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3 Lectures on "Vacua counting of 3d gauge theories"

Abstract: This is a brief summary of 3 Lectures given in "Strings, Branes and Gauge Theories", APCTP (webpage : https://www.apctp.org/plan.php/sbg2019/2606)

Contents

1 Lecture 1/2: 3D topological quantum field theory

In this section, we derive Verlinde formula (1.37) which counts ground states of a TQFT on general Riemann surface Σ_g. Refs : [1, 2]

1.1 Axiomatic approach to 3d TQFT

Basic axioms of unitary quantum field theory:

$$\begin{aligned}
&\text{Axiom I : an oriented closed Riemann surface } \Sigma_g \longrightarrow \text{a Hilbert-space } \mathcal{H}(\Sigma_g). \\
&\text{Axiom II : homomorphism } MCG(\Sigma_g) \to U\left(\mathcal{H}(\Sigma_g)\right) \ : \ \varphi \longrightarrow \hat{\varphi} \\
&\text{Axiom III : an oriented 3-manifold } M \text{ with } \partial M = \Sigma_g \longrightarrow \text{a partition vector } |M\rangle \in \mathcal{H}(\Sigma_g) \\
&\qquad\qquad \text{an oriented 3-manifold } M \text{ with } \partial M = \emptyset \longrightarrow \text{a partition function } Z(M) \in \mathbb{C} \\
&\text{Axiom IV : } Z(M = M_1 \cup_\varphi M_2) = \langle M_1|\hat{\varphi}|M_2\rangle \\
&\text{Axiom V : } Z(M = \Sigma_g \times_\varphi S^1) = \mathrm{Tr}_{\mathcal{H}(\Sigma_g)}\hat{\varphi}
\end{aligned} \tag{1.1}$$

1.2 Example: $u(1)_k$ pure CS theory

$u(1)_k$ Chern-Simons theory (A is purely imaginary 1-form field)

$$S = -\frac{ik}{4\pi}\int A \wedge dA\,. \tag{1.2}$$

Phase-space on $\Sigma_{g=1}$ Classically, the phase space $\mathcal{P}(\Sigma_{g=1})$ of the Chern-Simons theory on $\Sigma_{g=1}$ is given by

$$
\begin{aligned}
\mathcal{P}(\Sigma_{g=1}) &= \{dA = 0 \text{ on } \Sigma_{g=1}\}/(\text{gauge quotient}) \\
&= \{(X,P) \ : \ \frac{X}{i} \in \mathbb{R}/(2\pi\mathbb{Z}) \ , \ \frac{P}{i} \in \mathbb{R}/(2\pi\mathbb{Z})\} \\
&= S^1 \times S^1 \ .
\end{aligned}
\tag{1.3}
$$

More explicitly, the flat-connections can be written as

$$
A_{\mathbb{T}^2}(X,P) = X d\theta_1 + P d\theta_2 \ . \tag{1.4}
$$

Here θ_1 and θ_2 are two angular coordinates of the torus with periodicity 1:

$$
\theta_1 \sim \theta_1 + 1 \ , \quad \theta_2 \sim \theta_2 + 1 \ . \tag{1.5}
$$

Due to large gauge transformation $A \to A + \Lambda^{-1} d\Lambda$ with $\Lambda(\theta_1, \theta_2) = \exp\big(2\pi i (a\theta_1 + b\theta_2)\big)$, the (X,P) become periodic variables $(a, b \in \mathbb{Z})$,

$$
\Lambda \ : \ (X,P) \to (X + 2\pi i a, P + 2\pi i b) \tag{1.6}
$$

1D quantum mechanical action for a particle moving around the phase space is

$$
\begin{aligned}
&- \frac{ik}{4\pi} \int dt d\theta_1 d\theta_2 A_{\mathbb{T}^2}\left(X(t), P(t)\right) \wedge dA_{\mathbb{T}^2}\left(X(t), P(t)\right) \ , \\
&= -\frac{ik}{2\pi} \int dt X(t) \frac{dP(t)}{dt} \ .
\end{aligned}
\tag{1.7}
$$

From the action, we have following commutation relation

$$
[\hat{X}, \hat{P}] = -\frac{2\pi i}{k} \ . \tag{1.8}
$$

Or equivalently, we have following symplectic form ω on the phase space $\mathcal{P}(\Sigma_{g=1})$,

$$
\omega = \frac{k}{2\pi} dX \wedge dP \tag{1.9}
$$

Axiom I : $\mathcal{H}(\mathbb{T}^2)$ of $u(1)_k$ theory As a representation space of the commutation relation, we have following Hilbert-space (= space of ground states, since every states have $E = 0$ in a topological theory)

$$
\mathcal{H}(\mathbb{T}^2) = \text{Span}\{|X \in i\mathbb{R}\rangle \ : \ e^{2\pi i \partial_X}|X\rangle = |X\rangle \ , e^{2\pi i \partial_P}|X\rangle = |X\rangle\} \tag{1.10}
$$

In the above, we have take into account of the periodic property of X and P. Here $|X\rangle$ is the usual position-space basis,

$$
\langle X| e^{\hat{X}} = \langle X| e^{X} \ , \quad \langle X| e^{\hat{P}} = \langle X - \frac{2\pi i}{k}| \ . \tag{1.11}
$$

Using the relation $\exp(2\pi i \partial_P) = \exp(k\hat{X})$, we finally have

$$\begin{aligned}\mathcal{H}(\mathbb{T}^2) &= \text{Span}\{|X \in i\mathbb{R}\rangle \ : \ |X + 2\pi i\rangle = |X\rangle \ , \ e^{kX}|X\rangle = |X\rangle\} \ , \\ &= \text{Span}\{|X\rangle \ : \ X \sim X + 2\pi i, \ X \in \frac{2\pi i}{k}\mathbb{Z}\}\end{aligned} \tag{1.12}$$

Set $X = \frac{2\pi i n}{k}$,

$$\begin{aligned}&\mathcal{H}(\mathbb{T}^2) = \text{Span}\{|n\rangle : \ n \in \mathbb{Z}/(k\mathbb{Z})\} \ , \\ &\langle n|e^{\hat{X}} = \langle n|e^{\frac{2\pi i n}{k}} \ , \quad \langle n|e^{\hat{P}} = \langle n-1| \ .\end{aligned} \tag{1.13}$$

