Kerr-Schild DFT and Classical Double Copy

Kanghoon Lee

IBS-CFGS

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APCTP

Strings, Branes and Gauge Theories

- Double copy structure states that the scattering amplitudes of the Yang-Mills theory and gravity are related by exchanging the color and kinematic factors [Bern, Carrasco Johansson 2008,2010]

$$c_i \leftrightarrows n_i$$

color factor c_i : a polynomial of structure constants f^{abc},

kinematic factor n_i : a polynomial of Lorentz-invariant contractions of polarization vectors ϵ_i and momenta p_i.

- Gravity amplitudes can be obtained by just replacing the color factor to the kinematic factor without any knowledge of the gravity action or Feynman rules.

- Spectrum

$$\text{graviton}^{\pm 2}(p_i) = \text{gluon}^{\pm 1}(p_i) \otimes \text{gluon}^{\pm 1}(p_i)$$

$$\left.\begin{array}{l}\text{dilaton}\\ \text{axion}\end{array}\right\} = \text{gluon}^{\pm 1}(p_i) \otimes \text{gluon}^{\mp 1}(p_i)$$

- The double copy has the potential to provide a new way of quantum gravity

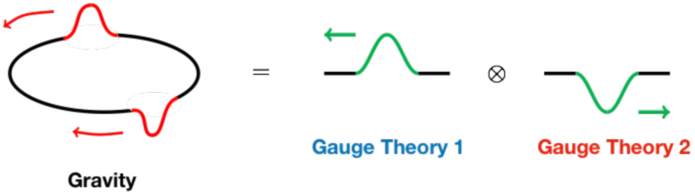
(perturbative) gravity = (Yang-Mills)^2

- For tree level amplitude, it is equivalent to the field theory limit, $\alpha' \to 0$, of the KLT relation in closed string theory.
- Tree level closed string and open string scattering amplitudes are related via the KLT relation [Kawai, Lewellen, Tye 1986]

$$M_n^{\text{tree}} = A_n^{\text{tree}} \mathcal{K}_n \tilde{A}_n^{\text{tree}}$$

where $\mathcal{K}_n$ is the KLT kernel.

- KLT relation provides the string theory origin of double copy structure.



- Tree level scattering amplitude $\longrightarrow$ on-shell, no quantum effects.

  It is possible to deduce its extension to the level of the classical equations of motion.

- Q: Can solutions of the Einstein field equations be represented by solutions of the Yang-Mills equations beyond perturbative level?

$$\text{Solution of GR} \underbrace{\Longleftrightarrow}_{?} \text{Solution of YM}$$

- Graviton $h_{\mu\nu}$ is given by the linearized perturbation of the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

  Recall the spectrum relation. Is it possible to represent $h_{\mu\nu} \sim A_\mu \tilde{A}_\nu$?

- One way is the so called classical double copy based on Kerr-Schild formalism in GR [Monteiro, O'Connell, White, 2014]
- The Kerr-Schild ansatz is an extension of linear perturbation around a background metric $\tilde{g}$.
- Einstein equation is nonlinear PDE $\Longrightarrow$ Hard to solve
- What is the condition

  Einstein equation becomes linear?

- Kerr and Schild proposed a metric ansatz which makes Einstein equation a linear equation [Kerr 1963], [Kerr, Schild 1965] .
- Meyers-Perry BH, (A)dS Kerr, (A)dS Kerr-Newman, Black string, branes, Waves in flat and (A)dS spaces (PP-wave, Kundt wave, Shock wave) etc.

- Kerr-Schild ansatz

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + \kappa\varphi\ell_\mu\ell_\nu$$

$\tilde{g}_{\mu\nu}$: a background metric satisfying Einstein equation

ℓ_μ : null vector

$$\ell_\mu \tilde{g}^{\mu\nu}\ell_\nu = \ell_\mu g^{\mu\nu}\ell_\nu = 0$$

- The main advantage of the Kerr-Schild ansatz is that it preserves some features of the linearized perturbation

$$g^{\mu\nu} = \tilde{g}^{\mu\nu} - \kappa\varphi\ell^\mu\ell^\nu\,, \qquad \det(g) = \det(\tilde{g})$$

- Suppose a vacuum Einstein equation, $R_{\mu\nu} = 0$. We get a consistency condition by contracting the null vectors ℓ^μ with $R_{\mu\nu}$

$$R_{\mu\nu}\ell^\mu\ell^\nu = -\kappa\varphi g^{\nu\sigma}\left(\ell^\mu\tilde{\nabla}_\mu\ell_\nu\right)\left(\ell^\rho\tilde{\nabla}_\rho\ell_\sigma\right) = 0\,,$$

where $\tilde{\nabla}_\mu$ is the covariant derivative with respect to the background metric $\tilde{g}$.

- Choosing affine parameter, ℓ^μ is null and geodesic

$$\ell^\mu \tilde{\nabla}_\mu \ell_\nu = 0 \,.$$

- The vacuum Einstein equation reduces to

$$R_{\mu\nu} = \kappa R^{(1)}{}_{\mu\nu} + \kappa^2 \varphi \ell_\mu \ell^\rho R^{(1)}{}_{\rho\nu} = 0 \,,$$

where $R^{(1)}$ is the linear terms with respect to κ,

$$R^{(1)}{}_{\mu\nu} = \kappa \tilde{\nabla}_\rho \Big(\tilde{\nabla}_{(\mu} \big(\varphi \ell_{\nu)} \ell^\rho \big) - \frac{1}{2} \tilde{\nabla}^\rho \big(\varphi \ell_\mu \ell_\nu \big) \Big) \,,$$

and the Einstein equation is the same as $R^{(1)}{}_{\mu\nu} = 0$.

