Quantum Loops in SL(2) Chern-Simons Theory on hyperbolic 3-manifold

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Based on

ArXiv : 1510.05011 with N. Kim (KHU), M. Romo (IPMU), M. Yamazaki (IPMU)

Work in progress with P. Agarwal (SNU), M. Romo (Tsinghua U) and S. Lee (SNU)

APCTP, July 2019

$$Z(K, M; \hbar, \tilde{\hbar}) = \int_\Gamma \frac{D\mathscr{A} D\widetilde{\mathscr{A}}}{\textbf{(gauge)}} e^{-\frac{1}{2\hbar}CS[\mathscr{A};M]-\frac{1}{2\tilde{\hbar}}CS[\widetilde{\mathscr{A}};M]}\, \mathbf{tr}\left(P \exp \oint_K \mathscr{A}\right)$$

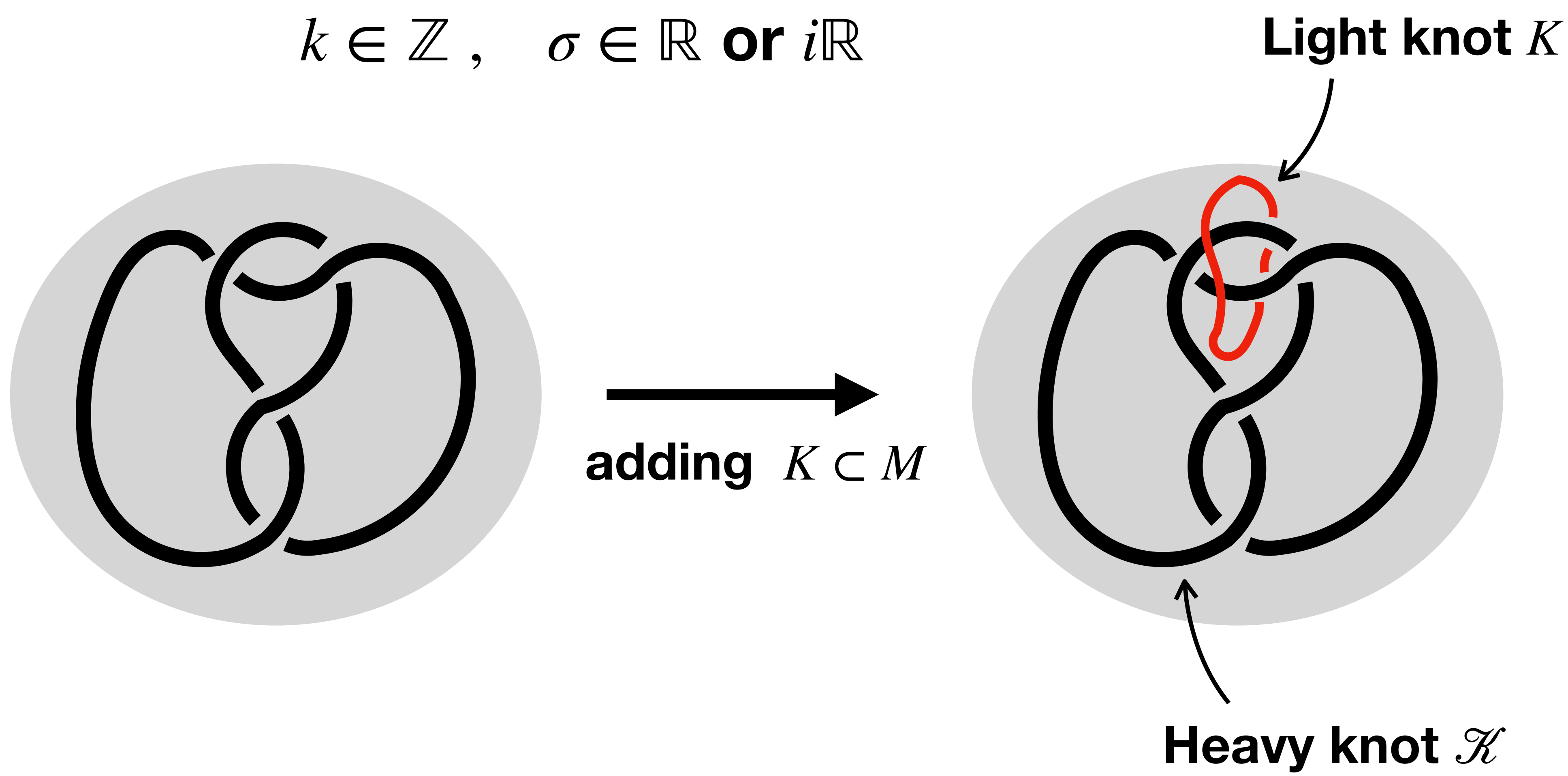
Vev of Wilson loop along a knot $K \subset M$ (hyperbolic 3-manifold)

$$CS[\mathscr{A}; M] := \int_M \mathbf{Tr}\left(\mathscr{A} d\mathscr{A} + \frac{2}{3}\mathscr{A}^3\right) \quad : \quad SL(2,\mathbb{C}) \textbf{ Chern-Simons}$$

$$\hbar = \frac{2\pi i}{k + i\sigma} \; : \; \textbf{holomorphic coupling}, \quad \tilde{\hbar} = \frac{2\pi i}{k - i\sigma} \; : \; \textbf{anti-hol coupling}$$

$$k \in \mathbb{Z}, \quad \sigma \in \mathbb{R} \textbf{ or } i\mathbb{R}$$

$M \; : \; S^3 \backslash \mathscr{K}$



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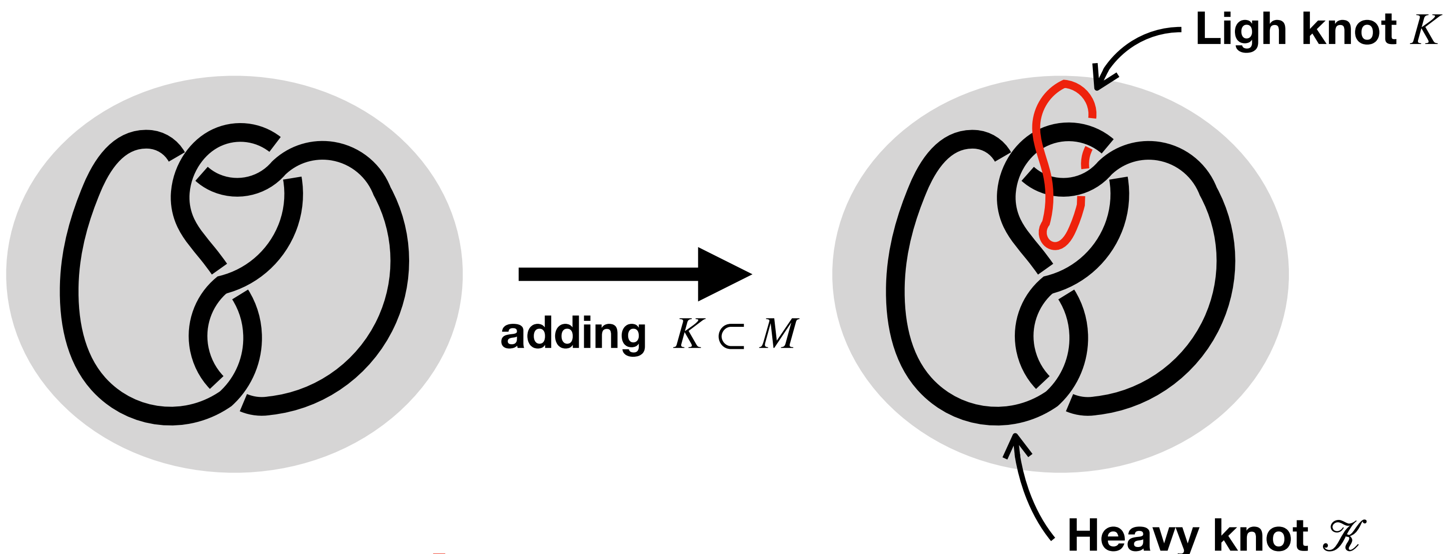

$M \;:\; S^3\backslash\mathscr{K}$

Why? 1. 3d-3d correspondence

[Terashimal, Yamazaki,'11] [Dimofte,Gukov, Gaiotto,'11]