Note that $\dim\mathcal{H}(\mathbb{T}^2) = |k|$. This is one way to see why the k should be an integer. The Hilbert-space is finite-dimensional since the phase space $\mathcal{H}(\mathbb{T}^2)$ is compact. In general, the dimension of a Hilbert-space obtained by quantizing a compact smooth phase space $(\mathcal{P}, \omega)$ of dimension $2n$ is given by

$$\dim\mathcal{H} = \left| \int_{\mathcal{P}} (\frac{\omega}{2\pi})^n \right| \ . \tag{1.14}$$

One can easily check that

$$\left| \frac{1}{2\pi} \int_{\mathcal{P}(\mathbb{T}^2)} \omega \right| = \left| \int \frac{k}{4\pi^2} dX \wedge dP \right| = |k| = \dim\mathcal{H}(\mathbb{T}^2) \ . \tag{1.15}$$

Let

$$W^q_{\theta_1} := \exp(q \oint_{\theta_1} A) = e^{qX} \ , \quad W^q_{\theta_2} := \exp(q \oint_{\theta_2} A) = e^{qP} \ . \tag{1.16}$$

After quantization, the loop operators satisfy following commutation relation

$$\hat{W}^{q_1}_{\theta_1} \hat{W}^{q_2}_{\theta_2} = \exp(-\frac{2\pi i q_1 q_2}{k}) \hat{W}^{q_2}_{\theta_2} \hat{W}^{q_1}_{\theta_1} \ . \tag{1.17}$$

The vacua are parametrized by vevs of the above non-local Wilson loop operators (there is no gauge-invariant local operators in the theory):

$$\langle n|\hat{W}^q_{\theta_1}|n\rangle = e^{\frac{2\pi i q n}{k}} \ . \tag{1.18}$$

From the matrix element computation $\langle n|\hat{W}^q|m\rangle$, we can check

$$\hat{W}^q_{\theta_1,\theta_2} = \hat{W}^{q+k}_{\theta_1,\theta_2} \ . \tag{1.19}$$

Ex 1. Show that dim $\mathcal{H}(\Sigma_g) = |k|^g$

Axiom II : $SL(2,\mathbb{Z})$ action on $\mathcal{H}(\mathbb{T}^2)$ For $g = 1$ case, the mapping class group is

$$MCG(\Sigma_{g=1}) = SL(2,\mathbb{Z}) = \{S, T \ : \ S^4 = 1, \ (ST)^3 = 1\} \ . \tag{1.20}$$

Matrix elements of $\hat{S}$ and $\hat{T}$ are

$$\begin{aligned}S^{n_1}_{\ n_2} &= \frac{1}{\sqrt{k}} e^{\frac{2\pi i}{k} n_1 n_2} \\ T^{n_1}_{\ n_2} &= \begin{cases} \delta^{n_1}_{n_2} \exp\left(\frac{\pi i n^2}{k} - \frac{2\pi i}{24}\right) \ , \ \text{even } k \\ \delta^{n_1}_{n_2} \exp\left(\frac{\pi i (n^2 - n)}{k} - \frac{2\pi i}{24}(1 - \frac{1}{k})\right) \ , \ \text{odd } k \end{cases}\end{aligned} \tag{1.21}$$

Ex 2. Show that

$$\begin{aligned} &\hat{S}^4 = 1\ , \quad (\hat{S}\hat{T})^3 = 1\ , \\ &\hat{S}^{-1} e^{\hat{P}} \hat{S} = e^{\hat{X}}\ , \quad \hat{S}^{-1} e^{\hat{X}} \hat{S} = e^{-\hat{P}}\ , \\ &\hat{T}^{-1} e^{\hat{X}} \hat{T} = e^{\hat{X}}\ , \quad \hat{T}^{-1} e^{\hat{P}} \hat{T} = \begin{cases} e^{\hat{X}+\hat{P}}\ , \text{ even } k \\ e^{\hat{X}} e^{\hat{P}}\ , \text{ odd } k \end{cases} \end{aligned} \tag{1.22}$$

Axiom III : Chern-Simons Wave function $\langle n_1, \ldots, n_h | \Sigma_{0,h} \times S^1 \rangle$ Let $\Sigma_{0,h}$ be a sphere with h holes. Then,

$$\partial(\Sigma_{0,h} \times S^1) = (\mathbb{T}^2)^h \ \Rightarrow \mathcal{H}(\partial(\Sigma_{0,h} \times S^1)) = \mathcal{H}(\mathbb{T}^2)^{\otimes h} \tag{1.23}$$

According to Axiom III, we can consider a wave function $\langle n_1, \ldots, n_h | \Sigma_{0,h} \times S^1 \rangle$. In path-integral point of view, the wave-function can be interpreted as

$$\begin{aligned} &\langle n_1, \ldots, n_h | \Sigma_{0,h} \times S^1 \rangle = \int \frac{DA}{(\text{gauge})}\bigg|_{b.c} e^{\frac{ik}{4\pi} \int_{\Sigma_{0,h} \times S^1} A \wedge dA}\ , \\ &\text{with boundary condition : } \exp\left(\oint_{\text{around } i\text{-th hole}} A \right) = e^{\frac{2\pi n_i}{k}} \end{aligned} \tag{1.24}$$

Or equivalently, the wave-function can be understood as ptn on $S^2 \times S^1$ with insertion of h Wilson loop operators of charge $\{n_i\}$ along $\{(i\text{-th hole}) \times S^1\}$:

$$\langle n_1, \ldots, n_h | \Sigma_{0,h} \times S^1 \rangle = \int \frac{DA}{(\text{gauge})} e^{\frac{ik}{4\pi} \int_{S^2 \times S^1} A \wedge dA} \prod_{i=1}^{h} \exp\left(n_i \oint_{(i\text{-th hole}) \times S^1} A \right)\ . \tag{1.25}$$

Here we use the fact that

$$(\text{Wilson loop of charge } n) = (\text{boundary condition fixing } \exp\left(\oint_{\text{around loop}} A \right) = e^{\frac{2\pi n_i}{k}}\ )$$

Ex 3. Check the above equivalence by solving the equation of motion with insertion of Wilson loop .