- Schwarzschild BH in Eddington-Finkelstein coordinate

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2GM}{r}\ell_\mu \ell_\nu$$

where

$$\ell^\mu = \left(1, \frac{x^i}{r}\right), \quad r^2 = x^i x_i, \quad i = 1 \dots 3$$

- Kerr BH in KS coordinate

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$
$$+ \frac{2mr^3}{r^4 + a^2 z^2}\left[dt + \frac{z}{r}dz + \frac{r}{r^2 + a^2}(xdx + ydy) - \frac{a}{r^2 + a^2}(xdy - ydx)\right]^2$$

and the null vector is given by

$$\varphi = \frac{2MGr^3}{r^4 + a^2 z^2}, \qquad \ell_\mu = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right)$$

- Consider KS ansatz on a flat background, $\tilde{g} = \eta$

$$g_{\mu\nu} = \eta_{\mu\nu} + \varphi \ell_\mu \ell_\nu$$

- Identify the null vector ℓ and φ with gauge field and the biadjoint scala field [Monteiro, O'Connell, White, 2014]

$$A_\mu = \varphi \ell_\mu$$

- Assume that spacetime is stationary (no time dependence) and choose ℓ^μ as $\ell^0 = 1$

$$R_{00} = \frac{1}{2}\nabla^2 \varphi$$
$$R_{0i} = \frac{1}{2}\partial^j \big(\partial_i(\varphi \ell_j) - \partial_j(\varphi \ell_i)\big) = -\frac{1}{2}\partial^j F_{ij}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$

Questions

- How can we include Kalb-Ramond field $B_{\mu\nu}$ and dilaton ϕ in Kerr-Schild formalism?

$$\square \otimes \square = \square\!\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array} \oplus \bullet$$

- Curved background generalization - It is not clear how to define scattering amplitude in curved background in general. (time-dependent backgrounds, nonasymptotic flat spaces)
- Classical double copy for non Kerr-Schild type geometries?
- non-abelian structure?

- Generalized Kerr-Schild method in DFT

  $\Longrightarrow$ Linearization of the equations of motion for supergravities (string NSNS sector).

  $\Longrightarrow$ Arbitrary on-shell background $\rightarrow$ More general spacetime

- Classical double copy for entire massless NSNS sector

- Classical double copy in Killing spinor equation

  $\Longrightarrow$ From the Killing spinor equation for gravitino, Yang-Mills BPS equation can be derived

- DFT is the best framework for describing the double copy structure.
- Double copy $\Longleftrightarrow$ Left-right decomposition of closed string theory
- Generalized metric is represented by the coset

$$\mathcal{H} \to \frac{O(d,d)}{O(d-1,1)\times O(1,1-d)}$$

and this implies there are two local Lorentz groups $\Longrightarrow \{e_\mu{}^m, \bar{e}_\mu{}^{\bar{m}}\}$

- These are related with local Lorentz groups for left-right sectors of closed string theory. [Arkani-Hamed,Kaplan, 2008], [Hohm, 2011]

$$\eta_{\mu\nu} + h_{\mu\nu} \to h_{m\bar{n}}$$

- Cheung and Remmen derived perturbative DFT action (without dilaton and $B_{\mu\nu}$) around an arbitrary curved background from Einstein-Hilbert action by assuming the two local Lorentz groups. [Cheung, Remmen, 2016]

Generalized Kerr-Schild ansatz

- Double Field Theory: String low energy effective field theory which is manifest under $O(d,d)$ T-duality. $\Longrightarrow$ Doubling the dimension
- Long History:
  - Sigma model side [Duff, 1990], [Tseytlin, 1990,1991], [Hull, 2004]
  - Low energy effective field theory side [Siegel, 1993], [Hull, Zwiebach, 2009]
- Geometrization of all the form fields in supergravities.
- Recall the Maxwell equation:

$$\partial_\mu F^{\mu\nu} = j^\nu \,, \qquad dF = 0$$

- Manifest under T-duality $\rightarrow O(d,d)$ tensors
- generalized metric $\mathcal{H}_{MN}$: rank 2 tensor wrt $O(d,d)$, which is an $O(d,d)$ element

$$\mathcal{H}_{MN}\mathcal{J}^{NP}\mathcal{H}_{PQ} = \mathcal{J}_{MQ}$$

where $\mathcal{J}_{MN}$ is the $O(d,d)$ metric parametrized

$$\mathcal{J}_{MN} = \begin{pmatrix} 0 & \delta^{\mu}{}_{\nu} \\ \delta_{\mu}{}^{\nu} & 0 \end{pmatrix}$$

- Parametrization in terms of supergravity fields $\{g, B, \phi\}$

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\rho}B_{\rho\nu} \\ B_{\mu\rho}g^{\rho\nu} & g_{\mu\nu} - B_{\mu\rho}g^{\rho\sigma}B_{\sigma\nu} \end{pmatrix}$$

- DFT scalar d : scalar wrt $O(d,d)$

$$e^{-2d} = \sqrt{-g}e^{-2\phi}$$

- First, we analyze the properties of linear perturbations of generalized metric around an on-shell background generalized metric $\mathcal{H}_0$ satisfying

$$\mathcal{H}_{0MN}\mathcal{J}^{NP}\mathcal{H}_{0PQ} = \mathcal{J}_{MQ}$$

- Split $\mathcal{H}$ into the background part and perturbation parts

$$\mathcal{H}_{MN} = \mathcal{H}_{0MN} + \kappa\hat{\gamma}_{MN}\,,$$

where $\hat{\gamma}$ describes perturbation and κ is a small expansion parameter.

