On SUSY Wilson Loops in 2d
and
their dualities

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Strings, Branes, and Gauge Theories 2019

APCTP, Pohang

KIAS KOREA INSTITUTE FOR ADVANCED STUDY

based on 1812.01315 in PRD with R. Panerai and D. Seminara

Plan of the Talk

1. SUSY Wilson Lines

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2. SUSY Wilson Loops

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1. SUSY Wilson Lines
2. SUSY Wilson Loops
3. Focus on S^2: localization results and Seiberg-like duality

SUSY Wilson Lines

Supersymmetry

$N = (2,2)$ with $\mathrm{U}(1)$ vector-like R-symmetry

- Given a Riemann surface with genus g the KSE are

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- Multiplets:

$$\boldsymbol{\mathcal{V}} = (A, \lambda, \tilde{\lambda}, \sigma, \tilde{\sigma}, D) \,, \qquad \boldsymbol{\chi} = (\varphi, \psi, F) \;\Rightarrow \text{twisted mass} \propto \tau.$$

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- $\frac{1}{2}$-BPS: Γ has to satisfy

$$\frac{\epsilon^-}{\epsilon^+}\frac{\tilde{\epsilon}^-}{\tilde{\epsilon}^+} = -\frac{\dot{x}^1 + \mathrm{i}\dot{x}^2}{\dot{x}^1 - \mathrm{i}\dot{x}^2} \ .$$

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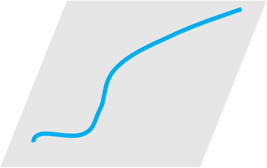

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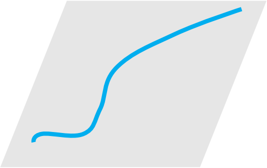

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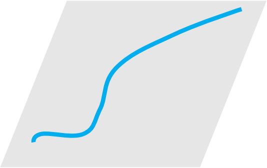
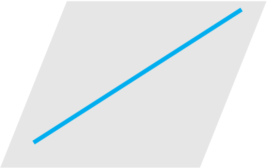


- $\frac{1}{2}$-BPS Wilson lines are straight segments.



on round S^2: background

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Killing Spinors

$$\epsilon = e^{\mathrm{i}\frac{\theta}{2}\gamma^1} e^{\mathrm{i}\frac{\varphi}{2}\gamma^3}\epsilon_0\,, \qquad \tilde{\epsilon} = e^{\mathrm{i}\frac{\theta}{2}\gamma^1} e^{\mathrm{i}\frac{\varphi}{2}\gamma^3}\tilde{\epsilon}_0\,.$$

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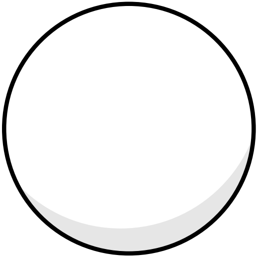

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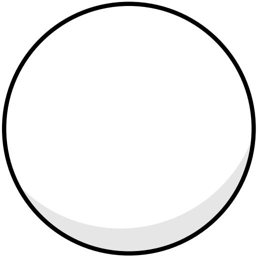
- Non-trivial $\epsilon(\epsilon_0)$ and $\tilde{\epsilon}(\tilde{\epsilon}_0)$.



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- Non-trivial $\epsilon(\epsilon_0)$ and $\tilde{\epsilon}(\tilde{\epsilon}_0)$.
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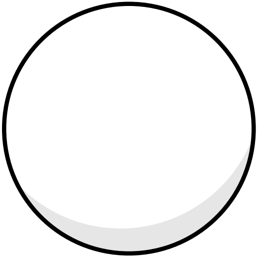
  - zero;
  - pole;



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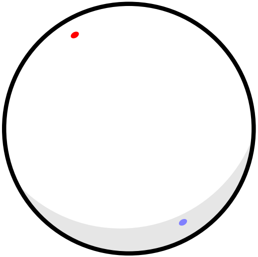

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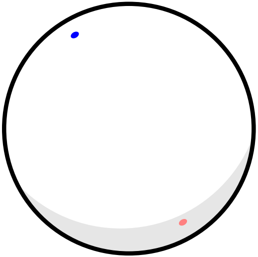

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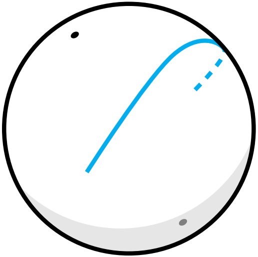