$6d$ A_1 (2,0) **theory on** $\mathbb{R}^{1,2}\times S^3$	$\xrightarrow{S^3\to 0}$	**3d** $\mathcal{N}=2$ $\mathscr{T}[M]$ **on** $\mathbb{R}^{1,2}$
codimension 2 defect along $\mathbb{R}^{1,2}\times\mathscr{K}$		
codimension 4 defect along $\mathbb{R}\;\times\;K$		**A loop operator** L_K **along** $\mathbb{R}$

$$Z(\mathscr{T}[M]+L_K \text{ on } S^3_b) = Z(K,M;\hbar,\tilde{\hbar})\Big|^{\hbar=2\pi i(1+b^2)}_{\tilde{\hbar}=2\pi i(1+b^{-2})}$$

$$Z(\mathscr{T}[M]+L_K \text{ on } S^2\times_q S^1) = Z(K,M;\hbar,\tilde{\hbar})\Big|^{\hbar=\log q}_{\tilde{\hbar}=-\log q}$$

$$Z(K,M;\hbar,\tilde{\hbar}) = \int_\Gamma \frac{D\mathscr{A}D\widetilde{\mathscr{A}}}{(\textbf{gauge})} e^{-\frac{1}{2\hbar}CS[\mathscr{A};M]-\frac{1}{2\tilde{\hbar}}CS[\widetilde{\mathscr{A}};M]} \textbf{tr}(P\exp\oint_K \mathscr{A})$$

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2. Quantum topology

♣. **3D Quantum holonomy cf) Quantum holonomy in 2D** [Bonahon, Wong,'11]

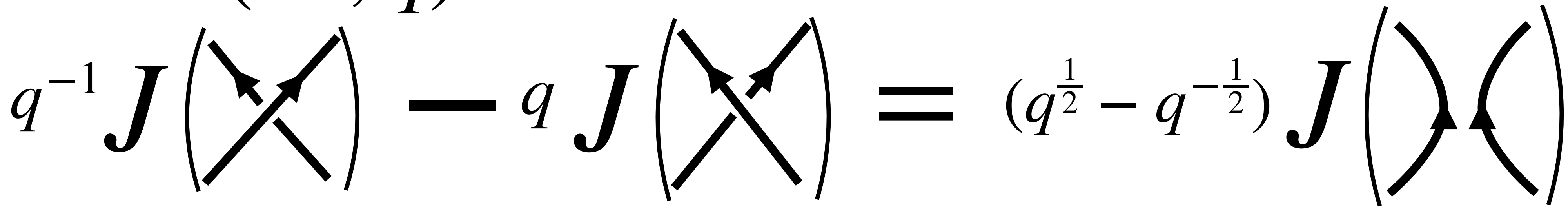
♣. **A generalization of volume conjecture**

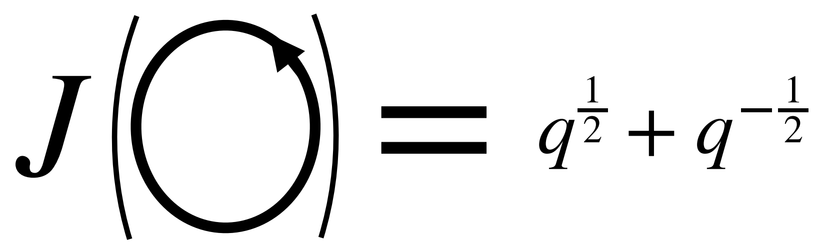
Volume Conjecture

[Kashaev]
[Murakami-Murakami]
[Gukov-Murakami]

Jones Polynomial $J(\mathscr{K};q)$

Skein Relation :





Witten's intepretation

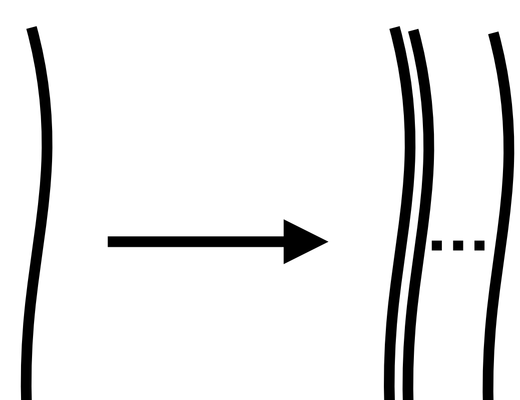
$$J(\mathscr{K};q) = \frac{Z_{su(2)}(\text{fund Wilson loop along } \mathscr{K}, S^3; q = e^{\frac{2\pi i}{k+2}})}{Z_{su(2)}(S^3; q = e^{\frac{2\pi i}{k+2}})}$$

Colored Jones Polynomial $J_n(\mathscr{K};q)$

$$J_R(\mathscr{K};q) = \frac{Z_{su(2)}(\text{Wilson loop in rep R along } \mathscr{K}, S^3; q = e^{\frac{2\pi i}{k+2}})}{Z_{su(2)}(S^3; q = e^{\frac{2\pi i}{k+2}})}$$

$$J_{R=\mathbf{2}^{\otimes n}}(\mathscr{K};q) = J(\underline{\mathscr{K}^n};q)$$

n **cabling of** $\mathscr{K}$



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$$J_R(\mathscr{K};q) \sim Z_{su(2)}(\textbf{Wilson loop in rep R along } \mathscr{K}, S^3; q = e^{\frac{2\pi i}{k+2}})$$

Volume conjecture

$$\lim_{\hat{k},n\to\infty;\frac{n}{\hat{k}}=1} \frac{2\pi}{\hat{k}} \log \left| \frac{J_n(\mathscr{K};q)}{J_n(\bigcirc;q)} \right|_{q=\exp(\frac{2\pi i}{\hat{k}})} = \textbf{vol}(S^3\backslash\mathscr{K})$$

Generalized all-order Volume conjecture

$$\frac{J_n(\mathscr{K};q)}{J_n(\bigcirc;q)}\Big|_{q=\exp(\frac{2\pi i}{\hat{k}})} \simeq Z^{\text{hyp}}_{SL(2,C)}\Big(M = S^3\backslash\mathscr{K}; \hbar = 2\pi i(1+b^2), \tilde{\hbar} = 2\pi i(1+\frac{1}{b^2})\Big)\Big|_{b^2=\hat{k}^{-1}}$$

Im$CS[\mathscr{A}^{\text{hyp}}] = -2\textbf{vol}(S^3\backslash\mathscr{K})$

Perturbative expansion around $\mathscr{A}^{\text{hyp}} = \omega + ie$

$$\simeq \exp\left(-\frac{1}{4\pi i b^2} CS[\mathscr{A}^{\text{hyp}}] + \ldots \right)$$

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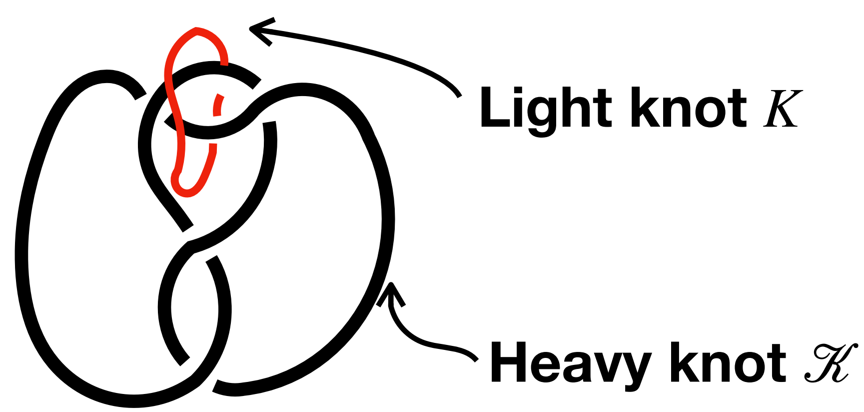
Our generalization

$$\frac{J_{n,\tilde{n}=2}(\mathscr{K}\cup K;q)}{J_n(\bigcirc;q)}\Big|_{q=\exp(\frac{2\pi i}{\hat{k}})} \simeq Z^{\text{hyp}}_{SL(2,C)}\big(K, M = S^3\backslash\mathscr{K}; \hbar = 2\pi i(1+b^2), \tilde{\hbar} = 2\pi i(1+\frac{1}{b^2})\big)\Big|_{b^2=\hat{k}^{-1}}$$



