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### Witten Index Revisited

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### APCTP Workshop on Quantum Field Theory and String Theory Pohang, Korea Thursday, November 21nd, 2019



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Plan:				

This talk will highlight two recent applications of the Witten index and the elliptic genus.

- Elliptic genus of 2d (0,2) SCFTs and chiral algebras.
- Witten Index of pure  $\mathcal{N} = 4$  supersymmetric quantum mechanics.

### Supersymmetric Localization

The common theme is to utilize and extend modern results in supersymmetric localization.

The elliptic genus of 2d gauged linear sigma models was derived in

- Benini, R. E., Hori, Tachikawa [arXiv:1305.0533, 1308.4896]
- Gadde, Gukov [arXiv:1305.0266]

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Applicatio	ns			



M-strings [arXiv:1305.6322]

Early applications from 2014 include

- M-Strings [Haghighat-lqbal-Kozcaz-Lockhart-Vafa]
- Surface operators [Gadde-Gukov]
- Mathieu moonshine [Harrison-Kachru-Paquette]
- Non-compact Kahler manifolds [Ashok-Doroud-Troost]
- *SL*(2, ℝ)/*U*(1) Cigar coset [Murthy]

## The Witten Index

Building upon the , several groups considered the computation of the Witten index of gauged supersymmetric quantum mechanics in 2014.

- C. Hwang, J. Kim, S. Kim, and J. Park
- Cordova, Shao
- Hori, Kim, and Yi

Considering M-theory compactified on  $S^1$  leads to the prediction that N D0-branes form a unique  $L^2$  normalizable bound state.

In particular, the Witten index of  $\mathcal{N} = 16$ , supersymmetric quantum mechanics with gauge group SU(N) is 1.

Intensive work went into studying the Witten index of  $\mathcal{N}=4,8,$  and 16 supersymmetric quantum mechanics.

Can modern methods tell us something new about these old problems?

Introduction

Examples

**2d**  $\mathcal{N} = (0, 2)$ 

Witten Index Revisited

### What is the Witten index?



The elliptic genus is a topological invariant that generalizes the Euler characteristic.

Theorem (Gauss-Bonet)

$$\chi(\Sigma) = \int_{\Sigma} K dA$$

# Higher dimensions

#### Theorem (Chern-Gauss-Bonnet)

$$\chi(X) = \int_X c_N(\mathcal{TX})$$
$$= \frac{1}{(2\pi)^N} \int_X \mathsf{Pf}\,\Omega$$

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The elliptic genus of a two-dimensional  $\mathcal{N}=(2,2)$  theory is

Definition (Elliptic Genus)

$$Z_{T^2}(\tau, z, u) = \operatorname{Tr}_{\mathsf{RR}}(-1)^F q^{H_L} \bar{q}^{H_R} y^J \prod_a x_a^{K_a},$$

- Trace taken in the RR sector.
- Flavor symmetry group K (with Cartan generators  $K_a$ )
- A left-moving U(1) R-symmetry J
- Fermions have periodic boundary conditions.
- F is the fermion number.

# Elliptic Genus – Definition

Definition (Elliptic Genus)

$$Z_{\mathcal{T}^2}(\tau, z, u) = \operatorname{Tr}_{\mathsf{RR}}(-1)^{\mathsf{F}} q^{\mathsf{H}_L} \bar{q}^{\mathsf{H}_R} y^J \prod_a x_a^{\mathsf{K}_a},$$

•  $q = e^{2\pi i \tau}$  is the complex structure of a torus.

• 
$$\tau = \tau_1 + i\tau_2$$

Left- and right-moving Hamiltonians  $H_L$  and  $H_R$  are  $2H_L = H + iP$ ,  $2H_R = H - iP$  in Euclidean signature. Since  $q^{H_L}\bar{q}^{H_R} = \exp(-2\pi\tau_2 H - 2\pi\tau_1 P)$ ,

The trace can be represented by a path integral on a torus of complex structure  $\boldsymbol{\tau}.$ 

For a superconformal theory, the operators  $H_L$ ,  $H_R$ , J equal the zero-mode generators  $L_0$ ,  $\overline{L}_0$ ,  $J_0$  of the superconformal algebra.