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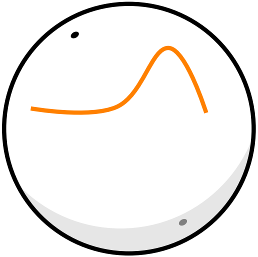

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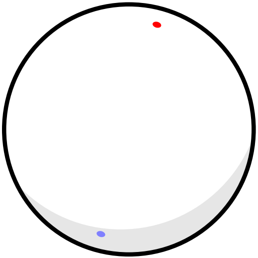

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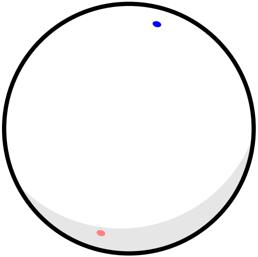

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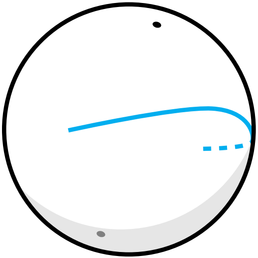

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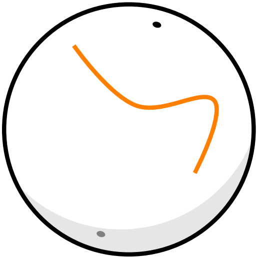

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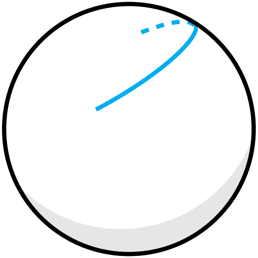
- $\frac{1}{2}$-BPS Wilson lines lie on circles.



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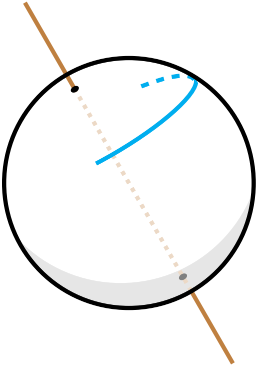

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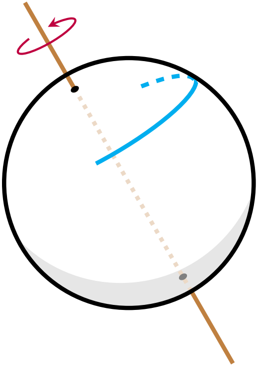

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- In a suitable polar coordinates one has:

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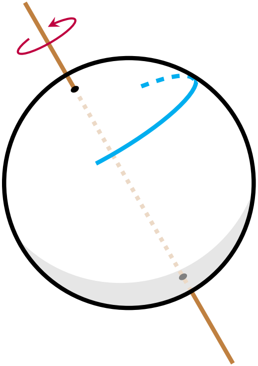
$$W_{\mathrm{b}} = \mathrm{P}\exp\int_{\varphi_0}^{\varphi_1}[\mathrm{i}A_\varphi + r(\sin^2\tfrac{\theta}{2}\,\sigma + \cos^2\tfrac{\theta}{2}\,\tilde{\sigma})]\,\mathrm{d}\varphi\ .$$



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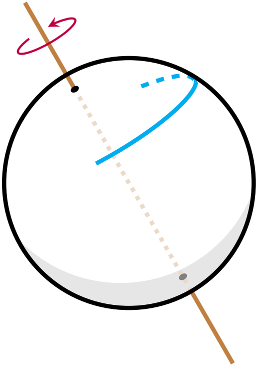

- "Local" limits:

$$\lim_{\theta\to 0} W_{\mathrm{a}} = e^{r(\varphi_0-\varphi_1)\sigma|_{\theta=0}}\ , \qquad \lim_{\theta\to\pi} W_{\mathrm{a}} = e^{r(\varphi_0-\varphi_1)\tilde{\sigma}|_{\theta=\pi}}\ ,$$

$$\lim_{\theta\to 0} W_{\mathrm{b}} = e^{r(\varphi_0-\varphi_1)\tilde{\sigma}|_{\theta=0}}\ , \qquad \lim_{\theta\to\pi} W_{\mathrm{a}} = e^{r(\varphi_0-\varphi_1)\sigma|_{\theta=\pi}}\ .$$

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- The $\frac{1}{2}$-BPS Wilson lines run along the $\mathrm{U}(1)$ action.
- The form of $\frac{1}{2}$-BPS Wilson line is the same as W_{a} or W_{b} (depending on the background).

