-B_{ij})\partial x^i\bar{\partial}x^j\,d^2z - 2\pi\int \Phi R^{(2)}\,d^2z + \dots$$

(where the $\dots$ involve fermions). But, in reality since we don't have the metric, this formulation is rather useless.

K3 Non-Linear Sigma Models

- This question is particularly well-motivated for K3 (as opposed to other Calabi-Yaus) because the β function of the non-linear sigma model is exactly 0 – not just to leading order in α'
- As an example of our ignorance, even for K3 the worldsheet partition function is not known at almost all points in moduli space.

We now can get K3 metric

Based on recent work with

Shamit Kachru, Arnav Tripathy

The key step is to realize the K3 surface as the moduli space of a little string theory.

Little string theory

- Type IIB NS5-brane has a decoupling limit [Seiberg '97]. (Small lie: need $N > 1$ NS5-branes.) At low energies, 6d $U(1)$, $g^2 = 1/M_s^2$.
- From supergravity perspective, this works because the corresponding soliton is so singular. In particular, an infinite throat with diverging g_s develops.

Physics of LST

- A crucial aspect of LSTs for us is that they exhibit Hagedorn behavior:

$$S(E) = E/T_H \ , \quad T_H \sim M_s \ .$$

  This is much faster than field theories, but slower than gravity.
- It is *not* a QFT – it has T-duality, for example, so there is no unique stress-energy tensor.

Moduli space on T^2

- Let's now compactify on T^2. In addition to the $\mathbb{R}^4$ free hypermultiplet moduli parametrizing the center of mass, there is a Coulomb branch coming from $U(1)$ Wilson lines. So, the Coulomb branch is T^2.

Moduli space on T^3

- Repeating this for T^3 would naïvely tell us the moduli space is T^3. Not hyper-Kähler. Dual photon $\to T^4$.
- Instead of thinking of the extra dimension as coming from the dual photon, it will be useful to think of it as a Wilson line for the magnetic 4d gauge field (a.k.a. a 't Hooft line).

Geometrizing the moduli space

- S-duality takes us to D5-brane. Now, to study the moduli space of the theory on T^2, use T-duality twice to replace D5 by D3. The Wilson lines of the D5 become the position of the D3 in the two new transverse dimensions!
- Similarly, to study the theory on T^3, use T-duality three times to replace D5 by D2. The extra circle parametrized by the dual photon is the M-theory circle!

Heterotic little string theory

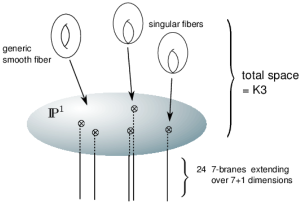
- To get from T^4 to K3, we replace the IIB NS5-brane by a $SO(32)$ heterotic 5-brane which is associated to a small instanton in the transverse dimensions. At low energies, this has a 6d $\mathcal{N} = (1,0)$ $SU(2)$ gauge theory with a free hyper and 16 fundamental hypers.

Moduli space

- On T^2, the moduli space $\mathcal{B}$ of $SU(2)$ Wilson lines is $T^2/Z_2 \sim S^2 \sim \mathbb{P}^1$, where the quotient comes from the Weyl group of $SU(2)$. On T^3, the moduli space $\mathcal{M}$ is $T^4/Z_2 \sim K3$. This is classical reasoning, and quantum corrections will generically smooth out the K3 surface.

Geometrizing the moduli space, I: heterotic / F-theory duality

- Strong-weak duality takes us to D5-brane in type I. Now, to study the moduli space of the theory on T^2, use T-duality twice to replace D5 by D3. Wilson lines $\to$ position.
- Heterotic $(T^2) \leftrightarrow$ type IIB orientifold on $T^2/Z_2 \to$ F-theory on K3



Geometrizing the moduli space, II: heterotic / M-theory duality

- Similarly, to study the theory on T^3, use T-duality three times to replace D5 by D2. The extra circle parametrized by the dual photon is the M-theory circle.
- Heterotic (T^3) $\leftrightarrow$ M-theory on K3

Parameters of LST

- Moduli of the heterotic string theory become parameters of the LST. Similarly, gauge symmetry in spacetime descends to global symmetry of LST.

Parameter and moduli spaces on T^2

- Moduli space of relevant K3 metrics:

$$\left[O(\Gamma^{18,2}) \backslash O(18,2)/(O(18) \times O(2)) \right] \times \mathbb{R}_+ \times \mathbb{R}_+ \, .$$

  F-theory is missing one $\mathbb{R}_+$ (fibers not part of spacetime), and LST is missing both (zero heterotic coupling / volume of base).
- Moduli space of the LST is the base $\mathbb{P}^1$ probed by the D3-brane. I'll use u as my homogeneous coordinate.
- Low energies: 4d $\mathcal{N} = 2$ gauge theory sees an infinitesimal piece of $\mathbb{P}^1$

Parameter and moduli spaces on T^3

- Moduli space of hyper-Kähler K3 metrics:

$$\left[O(\Gamma^{19,3})\backslash O(19,3)/(O(19)\times O(3))\right]\times\mathbb{R}_+$$

0 heterotic coupling $\rightarrow$ zero volume.

