Black Holes in $\mathcal{N} = 4$ Super-Yang-Mills

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Quantum gravity ?

⋆ STRING THEORY: Perturbative definition (+ some non-pert. objects)

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⋆ AdS/CFT: For gravity in asymptotically AdS space $\Rightarrow$ non-perturbative definition in terms of boundary ordinary QFT

Parameter map

- Large AdS$_D$ compared with Planck scale $\Rightarrow$ QFT with large "central charge" (large N)

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[Brown, Henneaux 86]

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E.g., in string theory: $$\frac{\ell_{\text{AdS}}^{4}}{\alpha'^2} \sim \lambda$$

$\uparrow$

PROBLEM!

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PROBLEM!

! Take advantage of modern non-perturbative methods !

Black holes have an entropy

$$S_{\mathsf{BH}} = \frac{\mathsf{Area}}{4G_N \hbar / c^3}$$

[Bekenstein 72, 73, 74; Hawking 74, 75]

Black hole $=$ Ensemble of states in quantum gravity $=$ Ensemble of states in boundary QFT

$$S_{\mathsf{micro}} = \log N_{\mathsf{micro}} = \frac{\mathsf{Area}}{4G_N} + \log \mathsf{Area} + \dots$$

(pert. and non-pert.)

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- Can we reproduce the Bekenstein-Hawking entropy?
- Can we go beyond and compute corrections?
- Can we determine the *exact integer number* N_{micro}?

Black holes in flat space

⋆ String theory reproduces the Bekenstein-Hawking entropy of BPS black holes in asymptotically flat spacetime [Strominger, Vafa 96]

- With enough SUSY, corrections can be computed as well

[G. Compere, A. Dabholkar, F. Denef, R. Dijkgraaf, J. Gomes, J. A. Harvey, M. Henneaux, F. Larsen, J. Maldacena, G. Moore, S. Murthy, B. Pioline, V. Reys, A. Sen, A. Strominger, E. Verlinde, H. Verlinde, E. Witten, D. Zagier, . . .]

(We still lack a non-perturbative definition)

- Computation done by exhibiting AdS_3 near horizon
  $\Rightarrow$ BTZ black holes understood as well

Black holes in AdS

- ⋆ Dual QFT reproduces the Bekenstein-Hawking entropy of magnetically-charged (dyonic) BPS black holes in asymptotically AdS_4 space [FB, Hristov, Zaffaroni 15]

  - Corrections are difficult [Liu, Pando Zayas, Rathee, Zhao 17; Jeon, Lal 17]

  - Generalized to magnetic black holes in other dimensions

[Azzurli, FB, Bobev, Cabo-Bizet, Crichigno, Hosseini, Hristov, Jain, Passias, Min, Nedelin, Pando Zayas, Willett, Yaakov, Zaffaroni, . . .]

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⋆ What about non-magnetic black holes in AdS?

The case of AdS_5 has remained a puzzle for a long time. . .

[Kinney, Maldacena, Minwalla, Raju 05]

BPS black holes in AdS$_5$

Setup:

Type IIB string theory on AdS$_5 \times S^5$	$\longleftrightarrow$	4d $SU(N)$ $\mathcal{N} = 4$ Super-Yang-Mills

BPS black holes in AdS_5

Setup:

Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ $\longleftrightarrow$ 4d $SU(N)$ $\mathcal{N}=4$ Super-Yang-Mills

Black hole solutions can be constructed in 5d gauged SUGRA in AdS_5

E.g.: 5d $\mathcal{N}=1$ gauged "STU model" (graviton mult. + 2 vector mult.)

⋆ Does $\mathcal{N}=4$ SYM contain BPS states that reproduce the black hole entropy?

Rotating & electrically-charged $\frac{1}{16}$-BPS black holes in AdS$_5$ [Gutokski, Reall 04]

[Chong, Cvetic, Lu, Pope 05; Kunduri, Lucietti, Reall 06]

$$
\begin{aligned}
ds^2 &= -(H_1H_2H_3)^{-\frac{2}{3}}(dt + \omega_\psi d\psi + \omega_\phi d\phi)^2 + (H_1H_2H_3)^{\frac{1}{3}}\left(f(r)dr^2 + r^2 ds^2_{S^3}\right) \\
A^I &= H_I^{-1}(dt + \omega_\psi d\psi + \omega_\phi d\phi) + U^I_\psi d\psi + U^I_\phi d\phi \\
\Phi^I &= (H_1H_2H_3)^{\frac{1}{3}} H_I^{-1}
\end{aligned}
$$