Wilson loops

Generalized field strength

- Let us define $\mathcal{F} = \mathrm{d}\mathcal{A} - \mathrm{i}\mathcal{A} \wedge \mathcal{A}$.

Generalized field strength

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- Using SUSY algebra on general background one can prove that

$$*\mathcal{F}_{\epsilon} = -\frac{\mathbf{Q}(\epsilon\lambda)}{2\epsilon^{+}\epsilon^{-}}\ , \qquad *\mathcal{F}_{\tilde{\epsilon}} = +\frac{\tilde{\mathbf{Q}}(\tilde{\epsilon}\tilde{\lambda})}{2\tilde{\epsilon}^{+}\tilde{\epsilon}^{-}}\ .$$

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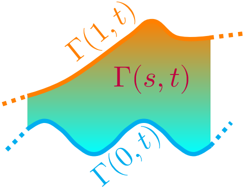

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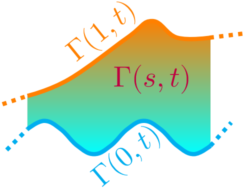

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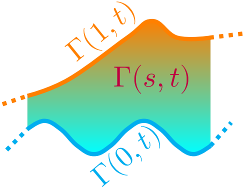

- It is possible to compute the variation

$$\partial_s L_{\mathcal{R}} = \mathrm{i} \operatorname{tr}_{\mathcal{R}} \int_0^1 \mathrm{d}t' \, W(s; t', 1) \mathcal{F}_{st}(s,t') W(s; 0, t') \; ,$$

with $\mathcal{F}_{st} = \mathcal{F}^{\mathsf{ab}} \partial_t x^{\mathsf{a}} \partial_s x^{\mathsf{b}}$.

Cohomological argument

- Using $\mathbf{Q}$ and $\tilde{\mathbf{Q}}$-exactness of the field strength

$$\partial_s L_{\mathcal{R},\epsilon} = \mathbf{Q}\,\mathrm{tr}_{\mathcal{R}} \int_0^1 \mathrm{d}t'\, \varepsilon^{\mathsf{ab}} \partial_t x^{\mathsf{a}}(t') \partial_s x^{\mathsf{b}}(t) \times W_\epsilon(s;t',1) \Big[-\frac{\epsilon\lambda}{2\epsilon^+\epsilon^-} \Big] W_\epsilon(s;0,t') \ ,$$

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- If Γ_1 is homotopic to Γ_2, we have

$$\langle L_{\mathcal{R}}(\Gamma_1)\rangle = \langle L_{\mathcal{R}}(\Gamma_2)\rangle \ .$$

Examples

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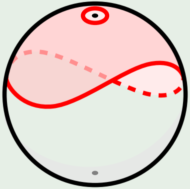

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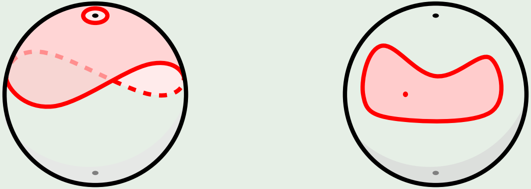

Localization and dualities on S^2

Localization on S^2 [Doroud–Gomis–Le Floch–S. Lee], [Benini–Cremonesi]

- $\mathrm{U}(N)$ SYM with matter (N_f fundamental, N_a anti-fundamental):

$$\mathscr{L} = \mathscr{L}_{\mathrm{vec}} + \mathscr{L}_{\mathrm{mat}} + \mathscr{L}_{\mathrm{FI}}\ .$$

with $N_\mathrm{f} > N_\mathrm{a}$ (or $N_\mathrm{f} = N_\mathrm{a}$ and $\xi > 0$) and $N_\mathrm{f} \geq N$.