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# Elliptic Genus – Definition

Definition (Elliptic Genus)

$$Z_{T^2}(\tau, z, u) = \operatorname{Tr}_{\mathsf{RR}}(-1)^F q^{H_L} \bar{q}^{H_R} y^J \prod_a x_a^{K_a},$$

We also define

$$y = e^{2\pi i z}$$
 and  $x_a = e^{2\pi i u_a}$ 

which specify the background R-symmetry and flavor-symmetry gauge fields  $A^{\rm R}$ ,  $A^{\rm flavor}$  via

$$z = \oint A_t^{\mathsf{R}} \mathrm{d}t - \tau \oint A_s^{\mathsf{R}} \mathrm{d}s \;, \qquad u_{\mathsf{a}} = \oint A_t^{\mathsf{a}\text{-th flavor}} \mathrm{d}t - \tau \oint A_s^{\mathsf{a}\text{-th flavor}} \mathrm{d}s \;,$$

where t, s are the temporal and spatial directions.

• Reduces to the Witten index

$$Z_{\mathcal{T}^2}(\tau,z=0)=\mathsf{Tr}(-1)^F$$

• Under the SL(2,  $\mathbb{Z}$ ) transformation au o au + 1

$$Z_{T^2}(\tau,z)=Z_{T^2}(\tau,z+1)$$

 $\bullet~{\rm Under}~\tau\to -1/\tau$ 

$$Z_{T^2}(-1/ au, z/ au) = e^{2\pi i (\hat{c}/2)(z^2/ au)} Z_{T^2}( au, z)$$

The CY/LG Correspondence

# Calabi-Yau/Landau Ginzburg Correspondence



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# The Quintic



The quintic Calabi-Yau is described by the IR limit of a U(1) gauge theory. The theory has one chiral multiplet P of charge -5 and five chiral multiplets  $X_i$  of charge 1, and super potential  $W = Pf(X_1, \ldots X_5)$  where f is a quintic polynomial.

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# The Quintic - Landau Ginzburg Phase



The 25 simple poles precisely match the twisted sector contributions to the elliptic genus of the the Landau-Ginburg phase.

$$Z_{T^2} = \frac{i\eta(q)^3}{\theta_1(q, y^{-1})} \sum_{k,l=0}^4 \oint_{u=(z+k+l\tau)/5} \frac{\mathrm{d}u}{\theta_1(q, y^{-5})} \left(\frac{\theta_1(q, y^{-1}x)}{\theta_1(q, x)}\right)^5$$
$$= \frac{1}{5} \sum_{k,l=0}^4 y^{-l} \left(\frac{\theta_1(q, y^{-4/5}e^{2\pi i(k+l\tau)/5})}{\theta_1(q, y^{1/5}e^{2\pi i(k+l\tau)/5})}\right)^5$$

Introduction

The Elliptic Genus

Examples

**2d**  $\mathcal{N} = (0, 2)$ 

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### Hidden exceptional symmetry in the pure spinor superstring



#### Guglielmo Lockhart



#### Eric Sharpe

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#### Based on [arXiv:1902.09504]

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# A simple $\mathcal{N} = (0,2)$ duality

#### Landau-Ginzburg Theory

6 (0, 2) chiral multiplets  $\phi_i$ 1 Fermi multiplet  $\Psi$ *J*-term  $J = \Psi \operatorname{Pf}(\phi_i)$ 

#### Gauge theory

G = SU(2)4 chiral multiplets in the fundamental representation

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- Putrov, Song, Yan [arXiv:1505.07110]
- Dedushenko, Gukov [arXiv:1712.07659]