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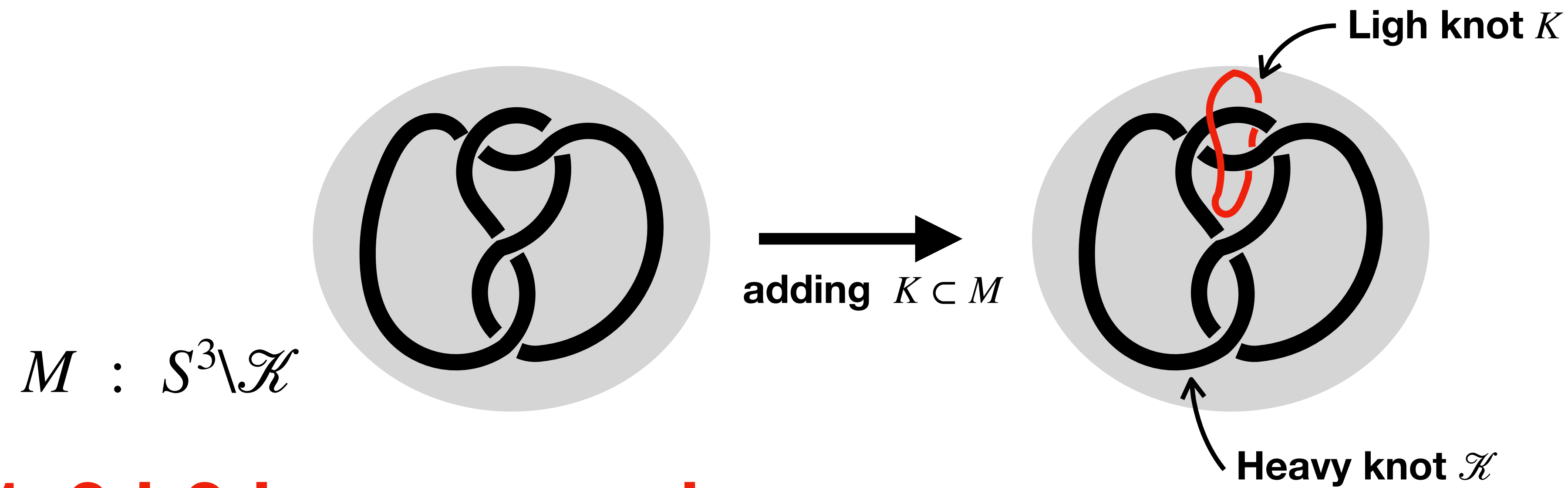

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$\int [dX]_{b.c} e^{-\frac{1}{\hbar}\int_I \dot{X}^2} = \langle X_0, X_1 \vert I \rangle \quad (\hbar \in i\mathbb{R})$ $b.c \ : \ X(t=0) = X_0 \,, \quad X(t=1) = X_1$ **Classically,** $P(\partial I) = \{X_0, P_0, X_1, P_1\}$ $\cup$ $\mathscr{L}(I) = \{X_1 = X_0 + P_0, P_1 = P_0\}$ **Quantization** $: \ \mathscr{H}(\partial I) = \{ \vert X_0, X_1 \rangle \ : \ X_i \in \mathbb{R}\}$ $\vert I \rangle \in \mathscr{H}(\partial I)$ $(\hat{X}_1 - \hat{X}_0 - \hat{P}_0)\vert I \rangle = (\hat{P}_0 - \hat{P}_1)\vert I \rangle = 0$ $\Rightarrow \langle X_0, X_1 \vert I \rangle = \frac{1}{\sqrt{2\pi i \hbar}} \exp\left(-\frac{1}{\hbar}(X_0 - X_1)^2\right)$	$\int \frac{[D\mathscr{A}][D\widetilde{\mathscr{A}}]}{(\textbf{gauge})} e^{-\frac{1}{2\hbar}CS[\mathscr{A}] - \frac{1}{2\tilde{\hbar}}CS[\widetilde{\mathscr{A}}]} = \langle m \vert M \rangle$ $b.c \ : \ P\exp\oint_{\text{merdian}} \mathscr{A} = \begin{pmatrix} e^{m/2} & 0 \\ * & e^{-m/2} \end{pmatrix}$ **Classically,** $P(\partial M) = \{x := e^m, p := e^\ell\}$ $\cup$ $\mathscr{L}(M) = \{A_K(x,p) = 0\}$
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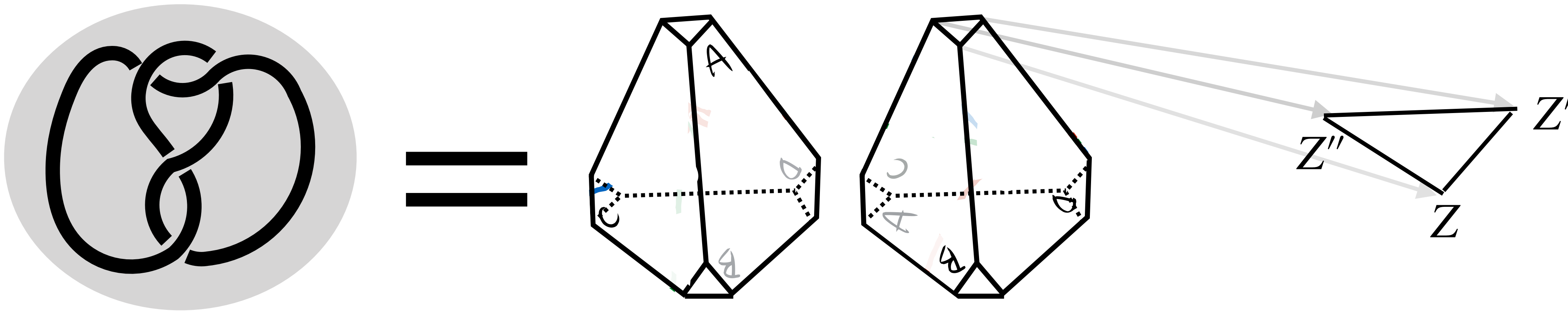
$\mathscr{L}(M) = \mathscr{M}^{\text{irred}}_{\textbf{flat}\ SL(2,\mathbb{C})}(M)$ from an ideal triangulation of M [Thurston]



$M\ :\ S^3 \backslash (\mathscr{K} = \mathbf{4}_1)$

Classical data for single tetrahedron

$$P(\partial \Delta_Z) = \{Z, Z', Z''\ :\ e^{Z+Z'+Z''} = -1\ ,\ Z \sim Z + 2\pi i\ ,\ Z' \sim Z' + 2\pi i\ ,\ Z'' \sim Z'' + 2\pi i\}$$

$$\mathscr{L}(\Delta_Z) = \{e^{-Z} + e^{Z''} - 1 = 0\} \subset P(\partial \Delta_Z)$$

Classical data for $M = S^3 \backslash (\mathscr{K} = \mathbf{4}_1)$ $\quad (z := e^Z\ .,\ldots)$

$$\mathscr{L}(M) = \mathscr{M}_{SL(2,\mathbb{C})} = \{y^{-1} + y'' - 1 = 0\ ,\ z^{-1} + z'' - 1 = 0\ ,\ yy'y'' = zz'z'' = -1,\ \underline{y'y^2z'z^2 = 1}\}\ .$$

Internal edge equation

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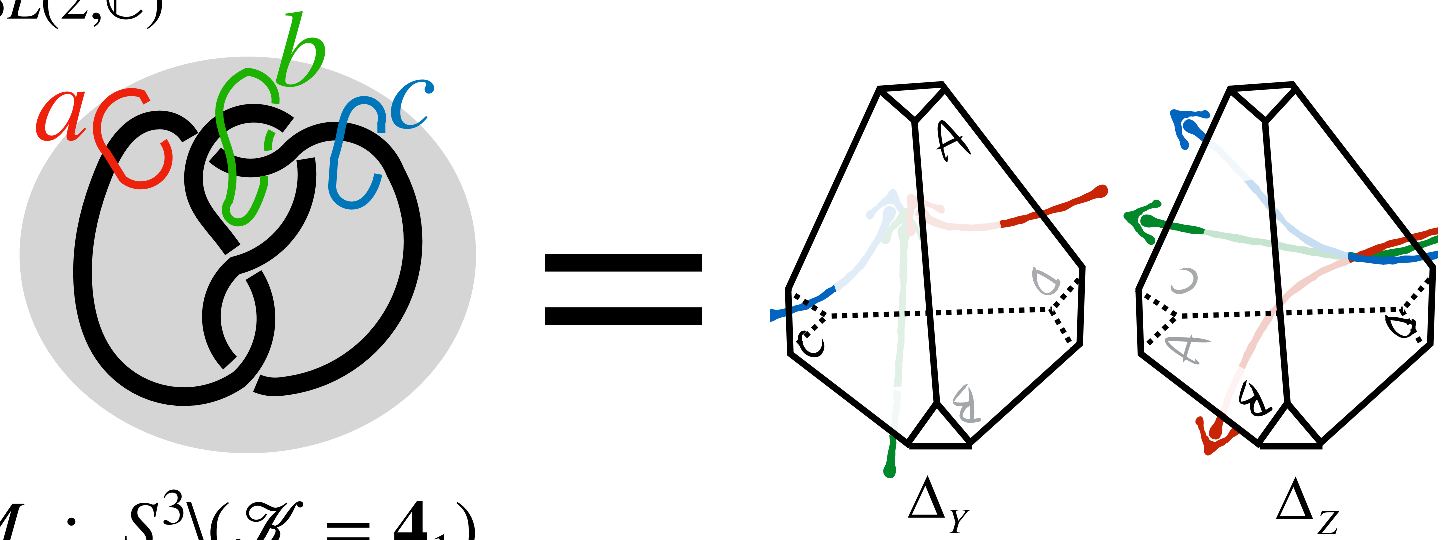