The wave-function, count the ground state of the Chern-Simons theory on S^2 with external particles of charge $\{n_i\}$. Since the S^2 is compact, the Hilbert-space is empty unless the total charge is zero. So, we expect that

$$\langle n_1, \ldots, n_h | \Sigma_{0,h} \rangle = \delta_{n_1 + \ldots + n_h, 0}\ . \tag{1.26}$$

Axiom IV : $Z(S^3) = \langle \Sigma_{0,1} | \hat{S}\hat{T}^L | \Sigma_{0,1} \rangle$ Topologically 3-sphere S^3 can be obtained by gluing two $\mathbb{T}^2$-boundaries of two solid tori $(\Sigma_{0,1} \times S^1)$ with a $SL(2,\mathbb{Z})$ twisted ST^L

$$S^3 = (\Sigma_{0,1} \times S^1) \cup_{\varphi = ST^L} (\Sigma_{0,1} \times S^1)\ . \tag{1.27}$$

The resulting manifold is topologically always S^3 regardless of the choice of L. Thus, according to the axiom IV (using Einstein's summation convention)

$$\begin{aligned} Z(S^3) &= \langle \Sigma_{0,1}|\hat{S}\hat{T}^L|\Sigma_{0,1}\rangle = \langle \Sigma_{0,1}|n_1\rangle (ST^L)^{n_1}_{\ n_2}\langle n_2|\Sigma_{0,1}\rangle \\ &= (ST^L)^{n_1=0}_{\ n_2=0} = \begin{cases} \frac{1}{\sqrt{k}}\exp(-\frac{2\pi i}{24}L), \\ \frac{1}{\sqrt{k}}\exp(-\frac{2\pi i}{24}(1-\frac{1}{k})L) \ . \end{cases} \end{aligned} \tag{1.28}$$

The phase factor of the $Z(S^3)$ depends on the choice of L. In Witten's original paper, it is noticed that the $Z(M)$ generally have following phase ambiguity due to framing anomaly

$$\text{Framing ambiguity}: \ \exp\big((\frac{2\pi i d}{24} + o(1/k))\mathbb{Z}\big) \ . \tag{1.29}$$

Here d is the dimension of gauge group, $d=1$ for our case. One interesting physical quantity of a 3d topological theory is so called 'topological entanglement entropy', which is given as the free energy on S^3

$$\mathcal{S}_{T.E} = (\text{topological entanglement entropy}) = -\log|Z(S^3)| = \frac{1}{2}\log k \ . \tag{1.30}$$

For $k>1$, the theory has non-trivial topological entanglement entropy.

Ex 4. Prove that for any unitary TQFT, $\mathcal{S}_{T.E}\geq 0$ (Hint: use the fact that $Z(S^3) = \langle 0|\hat{S}|0\rangle$ and the unitarity of $\hat{S}$, $|0\rangle = |\Sigma_{0,1}\rangle \in \mathcal{H}(\mathbb{T}^2)$)

Axiom V : Verlinde formula Let

$$\begin{aligned} S^3\backslash(\bigcirc^{\otimes h}) := &\text{ (3-manifold obtained by} \\ &\text{removing tubular neighborhoods of } h \text{ unknotted trivial knots from } S^3) \ . \end{aligned}$$

The manifold has $(\mathbb{T})^h$ boundary and the its wave-function $\langle n_1,\ldots,n_h|S^3\backslash(\bigcirc^{\otimes h})\rangle$ is the CS pth on S^3 with insertions of h Wilson loops along trivial knot $(\bigcirc)$ of charge $\{n_i\}$. There are two independent ways of computing the wave-function.

First method : Using the fact that $S^3\backslash(\bigcirc^{\otimes h}) = (\Sigma_{0,2}\times S^1)\times_{\varphi=S}(\Sigma_{0,h}\times S^1)$ (we are gluing a torus boundary of $\Sigma_{0,2}\times S^1$ to a torus boundary of $\Sigma_{0,h}\times S^1$ twisted by S),

$$\begin{aligned} &\langle n_1,\ldots,n_h|S^3\backslash(\bigcirc^{\otimes h})\rangle \\ &= \sum_{m_1,m_2}\langle \Sigma_{0,2}|n_1,m_1\rangle S^{m_1}_{\ m_2}\langle m_2,n_2,\ldots,n_s|\Sigma_{0,h}\times S^1\rangle \\ &= \sum_m S^{n_1}_{\ -m}\langle m,n_2,\ldots,n_h|\Sigma_{0,h}\times S^1\rangle \ . \end{aligned} \tag{1.31}$$

Second method : For simplicity, we focus on the case when $h=3$. From figure 1, we expect that

$$\begin{aligned} &Z\big(S^3\backslash(\bigcirc_1\cup\bigcirc_2)\big)\ Z\big(S^3\backslash(\bigcirc_2\cup\bigcirc_3)\big) \\ &= Z\big(S^3\backslash(\bigcirc_1\cup\bigcirc_2\cup\bigcirc_3)\big)Z\big(S^3\backslash\bigcirc_2\big) \end{aligned} \tag{1.32}$$

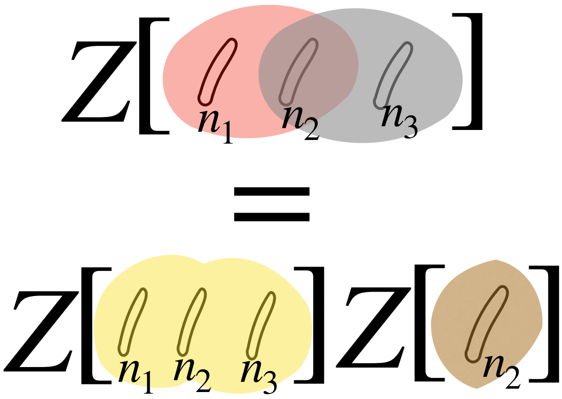

Figure 1. Graphical understanding of $Z\big(S^3\backslash(\bigcirc_1 \cup \bigcirc_2)\big)\ Z\big(S^3\backslash(\bigcirc_2 \cup \bigcirc_3)\big) = Z\big(S^3\backslash(\bigcirc_1 \cup \bigcirc_2 \cup \bigcirc_3)\big) Z\big(S^3\backslash \bigcirc_2\,\big)$