- Two angular momenta: J_1, J_2

  Three electric charges $U(1)^3 \subset SO(6)$: R_1, R_2, R_3

- Extremal, 1 complex supercharge $\mathcal{Q}$

  BPS relation: $2M = 2J_1 + 2J_2 + R_1 + R_2 + R_3$

  Large smooth horizon: non-linear relation among 5 charges $\rightarrow$ 4 parameters

- Near horizon: fibration AdS$_2 \rightarrow$ squashed S^3

  B-H entropy: $S_{\text{BH}} = \frac{\text{Area}}{4G_N} = \pi\sqrt{R_1R_2 + R_1R_3 + R_2R_3 - 2N^2(J_1+J_2)}$

- Angular momenta, charges and entropy scale $\sim N^2$

Superconformal index

[Romelsberger 05; Kinney, Maldacena, Minwalla, Raju 05]

- ⋆ Counts (with sign) BPS states on S^3 $=$ protected operators on flat space

Index of $\mathcal{N}=4$ SYM:

$$\mathcal{I}(p,q,y_1,y_2) = \mathrm{Tr}\,(-1)^F\, e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}\, p^{J_1+\frac{1}{2}R_3}\, q^{J_2+\frac{1}{2}R_3}\, y_1^{\frac{1}{2}(R_1-R_3)}\, y_2^{\frac{1}{2}(R_2-R_3)}$$

Write: $\quad p=e^{2\pi i\tau} \qquad q=e^{2\pi i\sigma} \qquad y_a=e^{2\pi i\Delta_a} \qquad F=R_3=2J_1=2J_2 \mod 2$

SUSY $\Rightarrow$ at most 4 independent fugacities

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⋆ *Exact* integral formula:

$$\mathcal{I} = \kappa_N \oint_{\mathbb{T}^{\mathrm{rk}(G)}} \prod_{i=1}^{\mathrm{rk}(G)} \frac{dz_i}{2\pi i z_i} \times \frac{\prod_{a=1}^3 \prod_{\rho\in\mathfrak{R}_{\mathsf{adj}}} \widetilde{\Gamma}\big(\rho(u)+\Delta_a;\tau,\sigma\big)}{\prod_{\alpha\in\mathfrak{g}} \widetilde{\Gamma}\big(\alpha(u);\tau,\sigma\big)}$$

with $\quad \Delta_1+\Delta_2+\Delta_3-\tau-\sigma \in \mathbb{Z}\,, \qquad z = e^{2\pi i u} \qquad$ and

$$\kappa_N = \frac{(p;p)_\infty^{\mathrm{rk}(G)}(q;q)_\infty^{\mathrm{rk}(G)}}{|\mathcal{W}_G|} \qquad\qquad \widetilde{\Gamma}(u;\tau,\sigma) = \prod_{m,n=0}^\infty \frac{1-p^{m+1}q^{n+1}/z}{1-p^m q^n z}$$

The index encodes (weighted) degeneracies:

$$\mathcal{I} = 1 + \#y + \#y^2 + \ldots + d(Q)\, y^Q + \ldots$$

To extract the degeneracies:

$$d(Q) = \frac{1}{2\pi i} \oint \frac{dy}{y^{Q+1}}\, \mathcal{I}(y) = \oint d\Delta \;\; e^{\log \mathcal{I}(\Delta) - 2\pi i Q \Delta}$$

Assuming large degeneracies, saddle-point approximation $\rightarrow$ Legendre transform

$$\log d(Q) \;\simeq\; \log \mathcal{I}(\Delta) - 2\pi i Q \Delta \,\Big|_{\Delta = \text{extremum}}$$

- We are interested in $Q \sim N^2$

Old attempt at large N limit

[Kinney, Maldacena, Minwalla, Raju 05]

Use plethystic representation of elliptic Γ function:

[Aharony, Marsano, Minwalla, Papadodimas, Van Raamsdonk 03]

$$\mathcal{I}_N = \kappa_N \oint \prod_{i=1}^{N} \frac{dz_i}{2\pi i z_i} \times \exp\left[- \sum_{i\neq j}^{N} \sum_{k=1}^{\infty} V_k \cos 2\pi k(u_i - u_j) \right]$$

with $\quad V_k(p,q,y_1,y_2,y_3) \quad$ and $\quad y_1 y_2 y_3 = pq \quad$ and $\quad z = e^{2\pi i u}$.