- From $O(d,d)$ constraint for $\mathcal{H}_0$, $\left(\mathcal{H}_0\right)^2 = \mathbf{1}$, one can define a background chirality and the corresponding projection operators

$$P_0 = \frac{1}{2}\left(\mathcal{J} + \mathcal{H}_0\right), \qquad \bar{P}_0 = \frac{1}{2}\left(\mathcal{J} - \mathcal{H}_0\right),$$

- One can show that $\hat{\gamma}$ has mixed chirality

$$\hat{\gamma} = P_0\hat{\gamma}\bar{P}_0 + \bar{P}_0\hat{\gamma}P_0\,, \qquad P_0\hat{\gamma}P_0 = \bar{P}_0\hat{\gamma}\bar{P}_0 = 0$$

- Following the conventional Kerr-Schild ansatz, we now assume that $\hat{\gamma}$ is a finite perturbation and κ is a formal finite parameter. The chirality condition is no longer a linearized approximation, but an exact relation.
- We introduce an ansatz for the generalized metric

$$\mathcal{H}_{MN} = \mathcal{H}_{0MN} + \kappa\varphi\big(K_M \bar{K}_N + \bar{K}_M K_N\big)\,,$$

where K and $\bar{K}$ are null vectors

$$K_M K^M = 0\,, \qquad \bar{K}_M \bar{K}^M = 0\,,$$

and satisfy the chirality conditions as

$$P_{0MN} K^N = K_M\,, \qquad \bar{P}_{0MN} \bar{K}^N = \bar{K}_M\,, \qquad K_M \bar{K}^M = 0\,,$$

- We refer this form as generalized Kerr-Schild ansatz. This ansatz satisfies the $O(d,d)$ constraint automatically without any approximation or truncation.

- Chirality condition $\Longrightarrow$ the K_M and $\bar{K}_M$ are parametrized in terms of the d-dimensional vectors l^μ and $\bar{l}^\mu$

$$K_M = \frac{1}{\sqrt{2}}\begin{pmatrix} l^\mu \\ (\tilde{g}+\tilde{B})_{\mu\nu} l^\nu \end{pmatrix}, \qquad \bar{K}_M = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{l}^\mu \\ (-\tilde{g}+\tilde{B})_{\mu\nu} \bar{l}^\nu \end{pmatrix}.$$

- Null condition $\Longrightarrow$ l and $\bar{l}$ are null vectors

$$l^\mu \tilde{g}_{\mu\nu} l^\nu = l^\mu l_\mu = 0\,, \qquad \bar{l}^\mu \tilde{g}_{\mu\nu} \bar{l}^\nu = \bar{l}^\mu \bar{l}_\mu = 0\,, \qquad l \cdot \bar{l} \neq 0$$

- More than one pair of null vectors?
- It is strictly forbidden in the Lorentzian signature metric! (Theory of quadratic form)

- Using the parametrization of generalized metric, we have

$$
\begin{aligned}
(g^{-1})^{\mu\nu} &= (\tilde{g}^{-1})^{\mu\nu} + \kappa\varphi l^{(\mu}\bar{l}^{\nu)} \,, \\
g_{\mu\nu} &= \tilde{g}_{\mu\nu} - \frac{\kappa\varphi}{1+\frac{1}{2}\kappa\varphi(l\cdot\bar{l})} l_{(\mu}\bar{l}_{\nu)} \,, \\
B_{\mu\nu} &= \tilde{B}_{\mu\nu} + \frac{\kappa\varphi}{1+\frac{1}{2}\kappa\varphi(l\cdot\bar{l})} l_{[\mu}\bar{l}_{\nu]} \,, \\
\det g &= (\det\tilde{g})\Big(1+\frac{1}{2}\kappa\varphi(l\cdot\bar{l})\Big)^{-2}
\end{aligned}
$$

- Though $\mathcal{H}$ is linear in κ, g and B are nonlinear.
- If we identify l^μ and $\bar{l}^\mu$ and ignore the B field, then it reduces to the conventional Kerr-Schild ansatz,

$$
g^{\mu\nu} = \tilde{g}^{\mu\nu} + \kappa\varphi l^\mu l^\nu \,, \qquad g_{\mu\nu} = \tilde{g}_{\mu\nu} - \kappa\varphi l_\mu l_\nu \,.
$$

Field equations and linear structure

- In GR, equations of motion is written in terms of curvature tensor

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

- In Riemannian geometry, Riemann tensor is given by commutator of covariant derivative

$$[\nabla_\mu, \nabla_\nu] V_\rho = R_{\mu\nu\rho\sigma} V^\sigma$$

- DFT covariant derivative and curvature?

- Generalized Lie derivative: Recast the diffeomorphism and one-form gauge transform of $B_{\mu\nu}$ in an $O(D,D)$ covariant way.
- "Semi" covariant derivative with respect to the gen. diffeomorphism [Jeon,KL,Park, 2011]

$$\nabla_M V_N = \partial_M V_N + \Gamma_{MNP} V^P$$

- Generalized curvature tensor and scalar

$$\mathcal{S}_{MN} = P_M{}^P \bar{P}_N{}^Q P^{RS} S_{RPSQ}, \qquad \mathcal{S} := 2P^{MN} P^{PQ} S_{MPNQ}$$

where

$$S_{MNPQ} = \frac{1}{2}\left(R_{MNPQ} + R_{PQMN} - \Gamma^R{}_{MN}\Gamma_{RPQ}\right)$$
$$R_{MNPQ} = \partial_M \Gamma_{NPQ} - \partial_N \Gamma_{MPQ} + \Gamma_{MP}{}^R \Gamma_{NRQ} - \Gamma_{NP}{}^R \Gamma_{MRQ}$$

DFT field equation in terms of supergravity fields

- Action

$$S_{\text{eff.}} = \int dx^D \sqrt{-g} e^{-2\phi} \left(R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right)$$

- EOM

$$\mathcal{G}_{\mu\nu} = R_{\mu\nu} + 2\nabla_{(\mu}\partial_{\nu)}\phi - \frac{1}{4} H_\mu{}^{\rho\sigma} H_{\nu\rho\sigma} = 0\,,$$

$$\mathcal{B}_{\mu\nu} = -\frac{1}{2}\nabla^\rho H_{\rho\mu\nu} + \partial^\rho \phi H_{\rho\mu\nu} = 0\,.$$

- For simplicity consider a flat background,

$$\mathcal{H}_{0\,MN} = \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}, \qquad d_0 = \text{const.}$$