Thus (no summation on n_2)

$$\begin{aligned}
&\langle n_1, n_2, n_3 | S^3\backslash(\bigcirc_1 \cup \bigcirc_2 \cup \bigcirc_3)\rangle \\
&= \frac{\langle n_1, n_2, | S^3\backslash(\bigcirc_1 \cup \bigcirc_2)\rangle \langle n_2, n_3, | S^3\backslash(\bigcirc_2 \cup \bigcirc_3)\rangle}{\langle n_2 | S^3\backslash\bigcirc_2\rangle} \\
&= \frac{S^{n_1}{}_{n_2} S^{n_2}{}_{n_3}}{S^{n_2}_{\ 0}} \text{ (or its permutations on } n_1, n_2, n_3)
\end{aligned} \tag{1.33}$$

Comparing the two computation for $h = 3$, we have (using the symmetric property of S-matrix)

$$\begin{aligned}
&\sum_m S^{n_1}{}_{-m} \langle m, n_2, n_3 | \Sigma_{0,3} \times S^1\rangle = \frac{S^{n_2}{}_{n_1} S^{n_3}{}_{n_1}}{S^{n_1}_{\ 0}} \ . \\
&\Rightarrow \langle n_1, n_2, n_3 | \Sigma_{0,3} \times S^1\rangle = \sum_s \frac{S^{n_1}{}_s S^{n_2}{}_s S^{n_3}{}_s}{S^s{}_0} \quad \text{(Verlinde formula)}
\end{aligned} \tag{1.34}$$

In the second line, we use

$$\sum_n S^n{}_{-m} S^l{}_n = \delta^l_m \ . \tag{1.35}$$

From the fact that $\Sigma_{g=2} \times S^1 = (\Sigma_{0,3} \cup_{\varphi = Id} \Sigma_{0,3})$, we have

$$\begin{aligned}
&Z(\Sigma_{g=2} \times S^1) = \langle \Sigma_{0,3} | \Sigma_{0,3}\rangle \\
&= \sum_{s,s',n_1,n_2,n_3} \frac{S^{n_1}_s (S^{n_1}_{s'})^* S^{n_2}_s (S^{n_2}_{s'})^* S^{n_3}_s (S^{n_3}_{s'})^* S^{n_4}_s (S^{n_4}_{s'})^*}{S^s_0 (S^{s'}_0)^*} \\
&= \sum_{s,s'} \frac{(\delta^s_{s'})^4}{S^s_0 (S^{s'}_0)^*} = \sum_s \frac{1}{(S^s_0)^2} \ . \quad \text{(Verlinde formula)}
\end{aligned} \tag{1.36}$$

Repeating the above computation for general g, we have following formula

$$Z(\Sigma_g \times S^1) = \sum_s \frac{1}{(S^s_0)^{2(g-1)}} = \sum_s (H_s)^{g-1} \ . \tag{1.37}$$

Here, we introduce 'handle gluing operator'

$$H_s := \frac{1}{(S^s_0)^2} \ . \tag{1.38}$$

For our case,

$$H_s = k \ , \quad \Rightarrow \ Z(\Sigma_g \times S^1) = \sum_{s=0}^{k-1} k^{g-1} = k^g \ . \tag{1.39}$$

It is compatible with the fact that $\dim\mathcal{H}(\Sigma_g) = k^g$.

Ex 5. Show that $u(1)_{k_1} \times u(1)_{k_2}$ theory is different from $u(1)_{s=k_1 k_2}$ theory. (Find a physical observable which distinguishes two topological theories.)

1.3 Example: $su(2)_k$ pure CS theory

In the case

$$\mathcal{H}(\mathbb{T}^2) = \{|j\rangle \ : j = 0, \dots, k\} \ . \tag{1.40}$$

S, T matrices are

$$S^j{}_{j'} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi(j+1)(j'+1)}{k+2} \ , \tag{1.41}$$

$$T^j{}_{j'} = \delta^j{}_{j'}(\text{phase factor depending } j) \tag{1.42}$$

Verlinde Formula

$$\dim(\Sigma_g) = (\frac{k+2}{2})^{g-1} \sum_j (\sin \frac{\pi(j+1)}{k+2})^{2-2g} \tag{1.43}$$

Ex 6. Check that the $\dim(\Sigma_g)$ is always natural number

2 Lecture 3: Witten/Twisted index of 3d $\mathcal{N} = 2$ abelian Chern-Simons Matter theories

Basic multiplets/interactions (3d $\mathcal{N} = 2$ multiplets) = (S^1-reduction of 4d $\mathcal{N} = 1$ multiplets)
Chiral multiplet $\Phi = \Phi + \theta\psi + F\theta^2$, Vector multiplet $V = 2i\sigma\theta\bar{\theta} + 2\theta\gamma^\mu\bar{\theta}A_\mu + \theta^2\bar{\theta}^2 D$
Bosonic action : $\mathcal{L} = \mathcal{L}_{maxwell} + \mathcal{L}_{CS} + \mathcal{L}_\Phi$

$$\begin{aligned} \mathcal{L}_{maxwell} &= -\frac{1}{e^2} F_{\mu\nu}F^{\mu\nu} + \frac{1}{e^2} D^2 + (\text{fermions}) \\ \mathcal{L}_{CS} &= \frac{k}{4\pi} (A \wedge dA + 2D\sigma) + (\text{fermions}) \\ \mathcal{L}_\Phi &= \int d^2\theta d^2\bar{\theta} \Phi^\dagger e^{-qV - V^{b.g}} \Phi = -D_\mu \Phi D^\mu \Phi^\dagger - q^2\sigma^2|\Phi|^2 - qD|\Phi|^2 + (\text{fermions}) \ . \end{aligned} \tag{2.1}$$

Supersymmetric deformations

$$
\begin{aligned}
&\text{real mass } m \text{ coupled to a flavor symmetry : } \langle V^{\text{b.g}} \rangle = 2im\theta\bar{\theta} \\
&\text{FI parameter } \zeta \text{ (= real mass for } u(1)_J \text{ topological symmetry) : } \frac{1}{2\pi}\zeta D \\
&\text{Superpotential deformation : } \int d^2\theta\, \mathcal{W} + (c.c) \\
&\quad \text{(gauge invariant chiral primary multiplet } \mathcal{W}\text{)}
\end{aligned}
\tag{2.2}
$$

The superpotential deformations and real mass (or FI parameter) deformations are mutually exclusive if the superpotential deformation breaks flavor symmetries associated to the real masses.