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with $V_k(p,q,y_1,y_2,y_3)$ and $y_1 y_2 y_3 = pq$ and $z = e^{2\pi i u}$.

- Saddle-point approximation, continuous distribution of eigenvalues $u_i \to \rho(u)$:

$$S\big[\rho(u)\big] = N^2 \sum\nolimits_{k=1}^{\infty} V_k |\rho_k|^2 \qquad\qquad \rho_k = \int du\, \rho(u)\, e^{2\pi i u}$$

For real fugacities, all $V_k > 0$. Then minimum at $\rho_k = 0$

$\Rightarrow$ homogeneous distribution of u_i on the unit circle

$$\mathcal{I}_{N=\infty} = \prod\nolimits_{k=1}^{\infty} \frac{(1-p^k)(1-q^k)}{(1-y_1^k)(1-y_2^k)(1-y_3^k)}$$

This result does not depend on N.

Old attempt at large N limit

Does the result describe black hole degeneracies?

- ⋆ Simplified setup: $p = q = t^3 \ , \quad y_1 = y_2 = y_3 = t^2$

$$\mathcal{I}_{N=\infty} \ \rightarrow \ \prod_{k=1}^{\infty} \frac{(1-t^{3k})^2}{(1-t^{2k})^3}$$

One can show that $\quad d(Q) \sim e^{\# \sqrt{Q}} \quad$ for $\quad Q \rightarrow \infty$

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For charges $Q \sim N^2$, we get entropy $S \sim N$ and not N^2 **No black holes!**

★ $\mathcal{I}_{N=\infty}$ matches the index of graviton-multiplet states in AdS_5

This saddle-point captures a gas of gravitons in AdS_5

Why the index does not capture BPS black holes?

- Maybe for $\frac{1}{16}$-BPS states there are huge cancelations due to $\mathrm{Tr}\,(-1)^F$ ...

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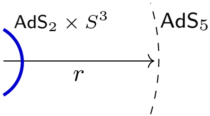
- Maybe for $\frac{1}{16}$-BPS states there are huge cancelations due to $\mathrm{Tr}\,(-1)^F$ ...

- ... but BPS black holes in $\mathrm{AdS}_4 \times S^7$ (dual to 3d $\mathcal{N} = 8$ ABJM theory) are also $\frac{1}{16}$-BPS, and in that case the index *does* capture black hole degeneracies [FB, Hristov, Zaffaroni 15]

Log Index = Black hole Entropy

[FB, Hristov, Zaffaroni 16]

⋆ Black hole solution is an holographic RG flow 4d → 1d



Near-horizon AdS_2: superconformal Quantum Mechanics

$$\mathfrak{su}(1,1|1) \supset \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{u}(1)_{R_{\text{sc}}}$$

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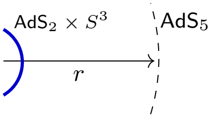

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Superconformal index $\to$ Witten index of QM, with respect to "trial" R-charge

$$\mathcal{I}(\Delta) = \mathrm{Tr}\,(-1)^{R_{\mathsf{trial}}(\mathbb{R}\mathrm{e}\,\Delta)} \;\; e^{-2\pi \sum \mathbb{I}\mathrm{m}\,\Delta \cdot Q} \;\; e^{-\beta \underbrace{\{\mathcal{Q},\mathcal{Q}^\dagger\}}_{H_{\text{near horizon}}}}$$

$$R_{\mathsf{trial}} = R_3 + 2\sum \mathbb{R}\mathrm{e}\,\Delta \cdot Q$$

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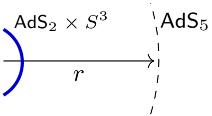

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Inputs from holography (large N):

- $\text{AdS}_2 \;\Rightarrow\; R_{\text{sc}} = 0\,.$ At $\widehat{\Delta}$ all states contribute with $+$ sign

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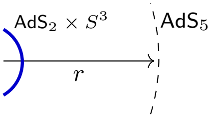

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Inputs from holography (large N):