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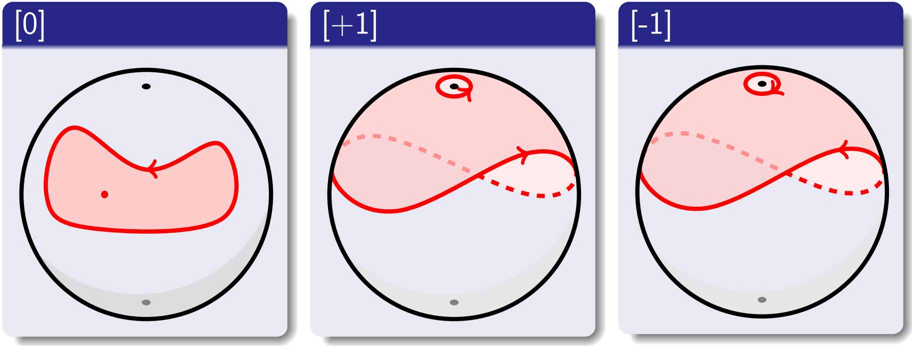
- We can use our $\mathbf{Q}$ or $\tilde{\mathbf{Q}}$-cohomological argument to deform a given WL to a $\frac{1}{2}$ BPS one.

Wilson loops on BPS locus

- Three types of non-self-intersecting paths



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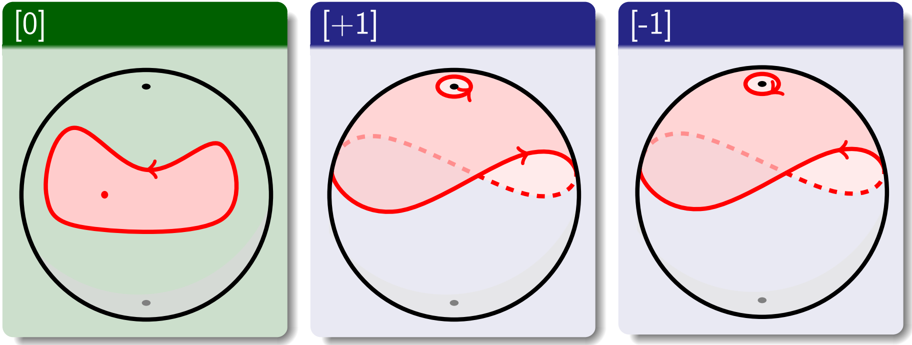

- Evaluation of local operator

$$\mathrm{tr}_{\mathcal{R}}\, e^{0} = \dim \mathcal{R}\ ,$$

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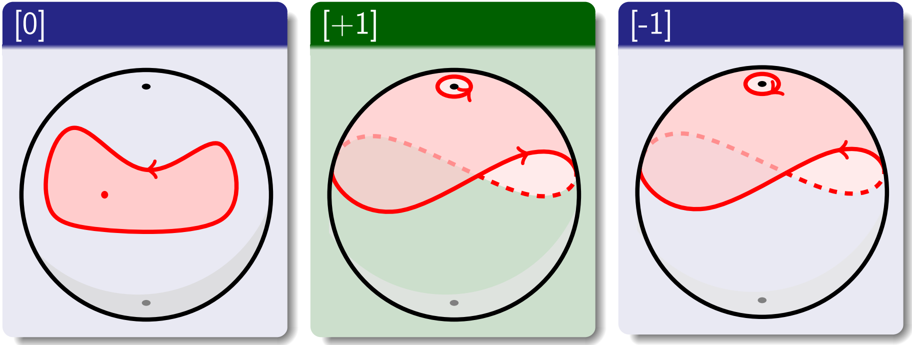

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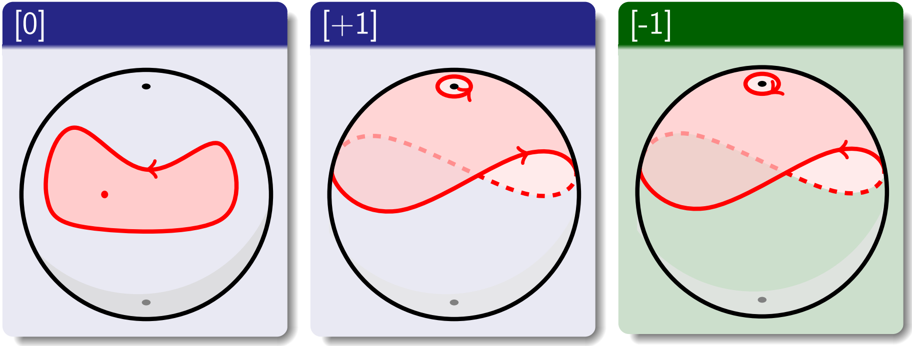