### Tests of the duality

- At low energies, the both theories are expected to flow to sigma models with targets complex cones on Gr(2, 4).
- There is a  $so(8)_{-2}$  chiral algebra on both sides
- The NS elliptic genus matches

$$Z_{\mathfrak{d}_4}(t,\mathbf{m}^{\mathfrak{a}_3},\tau)=\frac{i\eta(\tau)^5\theta_1(2\sigma,\tau)}{\prod_{w\in\mathbf{6}}\theta_1(\sigma+(\mathbf{m}^{\mathfrak{a}_3},w),\tau)}.$$

# Chiral Algebras

We can decompose the elliptic genus in terms of characters of the chiral algebra  $so(8)_{-2}$ 

$$Z_{\mathfrak{d}_4}(t, \mathbf{m}^{\mathfrak{a}_3}, \tau) = \widehat{\chi}_0^{(\widehat{\mathfrak{d}}_4)-2}(\mathbf{m}^{\mathfrak{d}_4}, \tau) - \widehat{\chi}_{-2\omega_4}^{(\widehat{\mathfrak{d}}_4)-2}(\mathbf{m}^{\mathfrak{d}_4}, \tau).$$

A simple "letter counting" approach to the index only sees one of the characters:

$$-\widehat{\chi}_{-2\omega_4}^{(\widehat{\mathfrak{d}}_4)_{-2}}(\mathbf{m}^{\mathfrak{d}_4}, au)$$

However, operators counted by the other character have yet to be constructed!

### Generalizations

We will study a family of examples arising from (0, 2) sigma models over complex cones. The elliptic genus is the partition function of the holomorphically twisted theory – the curved  $\beta\gamma$  system.

We are still lacking a GLSM description of the other members of the family.

### $\beta\gamma$ systems on complex cones

Curved  $\beta\gamma$  systems are two-dimensional sigma models of holomorphic maps  $\gamma:\Sigma\to\widehat{X}$  with action

$$S = \frac{1}{2\pi} \int_{\Sigma} \beta_i \bar{\partial} \gamma^i$$

$$\gamma^i(z)eta_j(w)\sim \delta^i_jrac{dw}{z-w}.$$

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We consider target spaces  $\widehat{X}_{\mathfrak{g}}$  with enlarged symmetry

$$\mathfrak{g} = \mathfrak{d}_4, \mathfrak{e}_6, \text{ and } \mathfrak{e}_7.$$

These varieties can be described as cones over:

- Grassmannian Gr(2,4)
- Orthogonal Grassmannian OG<sup>+</sup>(5, 10)
- Cayley plane  $\mathbb{OP}^2$

# Main Result:

#### The partition function is

$$Z = \widehat{\chi}_0^{(\widehat{\mathfrak{e}}_6)_{-3}} - \widehat{\chi}_{-3\omega_1}^{(\widehat{\mathfrak{e}}_6)_{-3}}.$$

This is the partition function for the  $\beta\gamma$  system describing the ghost sector of the pure spinor superstring. It reproduces the first five levels that were computed by Aisaka, Arroyo, Berkovits, and Nekrasov.

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The spaces  $\widehat{X}_{\mathfrak{g}}$  are Lagrangian submanifolds of the moduli spaces  $\widetilde{\mathcal{M}}_{\mathfrak{g},1}$  of a single centered  $\mathfrak{g}$ -instanton. These moduli spaces are in turn the Higgs branches of a family of  $4d \ \mathcal{N} = 2$  superconformal field theories (SCFTs)  $\mathcal{T}_{\mathfrak{g}}$ , whose associated chiral algebra is the vacuum module of  $\widehat{\mathfrak{g}}_k$  affine algebra, where k = -2, -3, -4 respectively for  $\mathfrak{g} = \mathfrak{d}_4, \mathfrak{e}_6$ , and  $\mathfrak{e}_7$ .