2.1 Witten index $\mathcal{I}_{g=1}$

Witten index of 3d $\mathcal{N}=2$ theories The index is defined to be

$$
\text{Witten index } \mathcal{I}_{g=1} = Z_{\Sigma_{g=1}\times S^1} = \text{Tr}_{\mathcal{H}(\Sigma_{g=0};m,\zeta,\mathcal{W})}(-1)^F e^{-\beta \hat{H}} \tag{2.3}
$$

In the path-integral on $\Sigma_{g=1} \times S^1$, periodic boundary conditions are imposed for boson/fermionic fields along the 3 S^1-directions in $\Sigma_{g=1}\times S^1 = \mathbb{T}^3$. The index gets contributions only from ground states ($\hat{H}|0\rangle = 0$) since there is cancellation between ($[\hat{H}, Q] = 0$)

$$
|E\rangle \;\text{ and }\; \hat{Q}|E\rangle\;, \quad \text{if } \hat{H}|E\rangle = (QQ^\dagger + Q^\dagger Q)|E\rangle \neq 0. \tag{2.4}
$$

So, the index is actually independent on β. The index is well-defined only when the Hilbert-space $\mathcal{H}(\Sigma_{g=0}; m; \zeta, \mathcal{W})$ has a mass gap. Upon a generic choice of supersymmetric deformations $(m, \zeta, \mathcal{W})$, 3d gauge theory has a mass gap. The index is invariant under continuous changes of $(m, \zeta, \mathcal{W})$ unless the spectrum becomes gapless during the change. Since the real mass m is real and there could be massless fields when $m = 0$, one may expect that there could be a wall crossing of the Witten index under the sign change of m. But actually it turns out that the 3d $\mathcal{N}=2$ Witten index does not experience wall-crossing. Upon a compactification along S^1 of radius R, a real scalar σ in the vector multiplet becomes a complex scalar $\Sigma = R(\sigma + ia)$ combined with gauge holonomy $e^{i2\pi Ra} = \exp \oint_{S^1} A$. So the real mass parameter m (vev of σ in a background vector multiplet coupled to a flavor symmetry) also becomes complex variable $m_{\mathbb{C}}$ upon S^1-compactification and one can continously connect $\text{Re}[m_{\mathbb{C}}] > 0$ to $\text{Re}[m_{\mathbb{C}}] < 0$ without crossing the singular point at $m_{\mathbb{C}} = 0$.

Two methods of computing the Witten index, which always give the same answer.

Method I : method used in [3, 4] Basic strategy is

$$
\begin{aligned}
&(\text{a } 3d\ \mathcal{N}=2 \text{ gauge theory}) \xrightarrow{\text{SUSY deformations } (m,\zeta,\mathcal{W})} \\
&\bigoplus_{i\in\{\text{Semiclassical SUSY vacua}\}} (\text{gapped theory described by a TQFT } \mathcal{T}_i) \\
&\text{then,} \\
&\mathcal{I}_{g=1}(\text{a } 3d\ \mathcal{N}=2 \text{ gauge theory}) = \sum_i Z(\text{TQFT } \mathcal{T}_i \text{ on } \Sigma_{g=1}\times S^1)\;.
\end{aligned}
\tag{2.5}
$$

Example : $U(1)_{k\in\mathbb{Z}+\frac{q^2}{2}}$ + (Φ of charge $q\neq 0$) The only supersymmetric deformation of the theory is the FI parameter deformation. The only flavor symmetry is $u(1)$ topological symmetry, usually denoted as $u(1)_J$, associate to the dynamical $U(1)$ gauge field. The FI parameter can be considered as real mass for the $u(1)_J$ symmetry. There is no gauge-invariant chiral primary operator in the theory. The semi-classical effective potential of the theory is

$$V_{eff} = \frac{e_{eff}^2}{32\pi^2}\left(2\pi q|\Phi|^2 - \zeta - k_{eff}\sigma\right)^2 + q^2\sigma^2|\Phi|^2 \,. \tag{2.6}$$

1-loop shift of Chern-Simons level from integrating out massive fermion ψ:

$$k_{eff} = k + \frac{1}{2}q^2\frac{m_\psi}{|m_\psi|} = k + \frac{1}{2}q^2\frac{q\sigma}{|q\sigma|} \,. \tag{2.7}$$

Quantization of k:

$$k_{eff}\in\mathbb{Z}\,, \quad k\in\mathbb{Z}+\frac{1}{2}q^2 \,. \tag{2.8}$$

Semiclassical vacua (we assume $k>\frac{1}{2}q^2$ and $q>0$) :

$$\begin{aligned}
&\text{when } \zeta>0\,, \quad \langle\sigma\rangle = -\frac{\zeta}{k_{eff}}\,, \quad \langle\Phi\rangle = 0 \;\Rightarrow\; \mathcal{T}_1^{\zeta>0} = (\text{pure } u(1)_{k_{eff}=k-\frac{q^2}{2}} \text{ theory}) \\
&\qquad\qquad \langle\sigma\rangle = 0, \quad |\langle\Phi\rangle| = \sqrt{\frac{\zeta}{2\pi q}} \;\Rightarrow\; \mathcal{T}_2^{\zeta>0} = (\text{pure } \mathbb{Z}_q \text{ gauge theory}) \\
&\text{when } \zeta<0\,, \quad \langle\sigma\rangle = -\frac{\zeta}{k_{eff}}\,, \quad \langle\Phi\rangle = 0 \;\Rightarrow\; \mathcal{T}_1^{\zeta<0} = (\text{pure } u(1)_{k_{eff}=k+\frac{q^2}{2}} \text{ theory})
\end{aligned} \tag{2.9}$$

So, the Witten index is (when $k>\frac{1}{2}q^2$ and $q>0$)

$$\begin{aligned}
&\mathcal{I}_{g=1}^{\zeta>0} = Z(\mathcal{T}_1^{\zeta>0} \text{ on } \Sigma_{g=1}\times S^1) + Z(\mathcal{T}_2^{\zeta>0} \text{ on } \Sigma_{g=1}\times S^1) = \left(k-\frac{q^2}{2}\right) + q^2 = k+\frac{q^2}{2}\,, \\
&\mathcal{I}_{g=1}^{\zeta<0} = Z(\mathcal{T}_1^{\zeta<0} \text{ on } \Sigma_{g=1}\times S^1) = k+\frac{q^2}{2}\,.
\end{aligned} \tag{2.10}$$