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Abelian Case

- The Abelian Matrix Model [DGLL], [BC] is

$$Z_{\mathrm{U}(1)}(\xi,\vartheta;\tau,\tilde{\tau}) = \sum_{\mathfrak{m}} \int \frac{\mathrm{d}y}{2\pi} e^{-4\pi\mathrm{i}\xi y - \mathrm{i}\mathfrak{m}\vartheta}$$

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- Then we have

$$\langle e^{-2\pi r\sigma}\rangle_{\mathrm{U}(1),\Lambda} = \frac{Z_{\mathrm{U}(1)}(\xi - \Lambda\frac{\mathrm{i}}{2}, \vartheta - \Lambda\pi; \tau, \tilde{\tau})}{Z_{\mathrm{U}(1)}(\xi,\vartheta;\tau,\tilde{\tau})} \,.$$

Non-Abelian Case

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$$Z(\xi,\vartheta;\tau,\tilde{\tau})_{\mathrm{U}(N)} = \langle\!\langle\mathbf{1}\rangle\!\rangle = \sum_{l\in\mathsf{C}(N_\mathrm{f},N)}e^{4\pi\mathrm{i}\sum_r\tau_{l_r}}\mathcal{Z}^{(l)}_{\text{1-loop}}(\tau,\tilde{\tau})\mathcal{Z}^{(l)}_{\mathrm{v}}(\xi,\vartheta;\tau,\tilde{\tau})\mathcal{Z}^{(l)}_{\mathrm{av}}(\xi,\vartheta;\tau,\tilde{\tau})\,.$$

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- Insertions of local operators are easy to write

$$\langle \mathrm{tr}_{\mathcal{R}}\, e^{-2\pi r\sigma}\rangle_{\mathrm{U}(N)} = \langle\!\langle \chi_{\mathcal{R}}(x_{l_1},\dots,x_{l_N})\rangle\!\rangle \,,$$

where $\chi_{\mathcal{R}}$ is the character of $\mathcal{R}$, and $x_l = e^{2\pi\tau_l}$.

$\mathrm{U}(N)$ irreps and characters

- $\mathrm{U}(N)$ irreps are labeled by a set iof N integers $\boldsymbol{\lambda}$

$$\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N \; .$$

For instance

  - the fundamental has label $(1, 0, \ldots, 0)$,
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- The character is defined:

$$\chi_{\boldsymbol{\lambda}}^{\mathrm{U}(N)}(x_1, \ldots, x_N) = \frac{a_{(\lambda_1+N-1, \lambda_2+N-2, \ldots, \lambda_N)}(x_1, \ldots, x_N)}{a_{(N-1, N-2, \ldots, 0)}(x_1, \ldots, x_N)} \ ,$$

where $a_{(\varrho_1, \ldots, \varrho_N)}(x_1, \ldots, x_N) = \det[x_i^{\varrho_j}]_{i,j=1}^N$.

Correlators of Wilson loops

- We can repeat our argument for (non-intersecting) WL:

$$\langle L_{\mathcal{R}_1}(\Gamma_1)\dots L_{\mathcal{R}_n}(\Gamma_n)\rangle_{\mathrm{U}(N)} = \prod_{\Gamma_i\in[0]} \dim\mathcal{R}_i \left\langle \prod_{\Gamma_j\in[+1]} \mathrm{tr}_{\mathcal{R}_j}\, e^{-2\pi r\sigma} \prod_{\Gamma_k\in[-1]} \mathrm{tr}_{\mathcal{R}_k}\, e^{+2\pi r\sigma} \right\rangle .$$

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$$\langle L_{\mathcal{R}_1}(\Gamma_1)\dots L_{\mathcal{R}_n}(\Gamma_n)\rangle_{\mathrm{U}(N)} = \prod_{\Gamma_i\in[0]} \dim\mathcal{R}_i \left\langle \prod_{\Gamma_j\in[+1]} \mathrm{tr}_{\mathcal{R}_j}\, e^{-2\pi r\sigma} \prod_{\Gamma_k\in[-1]} \mathrm{tr}_{\mathcal{R}_k}\, e^{+2\pi r\sigma} \right\rangle .$$

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- Homomorphism: algebra of WL and irreps of $\mathbf{G}$.

