# Witten Index Revisited 

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## Plan:

This talk will highlight two recent applications of the Witten index and the elliptic genus.

- Elliptic genus of 2d $(0,2)$ SCFTs and chiral algebras.
- Witten Index of pure $\mathcal{N}=4$ supersymmetric quantum mechanics.


## Supersymmetric Localization

The common theme is to utilize and extend modern results in supersymmetric localization.

The elliptic genus of 2d gauged linear sigma models was derived in

- Benini, R. E., Hori, Tachikawa [arXiv:1305.0533, 1308.4896]
- Gadde, Gukov [arXiv:1305.0266]


## Applications



M-strings [arXiv:1305.6322]

Early applications from 2014 include

- M-Strings [Haghighat-Iqbal-Kozcaz-Lockhart-Vafa]
- Surface operators [Gadde-Gukov]
- Mathieu moonshine [Harrison-Kachru-Paquette]
- Non-compact Kahler manifolds [Ashok-Doroud-Troost]
- $S L(2, \mathbb{R}) / U(1)$ Cigar coset [Murthy]


## The Witten Index

Building upon the, several groups considered the computation of the Witten index of gauged supersymmetric quantum mechanics in 2014.

- C. Hwang, J. Kim, S. Kim, and J. Park
- Cordova, Shao
- Hori, Kim, and Yi


## So What?

Considering M-theory compactified on $S^{1}$ leads to the prediction that $N$ D0-branes form a unique $L^{2}$ normalizable bound state.

In particular, the Witten index of $\mathcal{N}=16$, supersymmetric quantum mechanics with gauge group $S U(N)$ is 1 .

Intensive work went into studying the Witten index of $\mathcal{N}=4,8$, and 16 supersymmetric quantum mechanics.

Can modern methods tell us something new about these old problems?

## What is the Witten index?



$$
\chi=0
$$

$$
\chi=-2
$$

$$
\chi=-4
$$

The elliptic genus is a topological invariant that generalizes the Euler characteristic.

Theorem (Gauss-Bonet)

$$
\chi(\Sigma)=\int_{\Sigma} K d A
$$

## Higher dimensions

Theorem (Chern-Gauss-Bonnet)

$$
\begin{aligned}
& \chi(X)=\int_{X} c_{N}(\mathcal{T} \mathcal{X}) \\
= & \frac{1}{(2 \pi)^{N}} \int_{X} \operatorname{Pf} \Omega
\end{aligned}
$$

## Elliptic Genus - Definition

The elliptic genus of a two-dimensional $\mathcal{N}=(2,2)$ theory is
Definition (Elliptic Genus)

$$
Z_{T^{2}}(\tau, z, u)=\operatorname{Tr}_{R R}(-1)^{F} q^{H_{L}} \bar{q}^{H_{R}} y^{J} \prod_{a} x_{a}^{K_{a}}
$$

- Trace taken in the RR sector.
- Flavor symmetry group $K$ (with Cartan generators $K_{a}$ )
- A left-moving $U(1)$ R-symmetry J
- Fermions have periodic boundary conditions.
- $F$ is the fermion number.


## Elliptic Genus - Definition

## Definition (Elliptic Genus)

$$
Z_{T^{2}}(\tau, z, u)=\operatorname{Tr}_{\operatorname{RR}}(-1)^{F} q^{H_{L}} \bar{q}^{H_{R}} y^{J} \prod_{a} x_{a}^{K_{a}}
$$

- $q=e^{2 \pi i \tau}$ is the complex structure of a torus.
- $\tau=\tau_{1}+i \tau_{2}$

Left- and right-moving Hamiltonians $H_{L}$ and $H_{R}$ are $2 H_{L}=H+i P, 2 H_{R}=H-i P$ in Euclidean signature.
Since $q^{H_{L}} \bar{q}^{H_{R}}=\exp \left(-2 \pi \tau_{2} H-2 \pi \tau_{1} P\right)$,
The trace can be represented by a path integral on a torus of complex structure $\tau$.
For a superconformal theory, the operators $H_{L}, H_{R}, J$ equal the zero-mode generators $L_{0}, \bar{L}_{0}, J_{0}$ of the superconformal algebra.