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More explicitly,

$$\rho(a) = \begin{pmatrix} \sqrt{\frac{y''}{z}} & 0 \\ \frac{-1+y''}{\sqrt{y''z}} & \sqrt{\frac{z}{y''}} \end{pmatrix} \quad \rho(b) = \begin{pmatrix} \sqrt{\frac{z'}{y'}} & -\sqrt{\frac{z'}{y'}} \\ \sqrt{y'z'} & -\sqrt{\frac{y'}{z'}}(z'-1) \end{pmatrix} \quad \rho(c) = \begin{pmatrix} \frac{y+z''-1}{\sqrt{yz''}} & \frac{1-y}{\sqrt{yz''}} \\ \frac{z''-1}{\sqrt{yz''}} & \frac{1}{\sqrt{yz''}} \end{pmatrix}\ .$$

$$\pi_1(S^3\backslash\mathbf{4}_1) = \langle a, b, c : ac^{-1}ba^{-1}c = bc^{-1}b^{-1}a = 1\rangle\ .$$

boundary variables : (**meridian**) $\sim a \ \Rightarrow m = Y'' - Z\,,$

(**longitude**) $\sim ac^{-1}bca^{-1}b^{-1} \ \Rightarrow \ell = Z - Z''$

subjected to $A_{\mathbf{4}_1}(x = e^m, p = e^\ell) = p + p^{-1} + \dfrac{1}{x^2} - \dfrac{1}{x} + 2 - x + x^2 = 0$

How to compute $Z(K, M; \hbar, \tilde{\hbar})$?

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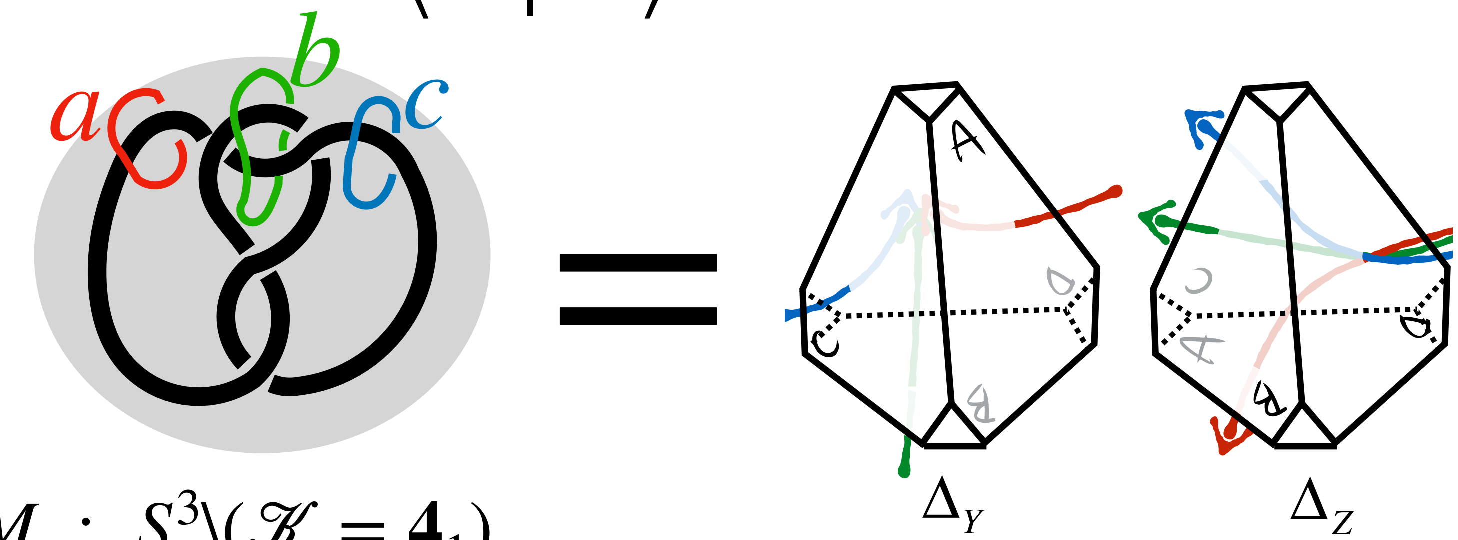
State-integral model $\langle m \,|\, M \rangle$ [Hikami] [Dimofte]



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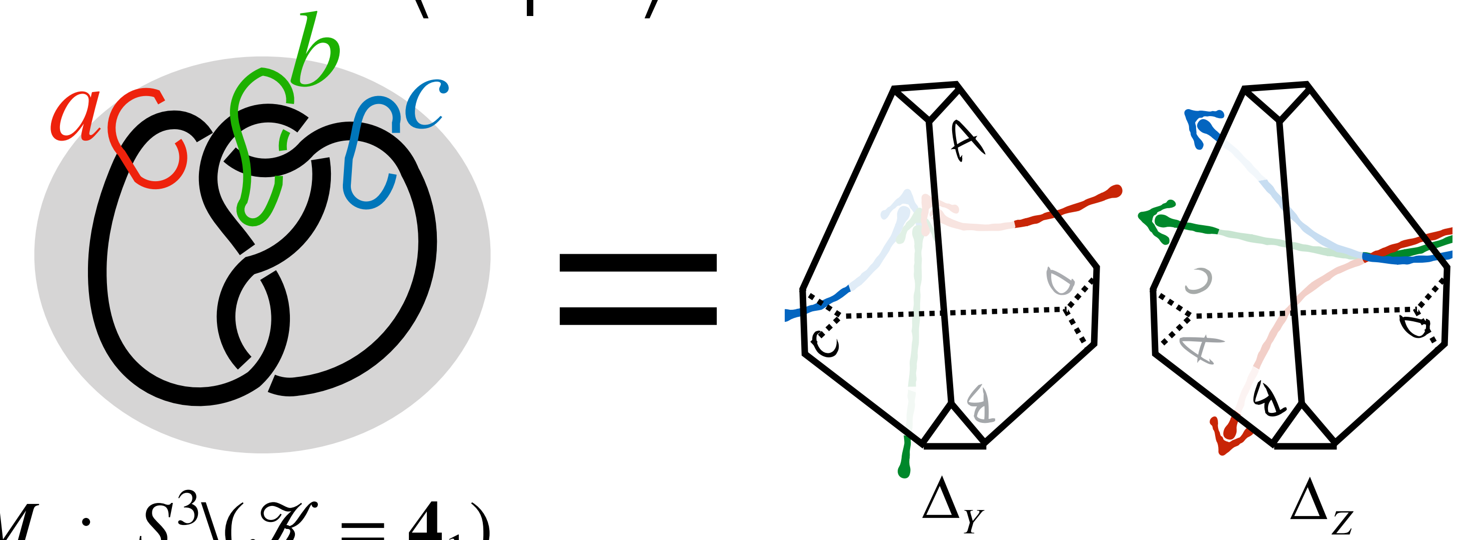

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Quantization for single ideal tetrahedron

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$$\langle m | M \rangle = \langle m | \Delta^{\otimes 2} \rangle = \int \frac{dY dZ}{(2\pi b^2)} \langle m | Y, Z \rangle \langle Y, Z | \Delta^{\otimes 2} \rangle$$

$$= \int \frac{dY dZ}{(2\pi b^2)} \psi_\hbar(Y) \psi_\hbar(Z) \exp \left(\frac{1}{4\pi i b^2} (m^2 - 2mY - 2YZ) \right)$$

$$\left(\langle m | e^{\hat{Y} + \hat{Z} - \hat{Y}'' - \hat{Z}''} = \langle m | \, , \ \langle m | e^{\hat{Y}'' - \hat{Z}} = \langle m | e^{m} \right) \to \langle m | Y, Z \rangle = \exp \left(\frac{1}{4\pi i b^2} (m^2 - 2mY - 2YZ) \right)$$

How to compute $Z(K, M; \hbar, \tilde{\hbar})$?