### Pure spinors

The pure spinor superstring is another story for another time. However I'll give a few teasers:

The lowest energy component of the non-vacuum term is the Hilbert series of the *Wallach representation* of the  $\mathfrak{e}_6$  finite-dimensional Lie algebra corresponding to highest weight  $-3\omega_1$ . Up to a constant, it is:

$$\sum_{\ell=0}^{\infty} t^{\ell} \chi^{\mathfrak{d}_5}_{V_{\ell\omega_5}}(\mathbf{m}^{\mathfrak{d}_5}).$$

### 10d Supersymmetric Yang–Mills

#### The infinite sum has a closed form:

$$\frac{1}{(1-t)^{16}}\left(1-10t^2+16t^3-16t^5+10t^6-t^8\right)$$

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The fields in the Batalin–Vilkovisky complex organize into the  $\bf 27$  and  $\bf \overline{27}$  of  $\mathfrak{e}_6$ 

Fields of ten-dimensional super Yang-Mills, with Spin(10) indices

BV:	1	0	-1	-2
$\theta^0$	c (1)			
$\theta^1$		$A_{\mu}$ (10)		
$\theta^2$		$\lambda_{lpha} \ (S_{+} = 16)$		
$\theta^3$			$\lambda_{lpha}^+ \ (S = \overline{f 16})$	
$\theta^4$			$A^+_{\mu}$ (10)	
$\theta^5$				$c^+$ (1)

A minuscule root defines a decomposition of  $\mathfrak{g}$  of the form

 $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ 

where  $\mathfrak{g}_0 = \mathfrak{u}_1 \oplus \mathfrak{s}$ , and  $\mathfrak{g}_1$  is a *minuscule* representation  $V_{\omega}$  associated to the highest weight  $\omega$  of the semi-simple Lie algebra  $\mathfrak{s}$ . Let  $G_s$  be the simply connected complex Lie group corresponding to  $\mathfrak{s}$ , and  $P_{\omega}$  be the parabolic subgroup corresponding to  $\omega$ .

$$\widehat{X}_{\mathfrak{g}}$$
 is defined to be  $\mathbb{P}(\widehat{X}_{\mathfrak{g}})=\mathcal{G}_{s}/\mathcal{P}_{\omega}.$ 

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### The spaces $\widehat{X}_{\mathfrak{g}}$ have the following homogeneous coordinate ring:

$$\mathbb{C}[\widehat{X}_{\mathfrak{g}}] \cong \bigoplus_{\ell \ge 0} V_{\ell \omega}.$$

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#### The geometries we consider are:

g	$\omega$	s	$dim\mathfrak{g}_1$	$\dim_{\mathbb{C}} \widehat{X}_{\mathfrak{g}}$	$\mathbb{P}(\widehat{X}_{\mathfrak{g}})$	$c_1(\mathbb{P}(\widehat{X}_\mathfrak{g}))$
$\mathfrak{d}_4$	$\omega_4$	$\mathfrak{a}_3$	6	5	Gr(2, 4)	4
$\mathfrak{e}_6$	$\omega_1$	$\mathfrak{d}_5$	16	11	$OG^{+}(5, 10)$	8
$\mathfrak{e}_7$	$\omega_7$	$\mathfrak{e}_6$	27	17	$\mathbb{OP}^2$	12

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# $\beta\gamma$ system from 4d/2d SCFT

To realize the (0,2) sigma models with the targets  $\widehat{X}_{g}$ , we begin with a triplet of four-dimensional SCFTs  $\mathcal{T}_{g}$ .

- $\mathcal{T}_{\mathfrak{d}_4}$  is the  $\mathcal{N}=2$  Super-Yang-Mills theory with gauge group SU(2) and four flavors
- $\mathcal{T}_{\mathfrak{e}_6}$  and  $\mathcal{T}_{\mathfrak{e}_7}$  are the rank-one  $\mathfrak{e}_6$  and  $\mathfrak{e}_7$ Minahan–Nemeschansky theories.

We next perform a partial N = -1 topological twist on the  $\mathcal{N} = 2$  SCFT along the lines of [Cecotti-Song-Vafa-Yan], and reduce the theory on a two-sphere  $S^2$ , leading to a two-dimensional theory preserving (0, 2) supersymmetry.