- Generalized Kerr-Schild ansatz

$$\mathcal{H}_{MN} = \mathcal{H}_{0\,MN} + \kappa\varphi\big(K_M \bar{K}_N + \bar{K}_M K_N\big)$$

$$d = d_0 + \kappa f\,.$$

- An on-shell condition from the DFT equations of motion, $\mathcal{S}_{KL} = 0$,

$$K^K \bar{K}^L \mathcal{S}_{KL} = 2K^K \bar{K}^L \partial_K \partial_L f - \frac{1}{2}\varphi\big(K^K \partial_K \bar{K}_M\big)\big(K^L \partial_L \bar{K}^M\big) + \frac{1}{2}\varphi\big(\bar{K}^K \partial_K K_M\big)\big(\bar{K}^L \partial_L K^M\big) = 0\,.$$

Recall that in GR, $R_{\mu\nu}\ell^\mu \ell^\nu = -\kappa\varphi g^{\nu\sigma}\big(\ell^\mu \tilde{\nabla}_\mu \ell_\nu\big)\big(\ell^\rho \tilde{\nabla}_\rho \ell_\sigma\big) = 0\,.$

- We shall impose stronger conditions

$$\bar{K}^M \partial_M K_P = 0\,, \qquad K^M \partial_M \bar{K}_P = 0\,, \qquad K^P \partial_P f = 0\,, \qquad \bar{K}^P \partial_P f = 0\,.$$

- DFT connection satisfies

$$K^P \Gamma_{PMN} \bar{K}^N = 0\,, \qquad \bar{K}^P \Gamma_{PMN} K^N = 0\,, \qquad \Gamma^P{}_{PM} K^M = \Gamma^P{}_{PM} \bar{K}^M = 0$$

and this implies

$$K^M \nabla_M \bar{K}_N = K^M \partial_M \bar{K}_N\,, \qquad \bar{K}^M \nabla_M K_N = \bar{K}^M \partial_M K_N$$

- Using the parametrization of K and $\bar{K}$ on a flat background,

$$K_M = \frac{1}{\sqrt{2}} \begin{pmatrix} l^\mu \\ l_\mu \end{pmatrix} , \qquad \bar{K}_M = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{l}^\mu \\ -\bar{l}_\mu \end{pmatrix}$$

The on-shell constraint on l and $\bar{l}$ is written as

$$l^\mu \partial_\mu \bar{l}_\nu = 0 \, , \qquad \bar{l}^\mu \partial_\mu l_\nu = 0 \, ,$$

$$l^\mu \partial_\mu f = 0 \, , \qquad \bar{l}^\mu \partial_\mu f = 0 \, ,$$

- Interestingly, these can be interpreted as the parallel transport equations along the l and $\bar{l}$ with the torsionful connections.

$$\bar{l}^\mu \nabla^+_\mu l_\nu = 0 \, , \qquad l^\mu \nabla^-_\mu \bar{l}_\nu = 0 \, ,$$

where $\nabla^\pm_\mu = \nabla_\mu \pm \frac{1}{2} H_\mu$ and $H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]}$.

- Substituting the KS ansatz into the equations of motion in the flat backgrounds

$$-2\kappa\partial_K\partial_L(\varphi K^K\bar{K}^L)+4\kappa\mathcal{H}_0^{KL}\partial_K\partial_L f-4\kappa^2\mathcal{H}_0^{KL}\partial_K f\partial_L f=0\,.$$

and

$$\kappa\Big[-\frac{1}{2}\mathcal{H}_0^{MN}\partial_M\partial_N\big(\varphi K_{(K}\bar{K}_{L)}\big)+\partial_M\partial_N\big(\varphi K^N\bar{K}_{(K}\big)P_{0\,L)}{}^M$$
$$-\partial_M\partial_N\big(\varphi K_{(K}\bar{K}^N\big)\bar{P}_{0\,L)}{}^M+4P_{0\,(K}{}^M\bar{P}_{0\,L)}{}^N\partial_M\partial_N f\Big]$$
$$+\kappa^2\mathcal{H}_0^{MN}\partial_M f\partial_N\big(\varphi K_{(K}\bar{K}_{L)}\big)=0\,.$$

- Unlike the conventional KS formalism in GR, the equations are quadratic in κ due to the presence of f. If we set $f=0$, field equations reduce to linear equations
- If we consider the power series expansion of f, then the linear terms are enough to determine φ, l and $\bar{l}$ or $g_{\mu\nu}$ and $B_{\mu\nu}$ completely.

- In terms of d-dimensional vector indices, the field equations reduces to

$$\Box(\varphi l_\mu \bar{l}_\nu) - \partial^\rho \partial_\mu (\varphi l_\rho \bar{l}_\nu) - \partial^\rho \partial_\nu (\varphi l_\mu \bar{l}_\rho) + \partial_\mu \partial_\nu (\varphi l \cdot \bar{l}) + \partial_\mu \partial_\nu H = 0\,.$$

- Note that $\mathcal{R}_{\mu\nu}$ is not symmetric tensor:
  - symmetric part $\rightarrow$ eom of g
  - antisymmetric part $\rightarrow$ eom of B
- It is interesting that the generalized KS ansatz for $g_{\mu\nu}$ and $B_{\mu\nu}$ is not linear in κ, l^μ and $\bar{l}^\mu$, but the field equations are linear in these fields.
- Curved background generalization is straightforward.

- So far we have considered a flat background with the Cartesian coordinates only. In a coordinate independent form in terms of the covariant derivative

$$\mathcal{R} = \kappa\Big[\,\tilde{\nabla}_{0\mu}\tilde{\nabla}_{0\nu}\big(\varphi l^{\mu}\bar{l}^{\nu}\big) - 4\tilde{\nabla}_{0}^{\mu}\partial_{\mu}f\,\Big] + 4\kappa^{2}\partial^{\mu}f\partial_{\mu}f = 0\,,$$

$$\mathcal{R}_{\mu\nu} = \frac{\kappa}{4}\Big[\,\tilde{\nabla}_{0\rho}\tilde{\nabla}_{0}^{\rho}\big(\varphi l_{\mu}\bar{l}_{\nu}\big) - \tilde{\nabla}_{0}^{\rho}\tilde{\nabla}_{0\mu}\big(\varphi l_{\rho}\bar{l}_{\nu}\big) - \tilde{\nabla}_{0}^{\rho}\tilde{\nabla}_{0\nu}\big(\varphi l_{\mu}\bar{l}_{\rho}\big) + 4\tilde{\nabla}_{0\mu}\partial_{\nu}f\,\Big]$$
$$- \frac{\kappa^{2}}{2}\partial^{\rho}f\tilde{\nabla}_{\rho}\big(\varphi l_{\mu}\bar{l}_{\nu}\big)\,.$$

where $\tilde{\nabla}_{0\mu}$ is a covariant derivative for a flat background in an arbitrary coordinate system.