- $AdS_2 \;\Rightarrow\; R_{\text{sc}} = 0\,.$ At $\widehat{\Delta}$ all states contribute with $+$ sign
- Single-center black hole in microcanonical ensemble: all states have charge Q

$$\frac{\partial\log\mathcal{I}}{\partial\Delta}\bigg|_{\widehat{\Delta}} = i\,\langle Q\rangle \qquad\qquad S_{\text{BH}} = \mathbb{R}\text{e}\Big[\log\mathcal{I} - 2\pi i\sum\Delta\,Q\Big]_{\widehat{\Delta}}$$

Assuming s.c. black hole dominates $\Rightarrow$ $\mathcal{I}$ captures the entropy [similar to Sen 09]

Three recent approaches

- Entropy from on-shell action [Cabo-Bizet, Cassani, Martelli, Murthy 18] [Cassani, Papini 19]
- Cardy limit [Choi, J. Kim, S. Kim, Nahmgoong 18] [M. Honda 19; Ardehali 19] [J. Kim, S. Kim, Song 19; Cabo-Bizet, Cassani, Martelli, Murthy 19]
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Bethe Ansatz formula for the superconformal index

For $\frac{\tau}{\sigma} = \frac{a}{b} \in \mathbb{Q}_+$ (*i.e.*, $\tau = a\omega$, $\sigma = b\omega$) alternative formula: [Closset, Kim, Willett 17] [FB, Milan 18]

$$\mathcal{I} = \kappa_N \sum_{\hat{u} \,\in\, \mathfrak{M}_{\text{BAE}}} \mathcal{Z}_{\text{tot}}(\hat{u}; \xi, \tau, \sigma)\, H(\hat{u}; \xi, \omega)^{-1}$$

1. $\mathfrak{M}_{\text{BAE}}$ are solutions to "Bethe Ansatz Equations" for $\mathrm{rk}(G)$ complexified holonomies $[\hat{u}_i]$ living on a complex torus T^2_ω of modular parameter ω:

$$Q_i = \prod_{\rho_a \in \mathfrak{R}} P\Big(\rho_a(u) + \omega_a(\xi) + r_a \tfrac{\tau+\sigma}{2}; \omega\Big)^{\rho_a^i} \qquad P(u;\omega) = \frac{e^{-\pi i \frac{u^2}{\omega} + \pi i u}}{\theta_0(u;\omega)}$$

$$\mathfrak{M}_{\text{BAE}} = \Big\{ [\hat{u}_i] \in T^2_\omega \;\Big|\; Q_i(u) = 1 \;, \quad w \cdot [\hat{u}] \neq [\hat{u}] \quad \forall\, w \in \mathcal{W}_G \Big\}$$

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2. κ_N and $\mathcal{Z}$ are the same prefactor and integrand as in the integral formula,

$$\mathcal{Z}_{\text{tot}}(u; \dots) = \sum_{\{m_i\}=1}^{ab} \mathcal{Z}(u - m\omega; \dots)$$

3. H is a Jacobian: $\qquad H = \det_{ij} \left(\frac{\partial Q_i}{\partial u_j} \right)$

Bethe Ansatz Equations for $\mathcal{N} = 4$ SYM

Specialize to 4d $SU(N)$ $\mathcal{N} = 4$ SYM, and $\tau = \sigma$ (*i.e.* $J_1 = J_2$). BAEs:

$$1 = Q_i = e^{2\pi i\left(\lambda + 3\sum_j u_{ij}\right)} \prod_{j=1}^{N} \prod_{\Delta \in \{\Delta_1, \Delta_2, -\Delta_1 - \Delta_2\}} \frac{\theta_0(u_{ji} + \Delta; \tau)}{\theta_0(u_{ij} + \Delta; \tau)}$$

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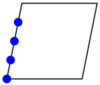
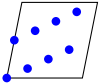
⋆ Class of <u>exact</u> solutions at finite N: [Hosseini, Nedelin, Zaffaroni 16; Hong, Liu 18]

- Basic solution: $u_{ij} = \frac{\tau}{N}(j - i)$



- T-transformed sol's: $u_{ij} = \frac{\tau + r}{N}(j - i)$ with $0 \leq r < N$

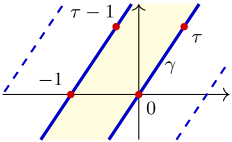


- Many other solutions — most related by $SL(2, \mathbb{Z})$

(This class does not exhaust all solutions)