Here we use the fact that

$$\begin{aligned}
&Z(\text{pure } G \text{ (dicrete) gauge theory on } \Sigma_g\times S^1) \\
&= \dim \mathcal{H}_{\Sigma_g}(\text{pure } G \text{ gauge theory}) \\
&= \sharp(\mathcal{P}(\Sigma_g) \text{ of pure } G \text{ gauge theory}) \quad (\text{a generalization of (1.14) when } \mathcal{P} \text{ is a finite set}) \\
&= \sharp(\text{flat } G \text{ connections on } \Sigma_g) \\
&= \sharp(\text{Hom}[\pi_1(\Sigma_g)\to G]/(\text{conj}))\,. \\
&\Rightarrow \\
&Z(\text{pure } G=\mathbb{Z}_q \text{ gauge theory on } \Sigma_g\times S^1) \\
&= \sharp\{a_1,b_1,\ldots,a_g,b_g\in\mathbb{Z}_q \;:\; [a_1,b_1]\ldots[a_g,b_g]=1\} \\
&= q^{2g}\,.
\end{aligned} \tag{2.11}$$

In the same way, one can check that

$$\mathcal{I}_{g=1}\big(u(1)_k + \Phi(\text{of charge } q)\big) = \begin{cases} |k| + \frac{q^2}{2}, & \text{for } |k| > \frac{q^2}{2} \\ q^2, & \text{for } |k| \le \frac{q^2}{2} \end{cases} . \tag{2.12}$$

Method II : Extremizing twisted superpotential Upon a compactification along S^1 of radius R,

$$\text{3d } \mathcal{N}=2 \text{ vector multiplet } V \longrightarrow \text{2d } \mathcal{N}=(2,2) \text{ vector } V \tag{2.13}$$

In 2d $\mathcal{N}=(2,2)$ theory, there is a following additional supersymmetric deformation term for vector multiplet:

$$\int d\theta_1 d\bar{\theta}_2 \widetilde{\mathcal{W}}(\Sigma) + (c.c) \ni \left(\frac{\partial \widetilde{\mathcal{W}}}{\partial \Sigma} + c.c\right) RD \tag{2.14}$$

Here Σ is a twisted chiral multiplet constructed from V

$$\begin{aligned} &\Sigma = R\bar{D}_1 D_2 V = \Sigma + \ldots + \theta_1\bar{\theta}_2 R(D + iF_{tx}) , \\ &\Sigma := R(\sigma + ia) , \quad \Sigma \sim \Sigma + i \end{aligned} \tag{2.15}$$

$\widetilde{\mathcal{W}}$ is a holomorphic function called twisted superpotential. To compute the effective twisted superpotential, we need to compute the term which contain D.

- Classical Chern-Simons term : $\mathcal{L} \ni 2\pi R \frac{k}{2\pi} D\sigma = \frac{k}{2\pi}(RD)(\Sigma + c.c)$

$$\Rightarrow \partial_\Sigma \widetilde{\mathcal{W}}_{\text{tree}} = \frac{k}{2R}\Sigma \quad \Rightarrow \quad \widetilde{\mathcal{W}}_{\text{tree}} = \frac{k}{4R}\Sigma^2 .$$

- 1-loop from Φ of charge q :

$$\begin{aligned} \mathcal{L} &\ni -qD \sum_{n\in\mathbb{Z}} \int \frac{d^2 k_E}{2\pi} \frac{1}{k_E^2 + R^{-2}(n+qRa)^2 + q^2\sigma^2} \\ &= \frac{qD}{4\pi} \log\left(2\pi q\Sigma \prod_{n\neq 0}(1 + iq\Sigma/n)\right) + (\Sigma \leftrightarrow \bar{\Sigma}) \\ &= \frac{qD}{4\pi}\log(2\sinh q\pi\Sigma) + (\Sigma \leftrightarrow \bar{\Sigma}) \end{aligned} \tag{2.16}$$

$$\Rightarrow \partial_\Sigma \widetilde{\mathcal{W}}_{\text{loop}} = \frac{q}{4\pi R}\log(2\sinh q\pi\Sigma) \quad \Rightarrow \quad \widetilde{\mathcal{W}}_{\text{loop}} = \frac{q^2}{8R}\Sigma^2 + \frac{1}{8\pi^2 R}\text{Li}_2(e^{-2\pi q\Sigma}) .$$

The 1-loop comes from a Feynman diagram in figure 2. After rescaling $Z := 2\pi\Sigma$ ($Z \sim Z + 2\pi i$) and $\widetilde{\mathcal{W}} \to 8\pi^2 R\widetilde{\mathcal{W}}$

$$\widetilde{\mathcal{W}}_{\text{tree}} = \frac{k}{2}Z^2 , \quad \widetilde{\mathcal{W}}_{\text{loop}} = \text{Li}_2(e^{-qZ}) + \frac{q^2}{4}Z^2 . \tag{2.17}$$

Witten index can be computed by counting solutions extremizing the twisted superpotential w.r.t $Z = 2\pi\Sigma = 2\pi R(\sigma + ia)$ in a dynamical vector multiplet V:

$$(\text{Witten index}) = \sharp\left\{ z \;:\; \exp\left(\frac{\partial \widetilde{\mathcal{W}}(Z)}{\partial Z}\right)\Bigg|_{Z\to\log z} = 1\right\} . \tag{2.18}$$

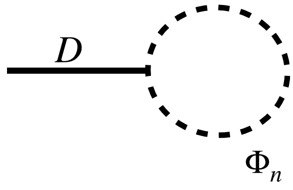

Figure 2. 1-loop contributions to $\widetilde{\mathcal{W}}$ from infinitely many KK-modes $\{\Phi_n\}$, $\Phi(t,x,y) = \frac{1}{\sqrt{2\pi R}}\sum_n \Phi_n(t,x)e^{\frac{iny}{R}}$, of mass $\{m_n^2 = R^{-2}(n+qRa)^2 + q^2\sigma^2\}$

Ex 7. We count solutions in terms of $z := e^Z$ since Z is a $(2\pi i)$-periodic variable due to a large gauge transformation along S^1_R. Then, why do we solve $\exp(\partial_Z\widetilde{\mathcal{W}}) = 1$ instead of $\partial_Z\widetilde{\mathcal{W}} = 0$? (Hint: consider a complexified FI parameter $U := 2\pi\zeta_{\mathbb{C}}$ which is also a $(2\pi i)$-periodic variable. As a side remark, we do not need to extremize the twisted superpotential w.r.t U since it is in a non-dynamical vector multiplet coupled to a flavor $u(1)_J$ symmetry.)