- Note that the DFT dilaton is not a scalar field, but a density that transform under a coordinate transform $x^{\mu} \to x'^{\mu}(x)$ as

$$e^{-2d} \to e^{-2d'} = \left|\frac{\partial x'}{\partial x}\right| e^{-2d}\,.$$

- We can find a new coordinate x'^{μ} that makes the new DFT dilaton d' vanish by requiring that the Jacobian is e^{2d}.

$$\left|\frac{\partial x'}{\partial x}\right| = e^{2d}\,.$$

  Thus, for a given d, we can make the DFT dilaton vanishes.

- As discussed, all the higher order terms in κ in the field equations include f. Using this fact, if we perform a coordinate transformation, the equations of motion reduces to linear

$$\check{\mathcal{R}} = \kappa \check{\nabla}_{0\mu}\check{\nabla}_{0\nu}\big(\varphi l^{\mu}\bar{l}^{\nu}\big) = 0\,,$$

$$\check{\mathcal{R}}_{\mu\nu} = \frac{\kappa}{4}\Big[\,\check{\nabla}_{0\rho}\check{\nabla}_{0}^{\rho}\big(\varphi l_{\mu}\bar{l}_{\nu}\big) - \check{\nabla}_{0}^{\rho}\check{\nabla}_{0\mu}\big(\varphi l_{\rho}\bar{l}_{\nu}\big) - \check{\nabla}_{0}^{\rho}\check{\nabla}_{0\nu}\big(\varphi l_{\mu}\bar{l}_{\rho}\big)\,\Big]\,,$$

  where $\check{\nabla}_{0\mu}$ is a covariant derivative for a flat space with the particular coordinate where $f' = 0$.

- However, it is not practical in solving eom, but useful for classical double copy

- The Killing spinor equation reduce the supergravity field equations to first order in derivatives. Combined with the generalized KS ansatz, it will lead to linear equations.
- The SUSY variation of fermions provides the Killing spinor equations, which are

$$\delta\rho = -\gamma^p \mathcal{D}_p \varepsilon = -\gamma^p V_p{}^M \partial_M \varepsilon - \frac{1}{4} V^M{}_p \Phi_{Mmn} \gamma^{pmn} \varepsilon - \frac{1}{2} V^{Mm} \Phi_{Mmn} \gamma^n \varepsilon = 0\,,$$

$$\delta\psi_{\bar{p}} = \bar{V}^M{}_{\bar{p}} \mathcal{D}_M \varepsilon = \bar{V}^M{}_{\bar{p}} \partial_M \varepsilon + \frac{1}{4} \bar{V}^M{}_{\bar{p}} \Phi_{Mmn} \gamma^{mn} \varepsilon = 0\,,$$

- For simplicity, let us choose ε as a Killing spinor for the background geometry satisfying

$$\partial_p \phi \gamma^p \varepsilon_0 + \frac{1}{12} \tilde{H}_{mnp} \gamma^{mnp} \varepsilon_0 = 0\,,$$

$$\tilde{D}^+_{\bar{p}} \varepsilon_0 = 0\,,$$

where ε_0 is the background Killing spinor.

- Then the Killing spinor equations are greatly simplified as

$$\Big(\partial_\mu\Psi + \frac{1}{2}\tilde{D}^+_\nu\big(\varphi' l_\mu \bar{l}^\nu\big)\Big)\gamma^\mu\varepsilon_0 = 0\,,$$

and

$$\Big(\tilde{D}_\mu\big(\varphi l_\nu \bar{l}_\rho\big) - \frac{1}{2}\tilde{H}_{\mu\rho\sigma}\big(\varphi l_\nu \bar{l}^\sigma\big)\Big)\gamma^{\mu\nu}\varepsilon_0 = 0\,.$$

where $\Psi = e^{-2\kappa f}$, $\varphi' = e^{-2\kappa f}\varphi$ and ε_0 is the background Killing spinor.

- For the flat background case

$$\Big(\partial_\mu\Psi + \frac{1}{2}\partial_\nu\big(\varphi' l_\mu \bar{l}^\nu\big)\Big)\gamma^\mu\varepsilon_0 = 0\,,$$

$$\partial_\mu\big(\varphi l_\nu \bar{l}_\rho\big)\gamma^{\mu\nu}\varepsilon_0 = 0\,,$$

where ε_0 is a constant spinor.

- These equations are remarkably simple, and much easier to solve than the full Killing spinor equations.

Cassical double copy

- The KLT and BCJ relations indicate that not only the pure Einstein equation, but also the field equations of entire massless NS-NS sector should be related to the gauge theory.
- Suppose that the full geometry admits at least one Killing vector ξ^{μ}.
- We can locally choose a coordinate system $x^{\mu} = \{x^{i}, y\}$ such that the Killing vector is a constant, $\xi^{\mu} = \partial x^{\mu}/\partial y = \delta^{\mu}_{y}$. The Killing vector ensures the following identities from the torsion free condition

$$\tilde{\nabla}_{\mu}\xi_{\nu} = \tilde{\nabla}_{[\mu}\xi_{\nu]} = \partial_{[\mu}\xi_{\nu]} = 0\,.$$