## Elliptic Genus - Definition

Definition (Elliptic Genus)

$$
Z_{T^{2}}(\tau, z, u)=\operatorname{Tr}_{\mathrm{RR}}(-1)^{F} q^{H_{L}} \bar{q}^{H_{R}} y^{J} \prod_{a} x_{a}^{K_{a}}
$$

We also define

$$
y=e^{2 \pi i z} \quad \text { and } \quad x_{a}=e^{2 \pi i u_{a}}
$$

which specify the background R-symmetry and flavor-symmetry gauge fields $A^{\mathrm{R}}, A^{\text {flavor }}$ via
$z=\oint A_{t}^{\mathrm{R}} \mathrm{d} t-\tau \oint A_{s}^{\mathrm{R}} \mathrm{d} s, \quad u_{a}=\oint A_{t}^{a-\text { th flavor }} \mathrm{d} t-\tau \oint A_{s}^{a \text {-th flavor }} \mathrm{d} s$,
where $t, s$ are the temporal and spatial directions.

## Properties of the elliptic genus

- Reduces to the Witten index

$$
Z_{T^{2}}(\tau, z=0)=\operatorname{Tr}(-1)^{F}
$$

- Under the $\operatorname{SL}(2, \mathbb{Z})$ transformation $\tau \rightarrow \tau+1$

$$
Z_{T^{2}}(\tau, z)=Z_{T^{2}}(\tau, z+1)
$$

- Under $\tau \rightarrow-1 / \tau$

$$
Z_{T^{2}}(-1 / \tau, z / \tau)=e^{2 \pi i(\hat{c} / 2)\left(z^{2} / \tau\right)} Z_{T^{2}}(\tau, z)
$$

## The CY/LG Correspondence

## Calabi-Yau/Landau Ginzburg Correspondence



## The Quintic



The quintic Calabi-Yau is described by the IR limit of a $U(1)$ gauge theory. The theory has one chiral multiplet $P$ of charge -5 and five chiral multiplets $X_{i}$ of charge 1, and super potential $W=\operatorname{Pf}\left(X_{1}, \ldots X_{5}\right)$ where $f$ is a quintic polynomial.

## The Quintic - Landau Ginzburg Phase

The 25 simple poles precisely match the twisted sector contributions to the elliptic genus of the the Landau-Ginburg phase.

$$
\begin{aligned}
Z_{T^{2}} & =\frac{i \eta(q)^{3}}{\theta_{1}\left(q, y^{-1}\right)} \sum_{k, l=0}^{4} \oint_{u=(z+k+l \tau) / 5} \frac{\mathrm{~d} u \frac{\theta_{1}\left(q, x^{-5}\right)}{\theta_{1}\left(q, y x^{-5}\right)}\left(\frac{\theta_{1}\left(q, y^{-1} x\right)}{\theta_{1}(q, x)}\right)^{5}}{} \\
& =\frac{1}{5} \sum_{k, l=0}^{4} y^{-1}\left(\frac{\theta_{1}\left(q, y^{-4 / 5} e^{2 \pi i(k+l \tau) / 5}\right)}{\theta_{1}\left(q, y^{1 / 5} e^{2 \pi i(k+l \tau) / 5}\right)}\right)^{5}
\end{aligned}
$$

## Hidden exceptional symmetry in the pure spinor superstring



Guglielmo Lockhart


Eric Sharpe
Based on [arXiv:1902.09504]

## A simple $\mathcal{N}=(0,2)$ duality

Landau-Ginzburg Theory
$6(0,2)$ chiral multiplets $\phi_{i}$
1 Fermi multiplet $\Psi$
$J$-term $J=\Psi \operatorname{Pf}\left(\phi_{i}\right)$

## Gauge theory

$G=S U(2)$
4 chiral multiplets in the fundamental representation

- Putrov, Song, Yan [arXiv:1505.07110]
- Dedushenko, Gukov [arXiv:1712.07659]


## Tests of the duality

- At low energies, the both theories are expected to flow to sigma models with targets complex cones on $\operatorname{Gr}(2,4)$.
- There is a so(8) $)_{-2}$ chiral algebra on both sides
- The NS elliptic genus matches

$$
Z_{\mathfrak{o}_{4}}\left(t, \mathbf{m}^{\mathfrak{a}_{3}}, \tau\right)=\frac{i \eta(\tau)^{5} \theta_{1}(2 \sigma, \tau)}{\prod_{w \in \mathbf{6}} \theta_{1}\left(\sigma+\left(\mathbf{m}^{\mathfrak{a}_{3}}, w\right), \tau\right)}
$$

## Chiral Algebras

We can decompose the elliptic genus in terms of characters of the chiral algebra so(8)-2

$$
Z_{\mathfrak{o}_{4}}\left(t, \mathbf{m}^{\mathfrak{a}_{3}}, \tau\right)=\widehat{\chi}_{0}^{\left.\widehat{\mathfrak{O}}_{4}\right)_{-2}}\left(\mathbf{m}^{\mathfrak{\delta}_{4}}, \tau\right)-\widehat{\chi}_{-2 \omega_{4}}^{\left.\widehat{\mathfrak{D}}_{4}\right)_{-2}}\left(\mathbf{m}^{\mathfrak{\delta}_{4}}, \tau\right) .
$$

A simple "letter counting" approach to the index only sees one of the characters:

$$
-\widehat{\chi}_{-2 \omega_{4}}^{\left(\hat{\delta}_{4}\right)-2}\left(\mathbf{m}^{\delta_{4}}, \tau\right)
$$

However, operators counted by the other character have yet to be constructed!