Example : $U(1)_k + (\Phi$ **of charge** $q)$ The twisted superpotential of the theory is

$$\begin{aligned}
&\widetilde{\mathcal{W}} = \mathrm{Li}_2(e^{-qZ}) + \frac{k+\frac{1}{2}q^2}{2}Z^2 \\
&\Rightarrow \exp\left(\frac{\partial\widetilde{\mathcal{W}}(Z)}{\partial Z}\right)\Bigg|_{Z\to\log z} = (1-\frac{1}{z^q})^q z^{k+\frac{1}{2}q^2} \\
&\Rightarrow (\text{Witten index}) = \sharp\left\{z : (z^q-1)^q z^{k+\frac{1}{2}q^2} = z^{q^2}\right\} = \begin{cases} |k| + \frac{q^2}{2}, & \text{for } |k| > \frac{q^2}{2} \\ q^2, & \text{for } |k| \le \frac{q^2}{2} \end{cases} .
\end{aligned} \tag{2.19}$$

2.2 Twisted index $\mathcal{I}_g$ on general Σ_g

Topological twisting on Σ_g 3d $\mathcal{N}=2$ theory has $U(1)$ R-symmetry under which supercharges have charge ± 1. The supercharge is also charged under local Lorentz transformation $U(1)_\Sigma = SO(2)_\Sigma$.

	$U(1)_R$	$U(1)_\Sigma$
$Q_{\epsilon_1,\epsilon_2}$	$\epsilon_1 \in \{+1,-1\}$	$\epsilon_2 \in \{+1,-1\}$

To preserve supersymmetry on Σ_g, we turn on background gauge field coupled to $U(1)_R$ symmetry

$$\text{topological twisting : } A^{b.g}_{U(1)_R} = \omega \text{ (spin-connection on } \Sigma_g) . \tag{2.20}$$

For $g \neq 1$, the topological twisting preserves 2 supercharges ($Q_{+1,-1}$ and $Q_{-1,+1}$) out of 4 Qs. Twisted index $\mathcal{I}_g$ is defined as[1]

$$\text{Twisted index } \mathcal{I}_g = Z_{\Sigma_g\times S^1} = \mathrm{Tr}_{\mathcal{H}(\Sigma_g)}(-1)^R e^{-\beta\hat{H}} \tag{2.21}$$

[1]Here we insert $(-1)^R$ instead of $(-1)^F = (-1)^{2j_3}$. Since Q has quantum number $(-1)^R = -1$, the quantity is also an index which is independent under continuous deformation. Localization computations are developed for both types ($(-1)^R$ or $(-1)^F$) of twisted index and they are almost same except some minor difference. Here, we only review the localization of twisted index with $(-1)^R$.

From a localization [5, 6], the twisted index can be expressed as

$$\mathcal{I}_g = \sum_{z:\exp\left(\frac{\partial \widetilde{\mathcal{W}}(Z)}{\partial Z}\right)\Big|_{Z=\log z}=1} \big(H(z)\big)^{g-1} \,, \qquad H(z) \;:\; \text{Handle gluing operator} \tag{2.22}$$

One way to obtain the handle gluing operator is considering $b^2 \to 0$ limit of S_b^3 partition function [7]. S_b^3 partition function can be computed using following rule ($\hbar := 2\pi i b^2$)

$$\begin{aligned} &\text{a chiral } \Phi \text{ of charge } q \;:\; \psi_\hbar(qZ) e^{\frac{q^2 Z^2}{4\hbar}} \,, \\ &\text{Chern-Simons level } k \;:\; \exp\left(k\frac{Z^2}{2\hbar}\right) \,, \\ &U(1) \text{ gauing } \;:\; \int \frac{dZ}{\sqrt{2\pi\hbar}} \end{aligned} \tag{2.23}$$

Here $\psi_\hbar(Z)$ is a special function called 'quantum dilogarithm' (QDL).

$$\begin{aligned} &\psi_{\hbar=2\pi i b^2}(Z) := \prod_{r=1}^{\infty} \frac{1-q^r e^{-Z}}{1-\tilde{q}^{-r+1} e^{-\tilde{Z}}} \\ &q := e^{2\pi i b^2} \,, \quad \tilde{q} := e^{2\pi i b^{-2}} \,, \quad \tilde{Z} = \frac{1}{b^2} Z \,. \end{aligned} \tag{2.24}$$

In the limit $\hbar \to 0$, the QDL function asymptotically behaves as

$$\log \psi_\hbar(Z) \xrightarrow{\hbar \to 0} \sum_{n=0}^{\infty} \frac{B_n \hbar^{n-1}}{n!} \mathrm{Li}_{2-n}(e^{-Z}) \,. \tag{2.25}$$

Here B_n is the n-th Bernoulli number with $B_1 = 1/2$. Using the localization rule, the S_b^3 ptn of a 3d $\mathcal{N}=2$ $U(1)$ gauge theory can be written as

$$\begin{aligned} &Z_{S_b^3} = \int \frac{dZ}{\sqrt{2\pi\hbar}} \mathcal{I}(Z;\hbar) \,, \\ &\log \mathcal{I}(Z;\hbar) \xrightarrow{\hbar\to 0} \frac{1}{\hbar}\mathcal{W}_0(Z) + \mathcal{W}_1(Z) + \ldots \hbar^{n-1}\mathcal{W}_n(Z) + \ldots \end{aligned} \tag{2.26}$$

Then,

$$\begin{aligned} &\widetilde{\mathcal{W}}(Z) = \mathcal{W}_0(Z) \,, \\ &H(z) = \frac{\partial^2 \mathcal{W}_0(Z)}{\partial Z \partial Z} \exp\big(-2\mathcal{W}_1(Z)\big)\Big|_{Z=\log z} \end{aligned} \tag{2.27}$$