- Consider the Lie derivative of an arbitrary rank-n tensor $F_{\mu_1\mu_2\cdots\mu_n}$ with respect to a constant Killing vector ξ^μ

$$\begin{aligned}\mathcal{L}_\xi F_{\mu_1\mu_2\cdots\mu_n} &= \xi^\rho\partial_\rho F_{\mu_1\mu_2\cdots\mu_n} + \sum_{i=1}^{n}\partial_{\mu_i}\xi^\rho F_{\mu_1\cdots\mu_{i-1}\rho\mu_{i+1}\cdots\mu_n} \\ &= \xi^\rho\tilde{\nabla}_\rho F_{\mu_1\mu_2\cdots\mu_n} + \sum_{i=1}^{n}\tilde{\nabla}_{\mu_i}\xi^\rho F_{\mu_1\cdots\mu_{i-1}\rho\mu_{i+1}\cdots\mu_n} = 0\,,\end{aligned}$$

- Since we are assuming that the Killing vector is covariantly constant, this shows that

$$\xi^\rho\tilde{\nabla}_\rho F_{\mu_1\mu_2\cdots\mu_n} = 0\,.$$

- We also normalize l_μ and $\bar{l}_\mu$ as

$$\xi\cdot l = \xi\cdot\bar{l} = 1$$

- Classical double copy is achieved by contracting the constant Killing vector ξ^μ with the generalized Ricci tensor in the specific coordinate

$$\check{\mathcal{R}}_{\mu\nu} = \frac{\kappa}{4}\Big[\, \check{\nabla}_{0\rho}\check{\nabla}_0^\rho(\varphi l_\mu \bar{l}_\nu) - \check{\nabla}_0^\rho\check{\nabla}_{0\mu}(\varphi l_\rho \bar{l}_\nu) - \check{\nabla}_0^\rho\check{\nabla}_{0\nu}(\varphi l_\mu \bar{l}_\rho) \,\Big] = 0$$

- Since $\mathcal{R}_{\mu\nu}$ is not symmetric tensor, we get three independent equations as follows:

$$\xi^\nu \check{\mathcal{R}}_{\mu\nu} = \frac{\kappa}{4}\Big[\, \tilde{\nabla}_0{}^\rho\tilde{\nabla}_{0\rho}(\varphi l_\mu) - \tilde{\nabla}_0{}^\rho\tilde{\nabla}_{0\mu}(\varphi l_\rho) \,\Big],$$
$$\xi^\mu \check{\mathcal{R}}_{\mu\nu} = \frac{\kappa}{4}\Big[\, \tilde{\nabla}_0{}^\rho\tilde{\nabla}_{0\rho}(\varphi \bar{l}_\nu) - \tilde{\nabla}_0^\rho\tilde{\nabla}_{0\nu}(\varphi \bar{l}_\rho) \,\Big],$$

- we identify φl_μ and $\varphi \bar{l}_\mu$ with gauge fields

$$A_\mu = \varphi l_\mu\,, \qquad \bar{A}_\mu = \varphi \bar{l}_\mu$$

- Then $\xi^\nu \mathcal{R}_{\mu\nu}$ and $\xi^\mu \mathcal{R}_{\mu\nu}$ reduce to a pair of Maxwell equations

$$\partial^\mu F_{\mu\nu} = 0\,, \qquad \partial^\mu \bar{F}_{\mu\nu} = 0\,,$$

where $F_{\mu\nu}$ and $\bar{F}_{\mu\nu}$ are the field strength of the Maxwell fields of A_μ and $\bar{A}_\mu$ respectively,

- Contracting ξ^μ with all the free indices of $\mathcal{R}^{(1)}_{\mu\nu}$, we make a scalar equation

$$\xi^\mu \xi^\nu \check{\mathcal{R}}_{\mu\nu} = \Box\varphi = 0\,,$$

- Monteiro, O'Connell and White identified φ as the biadjoint scalar field [Cachazo,He,Yuan 2013]

$$\Phi^{aa'} = \varphi c^a \bar{c}^{a'}$$

where c^a and $\bar{c}^{\bar{a}}$ are color index vectors for Lie group G_1 and G_2.

- It can be understood as a linearized equation of motion for $\Phi^{aa'}$

$$\partial^2 \Phi^{aa'} - g f^{abc} f^{a'b'c'} \Phi^{bb'} \Phi^{cc'} = 0$$

- This shows that the generalized KS type solution can be written in terms of the solutions of the two independent Maxwell equation and free scalar field equations

- On a flat background, Killing spinor equation for gravitino is given by

$$\kappa\partial_{[m}\left(\phi l_{n]}\bar{l}_{\mu}\right)\gamma^{mn}\varepsilon = 0$$

- contraction with a Killing vector ξ^{μ}

$$F_{\mu\nu}\gamma^{\mu\nu}\varepsilon = 0\,.$$

- This is the typical BPS equation of $N = 1$ SYM. This shows the classical double copy is still valid for supersymmetric backgrounds

- Recently Adamo, Casali, Mason, Nekova showed that BCJ color-kinematic duality can be extended pp-wave background.
- The classical double copy is well studied in a flat background, but curved background generalization was not obvious. (only for some simple cases, (A)dS background etc [Gonalez, Penco, Trodden, 2017], [Bahjat-Abbas, Luna, White, 2017])
- The KLT relation in a curved background - It is not clear how to define scattering amplitude in curved background in general. (time-dependent backgrounds, nonasymptotic flat spaces)
- Classical double copy in general background may give a clue, however, curved background generalization is an open problem.