## Generalizations

We will study a family of examples arising from ( 0,2 ) sigma models over complex cones. The elliptic genus is the partition function of the holomorphically twisted theory - the curved $\beta \gamma$ system.

We are still lacking a GLSM description of the other members of the family.

## $\beta \gamma$ systems on complex cones

Curved $\beta \gamma$ systems are two-dimensional sigma models of holomorphic maps $\gamma: \Sigma \rightarrow \widehat{X}$ with action

$$
S=\frac{1}{2 \pi} \int_{\Sigma} \beta_{i} \bar{\partial} \gamma^{i}
$$

- $\gamma_{i}, i=1, \ldots, \operatorname{dim} \widehat{X}$ serve as local coordinates
- $\beta_{i}$ are $(1,0)$-forms
$-$

$$
\gamma^{i}(z) \beta_{j}(w) \sim \delta_{j}^{i} \frac{d w}{z-w}
$$

We consider target spaces $\widehat{X}_{\mathfrak{g}}$ with enlarged symmetry

$$
\mathfrak{g}=\mathfrak{d}_{4}, \mathfrak{e}_{6}, \text { and } \mathfrak{e}_{7} .
$$

These varieties can be described as cones over:

- Grassmannian $\operatorname{Gr}(2,4)$
- Orthogonal Grassmannian $\mathrm{OG}^{+}(5,10)$
- Cayley plane $\mathbb{O P}^{2}$


## Main Result:

The partition function is

$$
Z=\widehat{\chi}_{0}^{\left(\widehat{\sigma}_{6}\right)-3}-\widehat{\chi}_{-3 \omega_{1}}^{\left(\widehat{\mathfrak{r}}_{6}\right)_{1}} .
$$

This is the partition function for the $\beta \gamma$ system describing the ghost sector of the pure spinor superstring. It reproduces the first five levels that were computed by Aisaka, Arroyo, Berkovits, and Nekrasov.

The spaces $\widehat{X}_{\mathfrak{g}}$ are Lagrangian submanifolds of the moduli spaces $\mathcal{M}_{\mathfrak{g}, 1}$ of a single centered $\mathfrak{g}$-instanton. These moduli spaces are in turn the Higgs branches of a family of $4 d \mathcal{N}=2$ superconformal field theories (SCFTs) $\mathcal{T}_{\mathfrak{g}}$, whose associated chiral algebra is the vacuum module of $\widehat{\mathfrak{g}}_{k}$ affine algebra, where $k=-2,-3,-4$ respectively for $\mathfrak{g}=\mathfrak{d}_{4}, \mathfrak{e}_{6}$, and $\mathfrak{e}_{7}$.

## Pure spinors

The pure spinor superstring is another story for another time. However I'll give a few teasers:

The lowest energy component of the non-vacuum term is the Hilbert series of the Wallach representation of the $\mathfrak{e}_{6}$ finite-dimensional Lie algebra corresponding to highest weight $-3 \omega_{1}$. Up to a constant, it is:

$$
\sum_{\ell=0}^{\infty} t^{\ell} \chi_{v_{\ell \omega_{5}}}^{\mathcal{O}_{5}}\left(\mathbf{m}^{\mathcal{D}_{5}}\right)
$$

## 10d Supersymmetric Yang-Mills

The infinite sum has a closed form:

$$
\frac{1}{(1-t)^{16}}\left(1-10 t^{2}+16 t^{3}-16 t^{5}+10 t^{6}-t^{8}\right)
$$

The fields in the Batalin-Vilkovisky complex organize into the $\mathbf{2 7}$ and $\overline{\mathbf{2 7}}$ of $\mathfrak{e}_{6}$