Example : Pure $\mathcal{N}=2$ $u(1)_k$ theory In the case, the squashed 3-sphere ptn is

$$\begin{aligned} &Z_{S_b^3}(\text{pure } u(1)_k) = \int \frac{dZ}{\sqrt{2\pi\hbar}} e^{\frac{kZ^2}{2\hbar}} \\ &\Rightarrow \mathcal{W}_0 = \frac{k}{2} Z^2 \,, \quad \mathcal{W}_{n\geq 1} = 0 \quad \Rightarrow \quad H(z) = k \,. \\ &\Rightarrow \mathcal{I}_g = \sum_{z:z^k=1} k^{g-1} = k^g \end{aligned} \tag{2.28}$$

The theory is actually equal to the pure bosonic $u(1)_k$ theory. The result is compatible with the result in (1.39). In the example, the S_b^3-ptn can be exactly computable

$$\mathcal{F}_b := -\log |Z_{S_b^3}| = \frac{1}{2}\log |k| \ . \tag{2.29}$$

Since the theory is a topological theory, the free-energy is b-independent and the free-energy is compatible with the result in (1.30).

Example : $u(1)_k + (\Phi$ **of charge 1) theory** The S_b^3-ptn is given by

$$Z_{S_b^3}\big(u(1)_k + (\Phi \text{ of charge } 1)\big) = \int \frac{dZ}{\sqrt{2\pi\hbar}} e^{\frac{(k+\frac{1}{2})Z^2 - 2Z\left(U+\nu(i\pi+\frac{\hbar}{2})\right)}{2\hbar}} \psi_\hbar(Z) \ . \tag{2.30}$$

Here, we introduce a real mass parameter (i.e. FI parameter) U associated to the $U(1)_J$ topological symmetry. The FI-term can be thought as mixed CS term between $U(1)_{\text{gauge}}$ and $U(1)_J$. We also introduce a R-symmetry mixing parameter ν. To preserve some supersymmetry on S_b^3 geometry, we need to turn on background gauge field coupled to a $U(1)_R$ symmetry. Choice of the $U(1)_R$-symmetry is not unique but can be mixed with flavor symmetry of the theory, $U(1)_J$ in this example:

$$u(1)_R^\nu = u(1)_R^{\nu=0} + \nu u(1)_J \ . \tag{2.31}$$

The S_b^3-ptn depends on the mixing parameter ν.

$$\nu \in \mathbb{R} \ , \quad \text{in } S_b^3 \text{ ptn computation} \tag{2.32}$$

Similarly, the twisted index $\mathcal{I}_g$ for $g \neq 1$ also depends on the mixing ν since we turn on a magnetic flux on Σ_g coupled to a $U(1)_R$ symmetry in (2.20). Unlike the S_b^3 case, the mixing parameter ν in twisted index $\mathcal{I}_g$ should be properly quantized to satisfy the Dirac quantization:

$$\begin{aligned} &(g-1) \times \big(u(1)_R^\nu \text{ charge of any states } |\psi\rangle \in \mathcal{H}(\Sigma_g)\big) \in \mathbb{Z} \ . \\ &\Rightarrow (g-1)\nu \in \mathbb{Z} \ , \quad \text{in } \mathcal{I}_g \text{ computation} \end{aligned} \tag{2.33}$$

Then,

$$\begin{aligned} &\mathcal{W}_0 = \frac{k+\frac{1}{2}}{2} Z^2 + \text{Li}_2(e^{-Z}) - Z(U + i\pi\nu) \ , \quad \mathcal{W}_1 = \frac{1}{2}\text{Li}_1(e^{-Z}) = -\frac{1}{2}\log(1-e^{-Z}) - \frac{\nu}{2} Z \ . \\ &\Rightarrow H(z) = \frac{(2k+1)z - 2k + 1}{2z^{1-\nu}} \\ &\Rightarrow \mathcal{I}_g(u;\nu) = \sum_{z: z^{k+1/2}(1-1/z) = u(-1)^\nu} \left(\frac{(2k+1)z - 2k + 1}{2z^{1-\nu}}\right)^{g-1} \end{aligned} \tag{2.34}$$

Here $u = e^U$ is the fugacity variable for $U(1)_J$ symmetry:

$$\begin{aligned} &\mathcal{I}_g(u;\nu) := \text{Tr}_{\mathcal{H}(\Sigma_g;\nu)} (-1)^R e^{-\beta \hat{H}} u^{\hat{J}} \ , \quad \text{where} \\ &\mathcal{H}(\Sigma_g;\nu) := \text{Hilbert-space on } \Sigma_g \text{ topologically twisted using } u(1)_R^\nu \text{ R-symmetry} \\ &\qquad \hat{J} \ : \text{charge of } U(1)_J \text{ flavor symmetry} \ . \end{aligned} \tag{2.35}$$

For example,

$$\begin{aligned}
&\mathcal{I}_{g}^{u(1)_{k=1/2}+\Phi}(u;\nu=0)=\mathcal{I}_{g}^{u(1)_{k=1/2}+\Phi}(u;\nu=1)=1\ , \quad \text{for all } g.\\
&\mathcal{I}_{g=0}^{u(1)_{k=3/2}+\Phi}(u;\nu=1)=\mathcal{I}_{g=0}^{u(1)_{k=-3/2}+\Phi}(u;\nu=0)=0\ , \quad \text{for all even } g.\\
&\mathcal{I}_{g=3}^{u(1)_{k=3/2}+\Phi}(u;\nu=1)=\mathcal{I}_{g=3}^{u(1)_{k=-3/2}+\Phi}(u;\nu=0)=2-8u\ ,\\
&\mathcal{I}_{g=5}^{u(1)_{k=3/2}+\Phi}(u;\nu=1)=\mathcal{I}_{g=5}^{u(1)_{k=-3/2}+\Phi}(u;\nu=0)=2-16u+32u^2\ ,
\end{aligned} \tag{2.36}$$

Ex 8. Check that the twisted index $\mathcal{I}_g(u;\nu)$ in (2.34) acutally have index structure, i.e $\mathcal{I}_g(u;\nu)=\sum_a a_n(-1)^{R_n}u^n$ with $a_n\in\mathbb{Z}, R_n\in\nu+\mathbb{Z}$

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