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<td>$\int \frac{[D\mathscr{A}][D\widetilde{\mathscr{A}}]}{(\textbf{gauge})} e^{-\frac{1}{2\hbar}CS[\mathscr{A}] - \frac{1}{2\tilde{\hbar}}CS[\widetilde{\mathscr{A}}]} = \langle m | M \rangle$
$b.c \ : \ P\exp\oint_{\text{merdian}} \mathscr{A} = \begin{pmatrix} e^{m/2} & 0 \\ * & e^{-m/2} \end{pmatrix}$
Classically, $P(\partial M) = \{x := e^m, p := e^{\ell}\}$
$\cup$
$\mathscr{L}(M) = \{A_K(x,p) = 0\}$
Quantization : $\mathscr{H}(\partial M) = \{ | m \rangle \}$
$| M \rangle \in \mathscr{H}(\partial M)$
$\hat{A}(\hat{x}, \hat{p}; q) | M \rangle = \hat{A}(\hat{\tilde{x}}, \hat{\tilde{p}}, \tilde{q}) | M \rangle = 0$
$\langle m | M \rangle = \int \frac{dYdZ}{(2\pi b^2)} \psi_{\hbar}(Y) \psi_{\hbar}(Z) \exp\left(\frac{1}{4\pi i b^2}(m^2 - 2mY - 2YZ) \right)$</td>
</tr>
<tr>
<td>Inclusion of loop operator
$\int [dX] e^{\frac{i}{\hbar}\int_I \dot{X}^2 + i\int_I \dot{X}} = \langle X_0, X_1 | e^{i(\hat{X}_1 - \hat{X}_0)} | I \rangle$</td>
<td></td>
</tr>
</table>

How to compute $Z(K, M; \hbar, \tilde{\hbar})$?

QM on $I = [0,1]$

$$\int [dX]_{b.c} e^{-\frac{1}{\hbar}\int_I \dot{X}^2} = \langle X_0, X_1 | I \rangle \qquad (\hbar \in i\mathbb{R})$$

$$b.c \;:\; X(t=0) = X_0\,, \quad X(t=1) = X_1$$

Classically, $P(\partial I) = \{X_0, P_0, X_1, P_1\}$

$$\cup$$

$$\mathscr{L}(I) = \{X_1 = X_0 + P_0, P_1 = P_0\}$$

Quantization : $\mathscr{H}(\partial I) = \{ |X_0, X_1\rangle \;:\; X_i \in \mathbb{R}\}$

$$|I\rangle \in \mathscr{H}(\partial I)$$

$$(\hat{X}_1 - \hat{X}_0 - \hat{P}_0)|I\rangle = (\hat{P}_0 - \hat{P}_1)|I\rangle = 0$$

$$\Rightarrow \langle X_0, X_1 | I \rangle = \frac{1}{\sqrt{2\pi i \hbar}} \exp\left(-\frac{1}{\hbar}(X_0 - X_1)^2\right)$$

$SL(2,\mathbb{C})$ Chern-Simons theory on M, $\partial M = \mathbb{T}^2$

$$\int \frac{[D\mathscr{A}][D\widetilde{\mathscr{A}}]}{(\textbf{gauge})} e^{-\frac{1}{2\hbar}CS[\mathscr{A}] - \frac{1}{2\tilde{\hbar}}CS[\widetilde{\mathscr{A}}]} = \langle m | M \rangle$$

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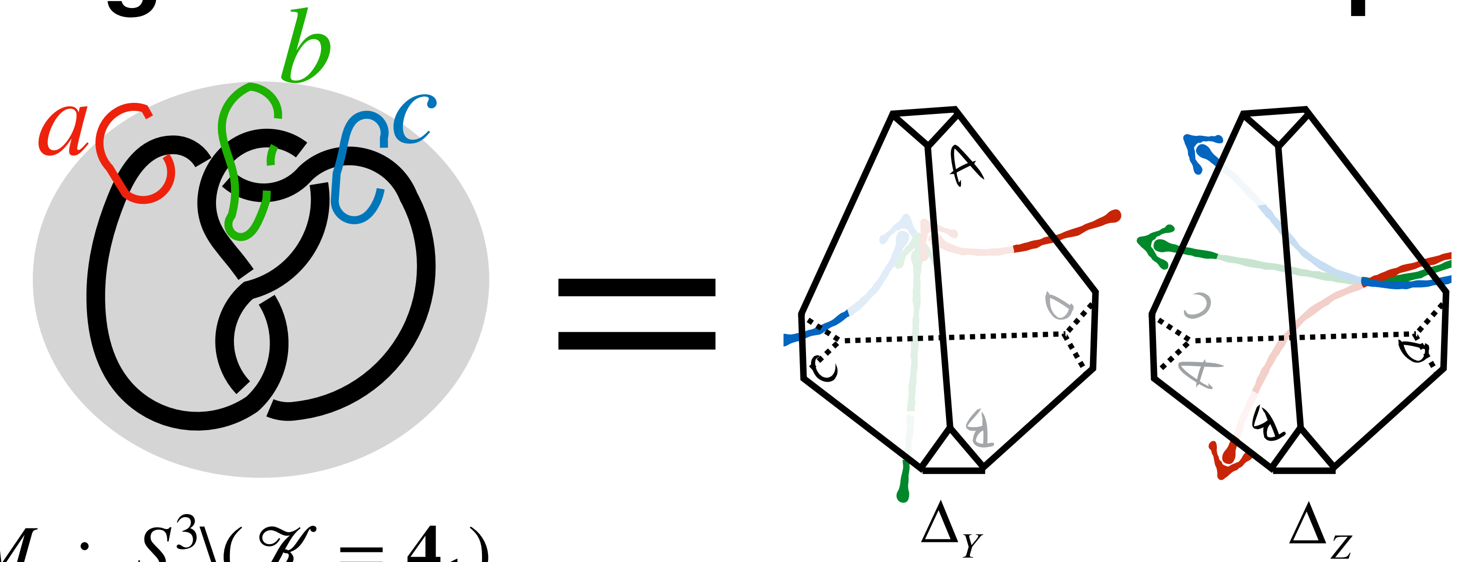
Inclusion of loop operator

State-integral model with Wilson loop



$$M \;:\; S^3\backslash(\mathscr{K} = \mathbf{4}_1)$$

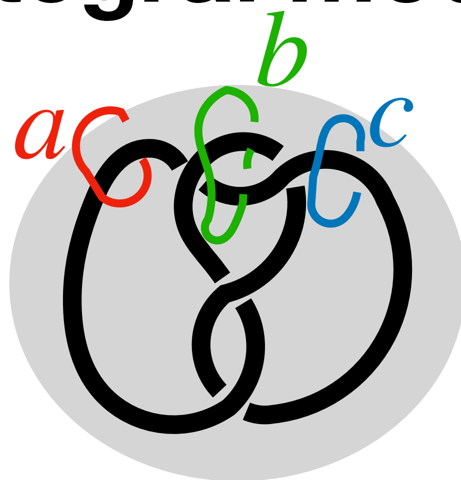
$$\mathscr{L}(M) = \{y^{-1} + y'' - 1 = 0 \;,\; z^{-1} + z'' - 1 = 0 \;,\; yy'y'' = zz'z'' = -1,\;\; y'y^2z'z^2 = 1\}\;.$$