- Heterotic supergravity: relaxed null condition

$$\mathcal{H}_{\hat{M}\hat{N}} = \mathcal{H}_{0\hat{M}\hat{N}} + \kappa\varphi\big(K_{\hat{M}}\bar{K}_{\hat{N}} + K_{\hat{N}}\bar{K}_{\hat{M}}\big)\,,$$

In terms of the heterotic supergravity fields

$$\begin{aligned}
g^{\mu\nu} &= \tilde{g}^{\mu\nu} + \kappa\varphi l^{(\mu}\bar{l}^{\nu)}\,,\\
g_{\mu\nu} &= \tilde{g}_{\mu\nu} - \frac{\kappa\varphi}{1+\frac{\kappa\varphi}{2}(l\cdot\bar{l})} l_{(\mu}\bar{l}_{\nu)} + \frac{1}{4}\left(\frac{\kappa\varphi}{1+\frac{\kappa\varphi}{2}(l\cdot\bar{l})}\right)^2 (\bar{l}\cdot\bar{l}) l_\mu l_\nu\,,\\
B_{\mu\nu} &= \tilde{B}_{\mu\nu} + \frac{\kappa\varphi}{1+\frac{\kappa\varphi}{2}(l\cdot\bar{l})}\left(l_{[\mu}\bar{l}_{\nu]} - \sqrt{\frac{\alpha'}{2}}\tilde{A}_{[\mu}{}^{\alpha} l_{\nu]} j_\alpha\right),\\
A_{\mu\alpha} &= \tilde{A}_{\mu\alpha} + \frac{1}{\sqrt{2\alpha'}}\frac{\kappa\varphi}{1+\frac{\kappa\varphi}{2}(l\cdot\bar{l})} l_\mu j_\alpha\,,
\end{aligned}$$

where l is a null vector, but $\bar{l}$ is not.

- It is possible to couple $U(1)$ gauge fields.

Examples

- A class of string backgrounds which have one conserved chiral null current on the world sheet. [Horowitz, Tseytlin, 1994]
- It is a generalization of the gravitational wave and fundamental string background and is exact in the α' expansion.
- In the target space they have a null Killing vector and unbroken supersymmetries.
- Special cases are the Taub-NUT geometry and rotating black holes.
- The explicit geometry is given by

$$\mathrm{d}s^2 = F(x^i)\mathrm{d}u\Big(\mathrm{d}v + K(u,x^i)\mathrm{d}u + 2V_i(u,x^i)\mathrm{d}x^i\Big) + \mathrm{d}x^i\mathrm{d}x^i\,,$$

$$B_{uv} = F(x^i)\,, \qquad B_{ui} = 2F(x^i)V_i(u,x^i)\,,$$

$$\phi = \phi(u) + \frac{1}{2}\log F(x^i)\,,$$

- This fits into the generalized Kerr-Schild ansatz in a flat background.

$$\mathrm{d}s^2 = \mathrm{d}u\mathrm{d}\tilde{v} + \mathrm{d}x^i\mathrm{d}x^i + (F-1)\mathrm{d}u\Big(\mathrm{d}\tilde{v} - \tilde{V}_i\tilde{V}^i\mathrm{d}u + \tilde{V}_i\mathrm{d}x^i\Big)\,,$$

where

$$V_i = \tilde{V}_i + \frac{1}{2}\partial_i X\,, \qquad v = \tilde{v} - X(x,u)\,,$$

$$X(x,u) = \int^u \Big(K + \frac{4F}{(F-1)}\tilde{V}_i\tilde{V}^i\Big)(\vec{x},u')\mathrm{d}u'\,,$$

- The associated φ and null vectors l and $\bar{l}$ can be easily read off

$$\kappa\varphi = F^{-1} - 1\,,$$

$$l_u = 1\,,$$

$$\bar{l}_u = -\Big(\frac{2F}{F-1}\Big)^2\tilde{V}_i\tilde{V}^i\,, \qquad \bar{l}_{\tilde{v}} = 1\,, \qquad \bar{l}_i = \frac{2F}{F-1}\tilde{V}_i\,,$$

and one can easily show that l and $\bar{l}$ are orthogonal with respect to the flat background metric.

- l and $\bar{l}$ satisfy the generalized geodesic constraint

$$l^\mu \partial_\mu \bar{l}_\nu = 0 \,, \qquad \bar{l}^\mu \partial_\mu l_\nu = 0$$

- Equations of motion imply, $\kappa f = \phi(u)$

$$\begin{aligned} \partial_i \partial^i F^{-1} &= 0 \,, \\ -\partial^i \partial_i K + 2\partial^i \partial_u V_i + 4F^{-1}\partial_u^2 \phi &= 0 \,, \\ -4\partial^j \mathcal{F}_{ji} + 4\partial_u \phi \partial_i F^{-1} &= 0 \,. \end{aligned}$$

  where $\mathcal{F}_{ij} = \partial_i V_j - \partial_j V_i$.

- This is the same exactly with the equation derived by Callan, Maldacena and Peet.

- To examine our formalism in the case of a curved background, let us consider superposition of a number Q_1 of fundamental strings and Q_5 of NS5 brane system.
- Wrap a number Q_5 of NS5-branes on T^5 along $x_5, \cdots x_9$. The fundamental strings wrap one of the directions of the torus along x^5 direction.

$$\begin{aligned} \mathrm{d}s^2 &= F_1^{-1}\big(-\mathrm{d}t^2 + \mathrm{d}x_5^2\big) + F_5\big(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_4^2\big) + \mathrm{d}x_6^2 + \cdots \mathrm{d}x_9^2 \\ e^{-2\phi} &= g_s F_1 F_5^{-1} \\ B_{05} &= F_1^{-1} - 1\,, \\ H_{ijk} &= \epsilon_{ijkl}\partial^l F_5\,, \qquad i,j,k,l = 1,2,3,4 \end{aligned}$$

where ϵ_{ijkl} is the flat space epsilon tensor and

$$F_1 = 1 + \frac{16\pi^4\alpha'^3 Q_1}{g_s^2 V_4 r^2}\,, \qquad F_5 = 1 + \frac{\alpha' Q_5}{r^2}\,,$$

here V_4 is the volume of the T^4.

- The NS5-brane background is treated as a background

$$
\begin{aligned}
\mathrm{d}\tilde{s}^2 &= -\mathrm{d}t^2 + F_5\left(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_4^2\right) + \mathrm{d}x_5^2 + \cdots \mathrm{d}x_9^2 \\
e^{-2\tilde{\phi}} &= g_s F_5^{-1} \\
\tilde{H}_{ijk} &= \epsilon_{ijkl}\partial^l F_5\,, \qquad i,j,k,l = 1,2,3,4
\end{aligned}
$$

where we put tilde for all the background quantities.