Compactification of the 4d theory

- Study little string theory on T^2, further compactified on S^1_R
- $R \to \infty$ limit is large complex structure / semi-flat limit studied by [Greene-Shapere-Vafa-Yau '90] and familiar from F-theory on K3
- Corrections away from this limit are determined by instantons in this theory
- These instantons are obtained by taking the worldlines of 4d BPS particles and wrapping them around S^1_R

Supersymmetric Wilson-'t Hooft lines, I

- To get the metric on this moduli space, we use some ideas from [Gaiotto-Moore-Neitzke '08, '10]. IR BPS Wilson-'t Hooft line operators:

$$\mathrm{Tr}_{\mathcal{R}}\ P\exp\oint\left(\frac{\varphi}{2\zeta}+A+\frac{\zeta\bar{\varphi}}{2}\right)$$

After compactifying on S^1_R, we obtain a function $\mathcal{X}_\gamma$ on the moduli space

Semi-flat

$$\mathcal{X}_\gamma^{\text{sf}} = \exp\left[\frac{\pi R}{\zeta} Z_\gamma + i\theta_\gamma + \pi R \zeta \overline{Z_\gamma}\right] \ , \quad \gamma \in \hat{\Gamma} \ .$$

$$R\, ds^2 = e^{\phi(u,\bar{u})} du d\bar{u} + \partial\bar{\partial} K(u, \bar{u}, z, \bar{z})$$

$$e^\phi = \tau_2 \left| \eta^2 \prod_{a=1}^{24} (u - u_a)^{-1/12} \right|^2 \ , \quad K = -(z - \bar{z})^2 / 2\tau_2 \ .$$

$$z = \frac{\theta_m - \tau\theta_e}{2\pi}$$

Supersymmetric Wilson-'t Hooft lines, II

- These functions are important because they have a number of nice mathematical properties.
- Hyper-Kähler manifolds have a whole $\mathbb{P}^1$ worth of complex structures. $\mathcal{X}_\gamma(\zeta)$ is holomorphic in the complex structure parametrized by $\zeta \in \mathbb{P}^1$, by virtue of SUSY.
- They satisfy the algebra

$$\mathcal{X}_\gamma \mathcal{X}_{\gamma'} = (-1)^{\langle \gamma, \gamma' \rangle} \mathcal{X}_{\gamma+\gamma'} \ .$$

  This reflects the shift in the fermion number of a bound state.
- They give canonical coordinates on $\mathcal{M}_\zeta$

Metric from Wilson lines

$$\mathcal{Y}^i(u,\theta;\zeta) = \log \mathcal{X}_{\gamma^i}(u,\theta;\zeta)$$

$$\varpi(\zeta) = \frac{1}{8\pi^2 R}\epsilon_{ij} d\mathcal{Y}^i(\zeta) \wedge d\mathcal{Y}^j(\zeta) = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\omega_-$$

(Compare to $\omega = dp \wedge dq = \frac{1}{2}\epsilon_{IJ} dQ^I \wedge dQ^J \;,\; Q^I = (p,q)$.)

$$\omega_\pm = \omega_1 \pm i\omega_2$$

$$g = -\omega_3 \omega_1^{-1} \omega_2$$

Supersymmetric Wilson-'t Hooft lines, III

- The $\mathcal{X}_\gamma$ are discontinuous in a precise way. These discontinuities are given by symplectomorphisms [Kontsevich-Soibelman '04, '08], so $\varpi(\zeta)$ is smooth!
- GMN: Riemann-Hilbert problem on $\mathbb{P}^1$ whose solution is $\mathcal{X}_\gamma(\zeta)$. Furthermore, can solve it at large R.
- Certain formulae in this solution only converge because of (at most) Hagedorn growth.

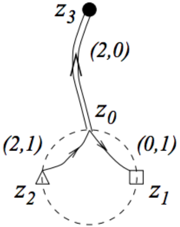
$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\mathrm{sf}}(\zeta) \exp\Bigg[-\frac{1}{4\pi i} \sum_{\gamma' \in \hat{\Gamma}'_u} \Omega(\gamma'; u) \langle \gamma, \gamma' \rangle \times \int_{\ell_{\gamma'}(u)} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 - \mathcal{X}_{\gamma'}(\zeta')) \Bigg]$$

Instanton corrections

- Summary: at large R, these $\mathcal{X}_\gamma$ take a universal form, up to corrections that result from 4d BPS states running around this circle. We have thus reduced the determination of a K3 metric to the simpler problem of counting BPS states in a little string theory on T^2.
- So, now we just need the BPS index (second helicity supertrace) $\Omega(\gamma; u)$ that counts 4d BPS states at a point in (4d) moduli space u.

LST BPS states in F-theory

- Recall heterotic on T^2 is dual to F-theory on (elliptically-fibered) K3. The 5-brane giving our little string theory maps to a D3-brane probing the K3 surface.
- BPS states in this frame are (p, q)-string webs running along the $\mathbb{P}^1$ base and ending on 7-branes.
- Related, via duality, to open string K3 Gromov-Witten invariants, furnishes proof of [Strominger-Yau-Zaslow '96] mirror symmetry conjecture for K3



Geometric engineering

- Can geometrically engineer LSTs using F-theory on a CY3
- LST on T^2 = IIA on CY3
- BPS state counting = Donaldson-Thomas theory
- Periods of mirror = open + closed periods of K3 (moduli + parameters of LST)!
- Very big moduli space of CY3s – includes lots of flopped geometries

Conclusions

- We've shown how to compute the metric given the BPS state counts. There are lots of ways to attack this problem: open string Gromov-Witten theory of K3 (Lin '14-'17), plus the tools physicists have for studying little string theories (holography, geometric engineering, DLCQ).
- Even without counts, we have some very accurate approximations
- This provides an interesting physical application of LST which makes use of many of its known properties, including its Hagedorn growth