Classically, Wilson loop is given as a function on $\mathscr{L}(M)$

$$\rho(a) = \begin{pmatrix} \sqrt{\frac{y''}{z}} & 0 \\ \frac{-1+y''}{\sqrt{y''z}} & \sqrt{\frac{z}{y''}} \end{pmatrix} \qquad \rho(b) = \begin{pmatrix} \sqrt{\frac{z'}{y'}} & -\sqrt{\frac{z'}{y'}} \\ \sqrt{y'z'} & -\sqrt{\frac{y'}{z'}}(z'-1) \end{pmatrix} \qquad \rho(c) = \begin{pmatrix} \frac{y+z''-1}{\sqrt{yz''}} & \frac{1-y}{\sqrt{yz''}} \\ \frac{z''-1}{\sqrt{yz''}} & \frac{1}{\sqrt{yz''}} \end{pmatrix}\;.$$

$$O_b = \mathbf{Tr}\rho(b) = z''/y + y''/z + 1/(yz)$$

State-integral model with Wilson loop



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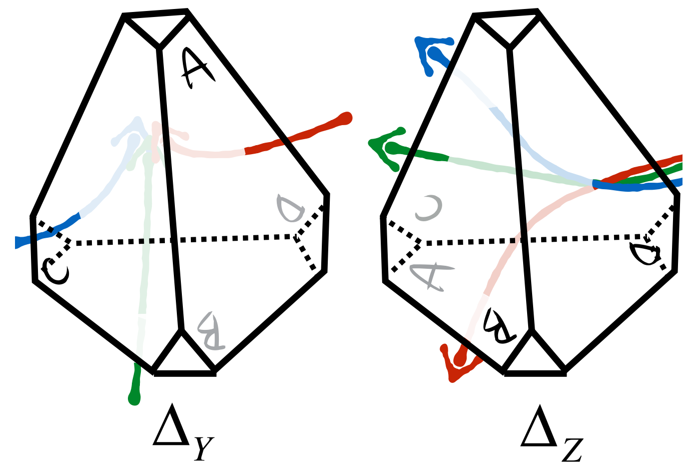

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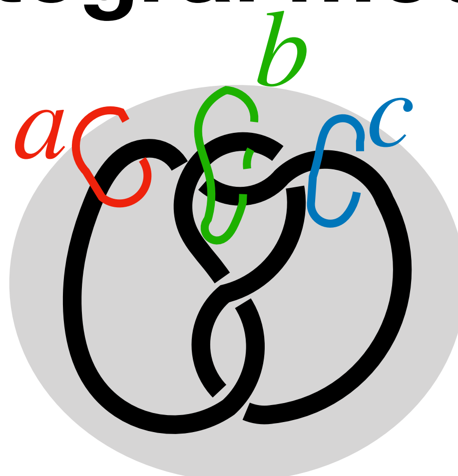
Quantization : $\quad O_K = \mathbf{Tr}(\rho(\gamma_K)) \to \hat{O}_K(\hat{y}, \hat{z}, \hat{y}'', \hat{z}''; q)$

$$Z(K, M = S^3\backslash\mathbf{4_1}; \hbar, \tilde{\hbar}) = \langle m | \hat{O}_K | \Delta^{\otimes 2} \rangle$$

$$\hat{Q}_K|_{q\to 1} \ = \ O_K \ (\textbf{on } \mathscr{L}(M))$$

State-integral model with Wilson loop



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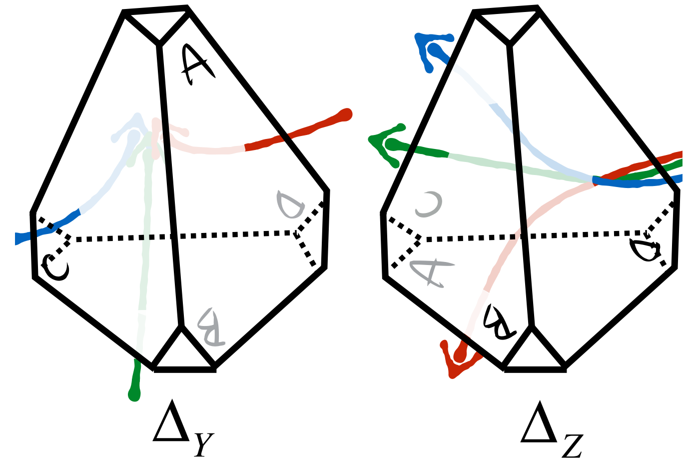

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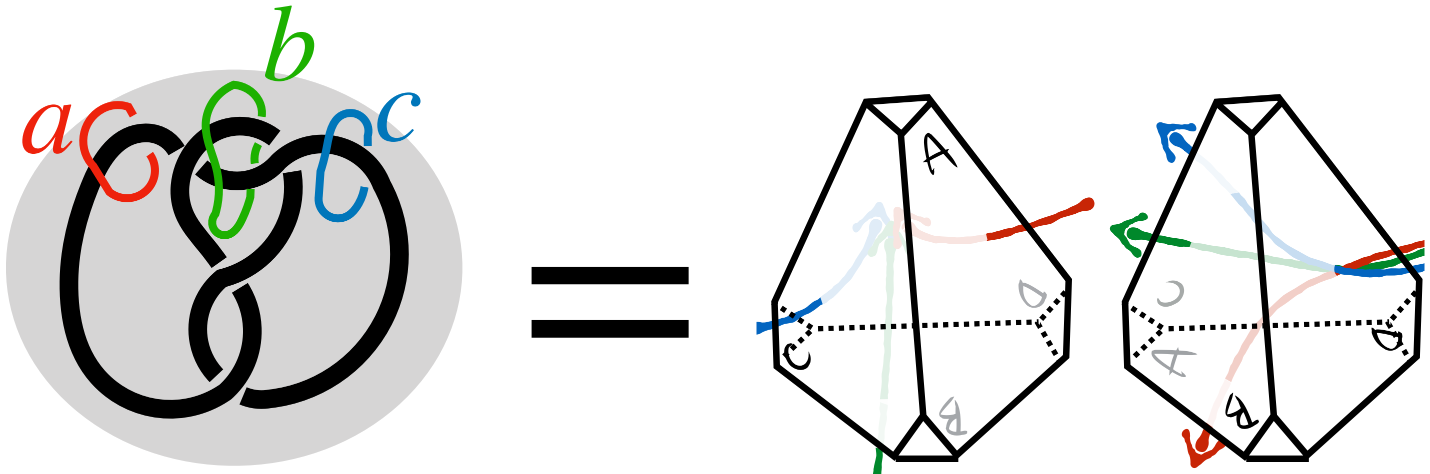
$$O_b = \mathbf{Tr}\rho(b) = z''/y + y''/z + 1/(yz)$$

Quantization : $\quad O_K = \mathbf{Tr}(\rho(\gamma_K)) \to \hat{O}_K(\hat{y}, \hat{z}, \hat{y}'', \hat{z}''; q)$ **"Ordering ambiguity"**

$$Z(K, M = S^3\backslash\mathbf{4_1}; \hbar, \tilde{\hbar}) = \langle m \,|\, \hat{O}_K \,|\, \Delta^{\otimes 2}\rangle$$

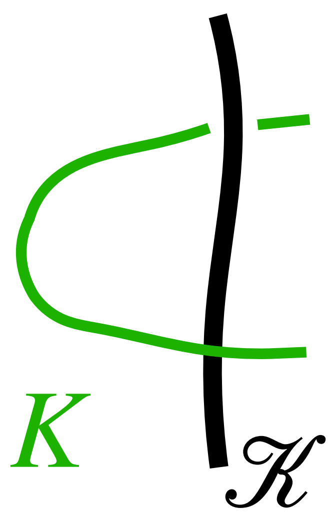
$$\hat{Q}_K|_{q\to 1} \;=\; O_K \;(\textbf{on}\; \mathscr{L}(M))$$

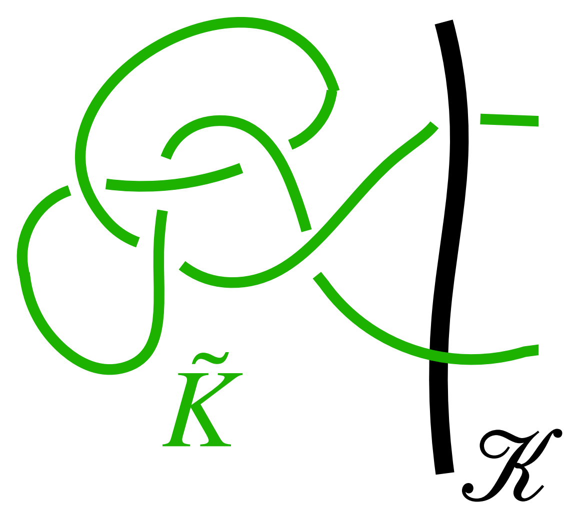
Geometrical meaning of Ordering ambiguity



$M \ : \ S^3 \backslash (\mathscr{K} = \mathbf{4}_1)$ Δ_Y Δ_Z

Tr$\rho(b) = z''/y + y''/z + 1/(yz)$





$\gamma_K = \gamma_{\tilde{K}}$ **(as an element of** $\pi_1(M)$**)**

$K \neq \tilde{K}$ **(as a knot in** $M = S^3 \backslash \mathscr{K}$**)**

Classically $O_K = O_{\tilde{K}} =$ **Tr**$\rho(\gamma_K)$, **but** $\hat{O}_K \neq \hat{O}_{\tilde{K}}$