### Back to the partition function

We consider the following partition function:

$$egin{aligned} & Z_{\mathfrak{g}}(t,\mathbf{m}^{\mathfrak{s}}, au)\!=\!\mathrm{Tr}_{\mathcal{H}}(-1)^{F}e^{2\pi i au H}t^{J-rac{1}{2}a_{\mathfrak{u}_{1}}}\!\exp(2\pi i\mathbf{m}^{\mathfrak{g}}), where \ & a_{\mathfrak{u}_{1}}=-c_{1}(\mathbb{P}(\widehat{X}_{\mathfrak{g}})) \end{aligned}$$

is the  $\mathfrak{u}_1$  symmetry anomaly appearing in the operator product expansion (OPE)

$$J(y)T(z)\sim rac{a_{\mathfrak{u}_1}}{(y-z)^3}+rac{J(z)}{(y-z)^2}.$$

The  $\mathfrak{u}_1$  level, which appears in the OPE

$$J(y)J(z)=\frac{k_{u_1}}{(y-z)^2},$$

is given in this class of models by

$$k_{\mathfrak{u}_1}=\frac{1}{2}a_{\mathfrak{u}_1}.$$

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The stress-energy tensor differs from the Sugawara stress tensor by a correction term:

$$T=T_{Sug}+\partial J,$$

this shifts the central charge of the (0,2) theory from the Sugawara value

$$c_{Sug} = \frac{\dim \mathfrak{g} \, k}{h^{\vee} + k}$$

to the effective central charge

$$c_{eff} = c_{Sug} - 24h_{min} = 2\dim \widehat{X}_{\mathfrak{g}},$$

which coincides with the central charge of the  $\beta\gamma$  system on  $\widehat{X}_{\mathfrak{g}}$ .

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Symmetr	v Enhancemer	nt		

#### A useful cartoon is:



The bottom row is the limit of 1-dimensional quantum mechanics.

Connections to surface operators?

The irreducible highest weight reprensetations for  $(\widehat{\mathfrak{d}}_4)_{-2}$  are

$$0, -2\omega_1, -2\omega_3, -2\omega_4.$$

and similarly for  $(\hat{\mathfrak{e}}_6)_{-3}$  [Arakawa-Moreau].

$$0, -3\omega_1, -3\omega_6, \omega_1 - 2\omega_3, \omega_6 - 2\omega_5, -2\omega_2, -\omega_4.$$

This gives a plausible construction for the operators that were unaccounted for by letter counting – they are defect operators that arise from wrapping a surface operator over  $S^2$ .

The Witten index of supersymmetric quantum mechanics reduces to a sum of matrix integrals. It is a beautiful story explained in [Hwang-Yi] and [Hwang-Li-Yi]. For now, I will just advertise some new results on the matrix integral [R.E. 1909.13798].

$$\mathcal{Z}^{G} = \frac{1}{\operatorname{Vol}(G/Z(G))} \int d\lambda \, \mathrm{d}A \, dD \, e^{-S_{YM}}$$

where

$$\mathcal{S}_{YM} = -\operatorname{Tr}\left(rac{1}{4}\left[\mathcal{A}_{\mu},\mathcal{A}_{
u}
ight]^{2}+\overline{\lambda}\overline{\sigma}^{\mu}\left[\mathcal{A}_{\mu},\lambda
ight]-2D^{2}
ight)$$

# The integral for exceptional groups

#### Exact evaluation using localization:



The integral reduces to a sum over *distinguished nilpotent orbits*.

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# The Witten Index

Joint work with P. Yi (to appear) assembles these matrix integrals into the Witten index.

The result recovers the Kac–Smilga smilga formula for the Witten index:

$$\frac{1}{|W|}\sum_{w\in W_G^{ell}}\frac{1}{\det(1-w)}=\frac{1}{Z(G)}\sum_{i\in\widehat{\Gamma_G}}\frac{1}{\delta_i}\mathcal{Z}^{H_i}.$$

Somewhat mysteriously, the contributions to the RHS appear are related to the formal degrees of unipotent discrete series representations of G over a p-adic field with  $q = p^n$  which play an important role in the local Langlands correspondence.