Fields of ten-dimensional super Yang-Mills, with Spin(10) indices

| BV: | 1 | 0 | -1 | -2 |
| :---: | :---: | :---: | :---: | :---: |
| $\theta^{0}$ | $c(\mathbf{1})$ |  |  |  |
| $\theta^{1}$ |  | $A_{\mu}(\mathbf{1 0})$ |  |  |
| $\theta^{2}$ |  | $\lambda_{\alpha}\left(S_{+}=\mathbf{1 6}\right)$ |  |  |
| $\theta^{3}$ |  |  | $\lambda_{\alpha}^{+}\left(S_{-}=\overline{\mathbf{1 6}}\right)$ |  |
| $\theta^{4}$ |  |  | $A_{\mu}^{+}(\mathbf{1 0})$ |  |
| $\theta^{5}$ |  |  |  | $c^{+}(\mathbf{1})$ |

## Constructing $\widehat{X}_{\mathfrak{g}}$

A minuscule root defines a decomposition of $\mathfrak{g}$ of the form

$$
\mathfrak{g}=\mathfrak{g}_{-1} \oplus \mathfrak{g}_{0} \oplus \mathfrak{g}_{1}
$$

where $\mathfrak{g}_{0}=\mathfrak{u}_{1} \oplus \mathfrak{s}$, and $\mathfrak{g}_{1}$ is a minuscule representation $V_{\omega}$ associated to the highest weight $\omega$ of the semi-simple Lie algebra $\mathfrak{s}$. Let $G_{s}$ be the simply connected complex Lie group corresponding to $\mathfrak{s}$, and $P_{\omega}$ be the parabolic subgroup corresponding to $\omega$.
$\widehat{X}_{\mathfrak{g}}$ is defined to be $\mathbb{P}\left(\widehat{X}_{\mathfrak{g}}\right)=G_{s} / P_{\omega}$.

The spaces $\widehat{X}_{\mathfrak{g}}$ have the following homogeneous coordinate ring:

$$
\mathbb{C}\left[\widehat{X}_{\mathfrak{g}}\right] \cong \bigoplus_{\ell \geq 0} V_{\ell \omega}
$$

The geometries we consider are:

| $\mathfrak{g}$ | $\omega$ | $\mathfrak{s}$ | $\operatorname{dim} \mathfrak{g}_{1}$ | $\operatorname{dim}_{\mathbb{C}} \widehat{X}_{\mathfrak{g}}$ | $\mathbb{P}\left(\widehat{X}_{\mathfrak{g}}\right)$ | $c_{1}\left(\mathbb{P}\left(\widehat{X}_{\mathfrak{g}}\right)\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathfrak{d}_{4}$ | $\omega_{4}$ | $\mathfrak{a}_{3}$ | 6 | 5 | $\operatorname{Gr}(2,4)$ | 4 |
| $\mathfrak{e}_{6}$ | $\omega_{1}$ | $\mathfrak{d}_{5}$ | 16 | 11 | $\operatorname{OG}^{+}(5,10)$ | 8 |
| $\mathfrak{e}_{7}$ | $\omega_{7}$ | $\mathfrak{e}_{6}$ | 27 | 17 | $\mathbb{O P}^{2}$ | 12 |

## $\beta \gamma$ system from 4d/2d SCFT

To realize the $(0,2)$ sigma models with the targets $\widehat{X}_{\mathfrak{g}}$, we begin with a triplet of four-dimensional SCFTs $\mathcal{T}_{\mathfrak{g}}$.

- $\mathcal{T}_{\mathcal{D}_{4}}$ is the $\mathcal{N}=2$ Super-Yang-Mills theory with gauge group $S U(2)$ and four flavors
- $\mathcal{T}_{\mathfrak{e}_{6}}$ and $\mathcal{T}_{\mathfrak{e}_{7}}$ are the rank-one $\mathfrak{e}_{6}$ and $\mathfrak{e}_{7}$ Minahan-Nemeschansky theories.

We next perform a partial $N=-1$ topological twist on the $\mathcal{N}=2$ SCFT along the lines of [Cecotti-Song-Vafa-Yan], and reduce the theory on a two-sphere $S^{2}$, leading to a two-dimensional theory preserving $(0,2)$ supersymmetry.