Therefore, the quantization Tr$\rho(\gamma_K) \to \hat{O}_K$ **is not well-defined procedure.**

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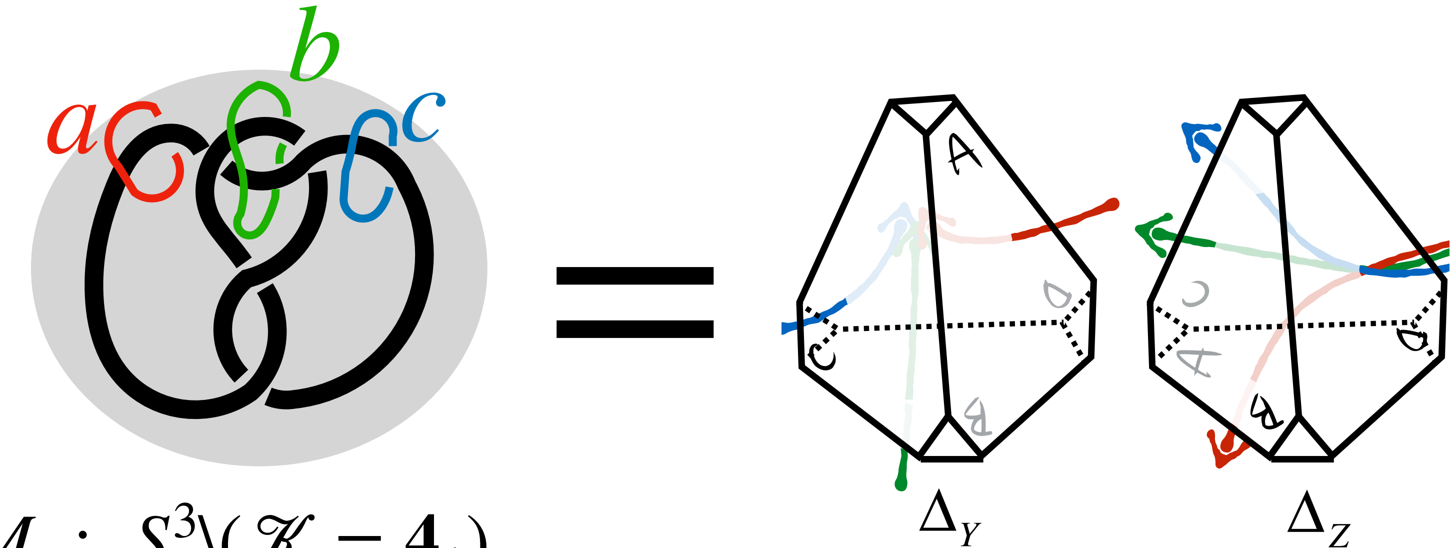

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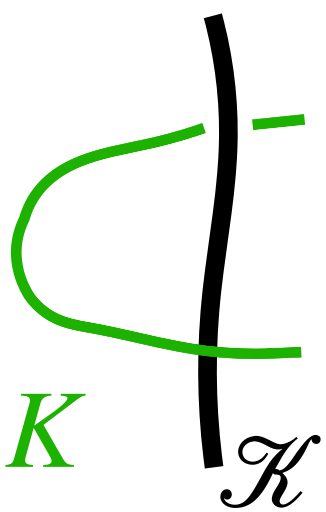

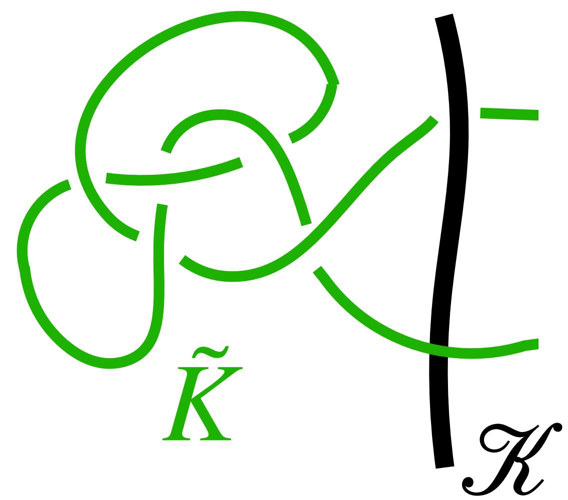

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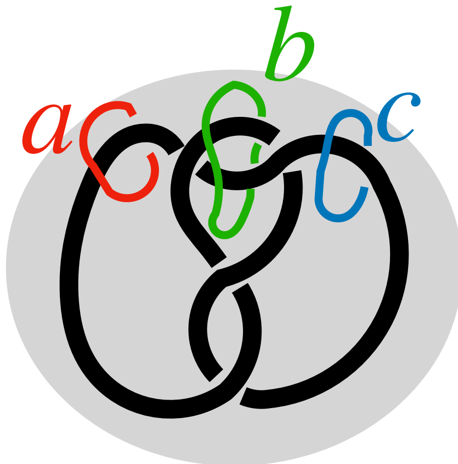
We need additional physical input :

Consistency conditions from 3d-3d correspondence

Consistency conditions from 3d-3d correspondence



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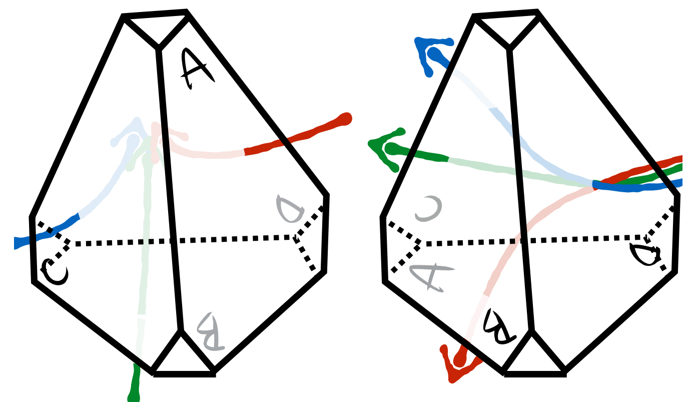


Δ_Y Δ_Z

Tr$\rho(b) = z''/y + y''/z + 1/(yz)$

$6d\ A_1$ (2,0) **theory on** $\mathbb{R}^{1,2} \times S^3$
codimension 2 defect along $\mathbb{R}^{1,2} \times \mathcal{K}$

$\xrightarrow{S^3 \to 0}$ **3d** $\mathcal{N} = 2 \;\; \mathcal{T}[M]$ **on** $\mathbb{R}^{1,2}$

codimension 4 defect along $\mathbb{R} \times K$

A loop operator L_K **along** $\mathbb{R}$

this is only for geodesic knot K

$\gamma \in \pi_1(M) \quad \to \quad K_\gamma^{\text{geod}} \quad \to$ **We fix ordering ambiguity of** $\hat{O}_{K_\gamma^{\text{geod}}}$ **by requiring**

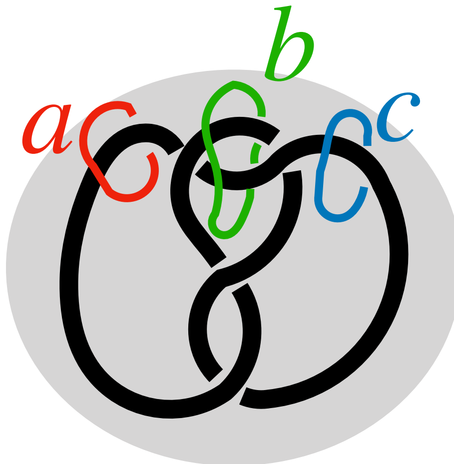
$$Z(\mathcal{T}[M] + L_{K_\gamma^{\text{geod}}} \text{ on } S_b^3) = \langle m \,|\, \hat{O}_{K_\gamma^{\text{geod}}} \,|\, \Delta^{\otimes 2} \rangle \Big|^{\hbar = 2\pi i(1+b^2)}_{\tilde{\hbar} = 2\pi i(1+b^{-2})}$$

Unique geodesic knot in the homopoty class γ

Once we find $\hat{O}_K$ **for a geodesic knot** K_γ^{geod}, **then**

$\hat{O}_K$ **for other knots** K_γ **in the same homotopy can be obatained by Skein relations**