Seiberg-like dualities

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Dictionary between Wilson loops

in the spirit of [Kapustin–Willett]

Partition function duality ($N_{\rm a} = 0$)

Electric Theory

- $\mathbf{G}^{\rm D} = \mathrm{U}(N)$;

Magnetic Theory

- $\mathbf{G} = \mathrm{U}(N_{\rm f} - N)$;

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$$\sum_f \tau_f = 0 \qquad \Rightarrow \qquad \prod_f x_f = 1 \, .$$

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Wilson loop duality: Idea ($N_{\mathrm{f}} = 3$, $N = 1$)

- At the level of partition functions one has

$$Z = \mathfrak{z}^{(3)} + \mathfrak{z}^{(2)} + \mathfrak{z}^{(1)} \,, \qquad Z^{\mathrm{D}} = \mathfrak{z}^{\mathrm{D},(12)} + \mathfrak{z}^{\mathrm{D},(13)} + \mathfrak{z}^{\mathrm{D},(23)} \,.$$

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- Evaluation of a non-trivial fundamental Wilson loop ($x_i = e^{2\pi\tau_i}$):

$$\begin{aligned}
\langle L_\mathrm{f}\rangle Z &= x_1\mathfrak{z}^{(1)} + x_2\mathfrak{z}^{(2)} + x_3\mathfrak{z}^{(3)} \\
&= [x_1 + x_2 + x_3]\mathfrak{z}^{(1)} - [x_2 + x_3]\mathfrak{z}^{(1)} \\
&+ [x_1 + x_2 + x_3]\mathfrak{z}^{(2)} - [x_1 + x_3]\mathfrak{z}^{(2)} \\
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\end{aligned}$$

- Therefore

$$\langle L_\mathrm{f}\rangle = \chi_\mathrm{a}^{\mathrm{U}(3)} - \langle L_\mathrm{a}\rangle^\mathrm{D} \,.$$

Fun with characters

- It is possible to express $\chi_{\boldsymbol{\lambda}}^{\mathrm{U}(N)}(x_1,\dots,x_n)$ in terms of power sums

$$p_{\boldsymbol{\nu}}(x_1,\dots,x_N) = p_{\nu_1}(x_1,\dots,x_N)p_{\nu_2}(x_1,\dots,x_N)\cdot\ldots\cdot p_{\nu_N}(x_1,\dots,x_N)\ ,$$

where $p_k(x_1,\dots,x_N) = x_1^k + \dots + x_N^k$.

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- Power sums are easy to manipulate

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- Every power sum can be expressed in terms of characters.
- All in all we have

$$\begin{aligned}\chi_{\boldsymbol{\lambda}}^{\mathrm{U}(N)}(x_{l_1},\dots,x_{l_N}) &= \sum_{\boldsymbol{\mu}} c_{\boldsymbol{\mu}}(x_1,\dots,x_{N_\mathrm{f}})\chi_{\boldsymbol{\mu}}^{\mathrm{U}(N_\mathrm{f}-N)}(x_{l_1^\mathrm{D}},\dots,x_{l^\mathrm{D}_{N_\mathrm{f}-N}}) \\ &= \sum_{\boldsymbol{\mu}} c_{-\boldsymbol{\mu}}(x_1^\mathrm{D},\dots,x^\mathrm{D}_{N_\mathrm{f}})\chi_{-\boldsymbol{\mu}}^{\mathrm{U}(N_\mathrm{f}-N)}(x^\mathrm{D}_{l_1^\mathrm{D}},\dots,x^\mathrm{D}_{l^\mathrm{D}_{N_\mathrm{f}-N}})\ ,\end{aligned}$$

  where $c_{-\boldsymbol{\mu}}$ are characters of $\mathrm{U}(N_\mathrm{f})$ (and of $\mathrm{SU}(N_\mathrm{f})$ once $\prod_f x_f = 1$).

Dictionary

$$
\begin{aligned}
L^{\mathrm{U}(N)}_{(0,\dots,0)} &\mapsto L^{\mathrm{U}(N_\mathrm{f}-N)}_{(0,\dots,0)}\,,\\
L^{\mathrm{U}(N)}_{(1,0,\dots,0)} &\mapsto \chi^{\mathrm{U}(N_\mathrm{f})}_{(0,\dots,0,-1)} - L^{\mathrm{U}(N_\mathrm{f}-N)}_{(0,\dots,0,-1)}\,,\\
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- The duality is an involution;
- More direct duality if we insert matter field in the connection.

Conclusion and outlook

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- Nontrivial WLs on T^2;

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Outlook

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- IR properties;
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Thank you for your attention!