## Back to the partition function

We consider the following partition function:

$$
\begin{gathered}
Z_{\mathfrak{g}}\left(t, \mathbf{m}^{\mathfrak{s}}, \tau\right)=\operatorname{Tr}_{\mathcal{H}}(-1)^{F} e^{2 \pi i \tau H} t^{J-\frac{1}{2} a_{\mathfrak{u}_{1}}} \exp \left(2 \pi i \mathbf{m}^{\mathfrak{g}}\right), \text { where } \\
a_{\mathfrak{u}_{1}}=-c_{1}\left(\mathbb{P}\left(\widehat{X}_{\mathfrak{g}}\right)\right)
\end{gathered}
$$

is the $\mathfrak{u}_{1}$ symmetry anomaly appearing in the operator product expansion (OPE)

$$
J(y) T(z) \sim \frac{a_{\mathfrak{u}_{1}}}{(y-z)^{3}}+\frac{J(z)}{(y-z)^{2}}
$$

## The Level

The $\mathfrak{u}_{1}$ level, which appears in the OPE

$$
J(y) J(z)=\frac{k_{u_{1}}}{(y-z)^{2}},
$$

is given in this class of models by

$$
k_{\mathfrak{u}_{1}}=\frac{1}{2} a_{\mathfrak{u}_{1}} .
$$

## The central charge

The stress-energy tensor differs from the Sugawara stress tensor by a correction term:

$$
T=T_{\text {Sug }}+\partial J,
$$

this shifts the central charge of the $(0,2)$ theory from the Sugawara value

$$
c_{\text {Sug }}=\frac{\operatorname{dim} \mathfrak{g} k}{h^{\vee}+k}
$$

to the effective central charge

$$
c_{e f f}=c_{S u g}-24 h_{\min }=2 \operatorname{dim} \widehat{X}_{\mathfrak{g}},
$$

which coincides with the central charge of the $\beta \gamma$ system on $\widehat{X}_{\mathfrak{g}}$.

## Symmetry Enhancement

A useful cartoon is:

$$
\begin{gathered}
\beta \gamma \text { system on } \widehat{X}_{\mathfrak{g}} \longleftrightarrow \text { affine VOA } V_{k}(\mathfrak{g}) \\
\mathcal{D}\left(\widehat{X}_{\mathfrak{g}}\right) \longleftrightarrow U(\mathfrak{g}) / \mathcal{J}_{0}
\end{gathered}
$$

The bottom row is the limit of 1-dimensional quantum mechanics.

## Connections to surface operators?

The irreducible highest weight reprensetations for $\left(\widehat{\mathfrak{d}}_{4}\right)_{-2}$ are

$$
0,-2 \omega_{1},-2 \omega_{3},-2 \omega_{4} .
$$

and similarly for $\left(\widehat{e}_{6}\right)_{-3}$ [Arakawa-Moreau].

$$
0,-3 \omega_{1},-3 \omega_{6}, \omega_{1}-2 \omega_{3}, \omega_{6}-2 \omega_{5},-2 \omega_{2},-\omega_{4} .
$$

This gives a plausible construction for the operators that were unaccounted for by letter counting - they are defect operators that arise from wrapping a surface operator over $S^{2}$.

## Matrix Integrals

The Witten index of supersymmetric quantum mechanics reduces to a sum of matrix integrals. It is a beautiful story explained in [Hwang-Yi] and [Hwang-Li-Yi]. For now, I will just advertise some new results on the matrix integral [R.E. 1909.13798].

$$
\mathcal{Z}^{G}=\frac{1}{\operatorname{Vol}(G / Z(G))} \int d \lambda \mathrm{~d} A d D e^{-S_{Y M}}
$$

where

$$
S_{Y M}=-\operatorname{Tr}\left(\frac{1}{4}\left[A_{\mu}, A_{\nu}\right]^{2}+\bar{\lambda} \bar{\sigma}^{\mu}\left[A_{\mu}, \lambda\right]-2 D^{2}\right)
$$

## The integral for exceptional groups

Exact evaluation using localization:

| $G$ | $\mathcal{Z}^{G}$ |
| :---: | ---: |
| $G_{2}$ | $\frac{151}{864}$ |
| $F_{4}$ | $\frac{493313}{3981312}$ |
| $E_{6}$ | $\frac{1417}{249952}$ |
| $E_{7}$ | $\frac{248265523}{254803968}$ |
| $E_{8}$ | $\frac{63048710459827}{37150418534400000}$ |

The integral reduces to a sum over distinguished nilpotent orbits.

## The Witten Index

Joint work with P. Yi (to appear) assembles these matrix integrals into the Witten index.

The result recovers the Kac-Smilga smilga formula for the Witten index:

$$
\frac{1}{|W|} \sum_{w \in W_{G}^{\text {el }}} \frac{1}{\operatorname{det}(1-w)}=\frac{1}{Z(G)} \sum_{i \in \widehat{\Gamma_{G}}} \frac{1}{\delta_{i}} \mathcal{Z}^{H_{i}} .
$$

Somewhat mysteriously, the contributions to the RHS appear are related to the formal degrees of unipotent discrete series representations of $G$ over a $p$-adic field with $q=p^{n}$ which play an important role in the local Langlands correspondence.