Consistency conditions from 3d-3d correspondence



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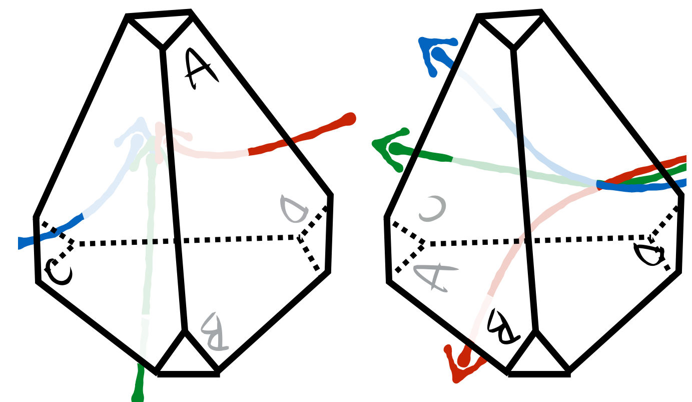


Δ_Y Δ_Z

$M \ : \ S^3 \backslash (\mathscr{K} = \mathbf{4}_1)$

Tr$\rho(b) = z''/y + y''/z + 1/(yz)$

$$\boxed{Z(\mathscr{T}[M] + L_{K_\gamma^{\text{geod}}} \text{ on } S_b^3) = \langle m \,|\, \hat{O}_{K_\gamma^{\text{geod}}} \,|\, \Delta^{\otimes 2} \rangle \Big|^{\hbar = 2\pi i (1+b^2)}_{\tilde{\hbar} = 2\pi i (1+b^{-2})}}$$

Ex) $\mathscr{T}[M] = (U(1)_0$ **coupled two** Φ**s**$)$ **[Dimofte,Gukov, Gaiotto,'11]**

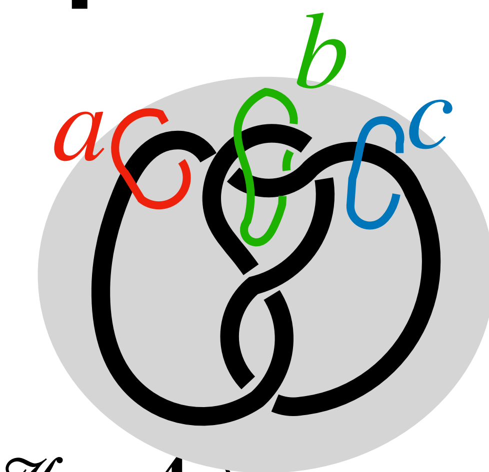
- **flavor symmetry** $: SU(2) \times U(1) \to SU(3)$ **[DG,Yonekura,'18]**
- $\dfrac{\mathscr{T}[M]}{SO(3)_P} \xrightarrow{IR}$ (**gapped and topological theory**) **for** $|P| < 4$

$$\Rightarrow Z\left(\frac{\mathscr{T}[M] + L_{K_\gamma^{\text{geod}}}}{SO(3)_{P;|P|<4}} \text{ on } S_b^3 \right) = \int \frac{dm}{\sqrt{2\pi b^2}} e^{\frac{P}{4\pi i b^2} m^2} \sinh(m) \sinh(\frac{m}{b^2}) \langle m \,|\, \hat{O}_{K_\gamma^{\text{geom}}} \,|\, M \rangle$$

$= (b -$ **independent modulo** $e^{i\mathbb{Q}\pi(b^2+b^{-2})+i\pi\mathbb{Q}})$

$$\Rightarrow \boxed{\hat{O}_{K_b^{\text{geod}}} = q^{\Delta} \left(\hat{z}'' \hat{y}^{-1} + \hat{y}'' \hat{z}^{-1} + \hat{y}^{-1} \hat{z}^{-1} \right)}, \quad \Delta \in \mathbb{Q} \, .$$

Test of quantization using a volume conjucture



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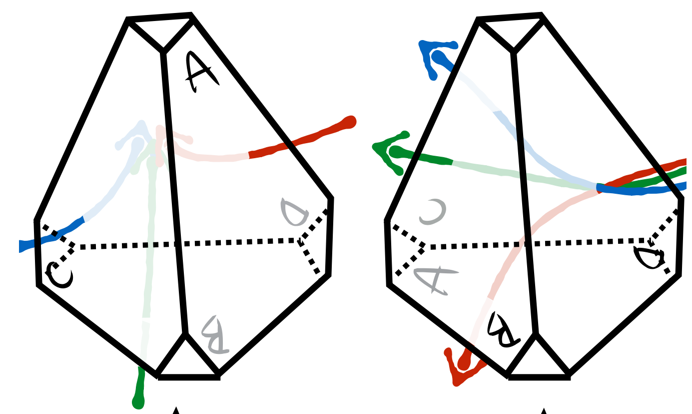

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$$Z(K_b^{\rm geod}, M; \hbar, \tilde{\hbar})\Big|^{\hbar=2\pi i(1+b^2)}_{\tilde{\hbar}=2\pi i(1+b^{-2})} = \int \frac{dYdZ}{(2\pi b)^2} \exp\left(\frac{1}{4\pi i b^2}(m^2 - 2mY - 2YZ)\right)\psi_\hbar(Y)\psi_\hbar(Z) \times e^{\pi i b^2}\left(e^{-Y}(1-e^{-Z}) + e^{-Z}(1-e^{-Y}) + e^{-Y-Z}\right)$$

Saddle point : $\mathscr{A}^{\rm hyp} = \omega + ie \quad \Leftrightarrow \quad m = 0 \, , (Y, Z)^{\rm hyp} = (-i\pi/3, -i\pi/3)$

$$\log \frac{Z(K_b^{\rm geod}, M; \hbar, \tilde{\hbar})}{Z(M; \hbar, \tilde{\hbar})} \xrightarrow{b^2 \to 0} \log(2 + e^{2i\pi/3}) + \frac{1}{18}(6 + i\sqrt{3})(2\pi i b^2) + \frac{i}{18\sqrt{3}}(2\pi i b^2)^2 + o(b^6) \ .$$

numerically check that the expansion is equal to the asymptotic expansion of

$$\log \frac{J_{n,\tilde{n}=2}(\mathscr{K} \cup K_b^{\rm geod}; q)}{J_n(\mathscr{K}; q)}\Big|_{q=\exp(\frac{2\pi i}{\hat{k}}) \, , \ \hat{k}=n=1/b^2 \to \infty}$$

Conclusion

We compute

$$Z(K, M = S^3\backslash\mathscr{K}; \hbar, \tilde{\hbar}) = \int_\Gamma \frac{D\mathscr{A}D\widetilde{\mathscr{A}}}{(\textbf{gauge})} e^{-\frac{1}{2\hbar}CS[\mathscr{A};M]-\frac{1}{2\tilde{\hbar}}CS[\widetilde{\mathscr{A}};M]}\, \textbf{tr}(P\exp\oint_K \mathscr{A})$$

Using a state-integral model

$$S^3\backslash\mathscr{K} = \left(\bigcup_{i=1}^{k} \Delta_i\right)/\sim$$

$$Z(K, M = S^3\backslash\mathscr{K}; \hbar, \tilde{\hbar}) = \langle m\,|\,\hat{O}_K(\hat{z}_i, \hat{z}_i''; q)\,|\,\Delta^{\otimes k}\rangle$$

We fix ordering ambiguity by requring (for geodesic knot K^{geod})

$$Z(\mathscr{T}[M] + L_{K_\gamma^{\text{geod}}} \textbf{ on } S_b^3) = \langle m\,|\,\hat{O}_{K_\gamma^{\text{geod}}}\,|\,\Delta^{\otimes 2}\rangle\Big|^{\hbar=2\pi i(1+b^2)}_{\tilde{\hbar}=2\pi i(1+b^{-2})}$$

(quantization of geodesic knot) + (**Skein relation**) → **Fix for all** K

We test the quantization using a newly proposed generalized VC

$$\frac{J_{n,\tilde{n}=2}(\mathscr{K}\cup K; q)}{J_n(\mathbf{0}; q)}\Big|_{q=\exp(\frac{2\pi i}{\hat{k}})} \simeq Z^{\text{hyp}}_{SL(2,C)}\big(K, M = S^3\backslash\mathscr{K}; \hbar = 2\pi i(1+b^2), \tilde{\hbar} = 2\pi i(1+\frac{1}{b^2})\big)\Big|_{b^2=\hat{k}^{-1}}$$

Thank you !!