- We can read off the corresponding generalized KS ansatz

$$
\varphi = F_1 - 1\,, \qquad l = \mathrm{d}t + \mathrm{d}x_5\,, \qquad \bar{l} = -\mathrm{d}t + \mathrm{d}x_5\,.
$$

Note that l_μ and $\bar{l}_\mu$ are orthogonal to the background 3-form flux, $\tilde{H}_{\mu\nu\rho}$

$$
l^\mu \tilde{H}_{\mu\nu\rho} = \bar{l}^\mu \tilde{H}_{\mu\nu\rho} = 0\,,
$$

and $\phi = \frac{1}{\sqrt{-g}}$ of

$$
f = 0\,.
$$

- Internal momenta and axion charges are interchangeable via T-duality [Horne, Horowitz, Steif, 1991]
- Consider a 5-dimensional uncharged black string, Schwarzschild $\times\, S^1$.
- 1. Boost along the circle direction, $y \Longrightarrow$ Generates Off diagonal terms in metric
  2. Take a T-duality along the y direction
- The explicit geometry is given by

$$\mathrm{d}s^2 = \left(1 + \frac{2mS^2}{r}\right)^{-1}\left[-\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \mathrm{d}y^2\right] + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega\,,$$

$$B_{yt} = \frac{C}{S}\left(1 + \frac{2mS^2}{r}\right)^{-1}, \qquad e^{-2\phi} = 1 + \frac{2mS^2}{r}\,,$$

where $C = \cosh\alpha$ and $S = \sinh\alpha$, and α is a boost parameter.

- Uncharged black string solution:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}\tilde{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 + \mathrm{d}y^2\,. \tag{1}$$

- Using the conventional Kerr-Schild ansatz for Schwarzschild BH in Eddington-Finkelstein coordinate, we have

$$\mathrm{d}s^2 = -\mathrm{d}\hat{t}^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 + \mathrm{d}y^2 - \kappa\varphi\big(\mathrm{d}\hat{t} + \mathrm{d}r\big)^2\,, \qquad \kappa\varphi = -\frac{2M}{r}\,, \tag{2}$$

where $\hat{t} = \tilde{t} + \big(r_* - r\big)$ and r_* is the tortoise coordinate defined

$$r_* = r + 2M\log\left|\frac{r}{2M} - 1\right|\,, \qquad \mathrm{d}r_* = \mathrm{d}r\left(1 - \frac{2M}{r}\right)^{-1}. \tag{3}$$

- Applying a boost along the y-direction, $\hat{t} \to \hat{t}\cosh\alpha + y\sinh\alpha$, $y \to \hat{t}\sinh\alpha + y\cosh\alpha$, we get

$$\mathrm{d}s^2 = -\mathrm{d}\hat{t}^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 + \mathrm{d}y^2 - \kappa\varphi\big(C\mathrm{d}\hat{t} + S\mathrm{d}y + \mathrm{d}r\big)^2\,, \tag{4}$$

and the null vectors are identical, $l = \bar{l} = C\mathrm{d}\hat{t} + S\mathrm{d}y + \mathrm{d}r$.

- Let us take a T-duality along the y-direction. According to the Buscher's rule for the null vectors, the null vectors split into

$$l \to l' = C\mathrm{d}\hat{t} + S\mathrm{d}y + \mathrm{d}r\,, \qquad \bar{l} \to \bar{l}' = C\mathrm{d}\hat{t} - S\mathrm{d}y + \mathrm{d}r\,,$$

and $l \cdot \bar{l} = -2S^2$. The corresponding metric and Kalb-Ramond field are

$$\mathrm{d}s'^2 = \Big(1 + \frac{2MS^2}{r}\Big)^{-1}\Big[-\Big(1 - \frac{2M}{r}\Big)\mathrm{d}\hat{t}^2 + \frac{4MC}{r}\mathrm{d}\hat{t}\mathrm{d}r + \Big(1 + \frac{2MC^2}{r}\Big)\mathrm{d}r^2 + \mathrm{d}y^2\Big] + \cdots$$
$$B' = \Big(-\frac{C}{S} + \frac{2MCS}{r}\Big(1 + \frac{2MS^2}{r}\Big)^{-1}\Big)\mathrm{d}\hat{t}\wedge\mathrm{d}y - \frac{2MS}{r}\Big(1 + \frac{2MS^2}{r}\Big)^{-1}\mathrm{d}r\wedge\mathrm{d}y\,.$$

- Finally, we make a further coordinate transform $\hat{t} = t + C\big(r_* - r\big)$

$$\mathrm{d}s^2 = -\mathrm{d}\hat{t}^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 + \mathrm{d}y^2 + \frac{2M}{r + 2MS^2}\big(C\mathrm{d}\hat{t} + S\mathrm{d}y + \mathrm{d}r\big)\big(C\mathrm{d}\hat{t} - S\mathrm{d}y + \mathrm{d}r\big)$$
$$B = \frac{2M}{r + 2MS^2}\big(C\mathrm{d}\hat{t} + S\mathrm{d}y + \mathrm{d}r\big)\wedge\big(C\mathrm{d}\hat{t} - S\mathrm{d}y + \mathrm{d}r\big)\,.$$

- In this example, the DFT dilaton vanishes, thus the generalized KS field equations become linear.

- A novel solution generating technique in supergravities via generalized Kerr-Schild method in DFT
- Classical double copy including $B_{\mu\nu}$ and dilaton
- Classical double copy in Killing spinor equation
- Including RR sector, Introducing U(1) gauge fields using Kaluza-Klein reduction, Gauged supergravity extension via Scherk-Schwarz reduction.
- M-theory extension: Exceptional field theories ($SL(5)$, $SO(5,5)$, E_6, E_7 and E_8)
- Finding the most general solutions in a flat or curved backgrounds and their physical interpretations. Applications to AdS/CFT?
- Scattering amplitude computation in the DFT language. Extension double copy structure to curved backgrounds.

Thank you