Contribution of BASIC SOLUTION at large N

Define $\quad [\Delta]_\tau \equiv \Delta + n \quad$ s.t. $\in$ STRIP



Contribution of the BASIC SOLUTION at large N:

$$\lim_{N\to\infty} \log \mathcal{I}\Big|_{\substack{\text{BASIC}\\ \text{SOLUTION}}} = -i\pi N^2\, \Theta(\Delta_1, \Delta_2, \tau)$$

$$\Theta = \begin{cases} \frac{[\Delta_1]_\tau\, [\Delta_2]_\tau \left(2\tau - 1 - [\Delta_1]_\tau - [\Delta_2]_\tau\right)}{\tau^2} & \text{if } [\Delta_1]_\tau + [\Delta_2]_\tau \in \text{STRIP} \\ \frac{\left([\Delta_1]_\tau + 1\right)\left([\Delta_2]_\tau + 1\right)\left(2\tau - 1 - [\Delta_1]_\tau - [\Delta_2]_\tau\right)}{\tau^2} & \text{if } [\Delta_1]_\tau + [\Delta_2]_\tau + 1 \in \text{STRIP} \end{cases}$$

This limit is a discontinuous analytic function: Stokes phenomenon

Black hole entropy

Extract entropy from $\log \mathcal{I}\big|_{\text{BASIC SOLUTION}}$

- Caveat: the theory has 5 charges, but the index only 4 fugacities

$$\int d\tau\, d\sigma\, d\Delta_1\, d\Delta_2\, \mathcal{I}(\tau, \sigma, \Delta_1, \Delta_2)\, p^{-J_1}\, q^{-J_2} \prod_{a=1}^{3} y_a^{-\frac{R_a}{2}} = \sum_{R_3} d(J, R)\Big|_{\substack{\text{other charges}\\ \text{fixed}}}$$

  SUGRA: at most one s.c. black hole contributes to the sum

Black hole entropy

Extract entropy from $\log \mathcal{I}\big|_{\text{BASIC SOLUTION}}$

- Caveat: the theory has 5 charges, but the index only 4 fugacities

$$\int d\tau\, d\sigma\, d\Delta_1\, d\Delta_2\, \mathcal{I}(\tau, \sigma, \Delta_1, \Delta_2)\, p^{-J_1} q^{-J_2} \prod\nolimits_{a=1}^{3} y_a^{-\frac{R_a}{2}} = \sum\nolimits_{R_3} d(J, R)\Big|_{\substack{\text{other charges}\\ \text{fixed}}}$$

SUGRA: at most one s.c. black hole contributes to the sum

⋆ Set $X_1 = [\Delta_1]_\tau \quad X_2 = [\Delta_2]_\tau$. Obtain "entropy function":

$$\boxed{\log \mathcal{I} = -i\pi N^2 \frac{X_1 X_2 X_3}{\tau^2}} \qquad \text{with} \qquad \sum_{a=1}^{3} X_a - 2\tau + 1 = 0$$

Its (constrained) Legendre transform *exactly* gives the black hole entropy:

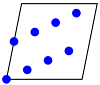
[Hosseini, Hristov, Zaffaroni 17]

$$S_{\text{BH}} = \log \mathcal{I} - 2\pi i \left(\sum X_a \frac{R_a}{2} + 2\tau J \right) \Big|_{\text{constrained extremum}}$$

Extract X, τ from R, J and check that satisfy strip inequality $\Rightarrow$ self-consistency

What about other solutions? They play the role of multiple "saddle points"

- ⋆ T-TRANSFORMED SOL's with $-\frac{N}{2} \lesssim r \lesssim \frac{N}{2}$



All contributions are of order N^2: the one with largest real part dominates $\mathcal{I}$

$$\lim_{N\to\infty} \log \mathcal{I}\Big|_{\text{T-TRANSF}} = \widetilde{\max_{r\in\mathbb{Z}}}\Big(-i\pi N^2\, \Theta(\Delta_1, \Delta_2, \tau + r)\Big)$$

This ensures periodicity under $\tau \to \tau + 1$

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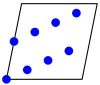

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⋆ Stokes phenomenon

In the limit, multiple exponential contributions compete (as in phase transitions)

$$\lim = e^{a_1(\Delta,\tau)N^2} + e^{a_2(\Delta,\tau)N^2} + \dots$$

$\rightarrow$ Different regions with different analytic limits,
separated by (real-codimension-1) "Stokes lines"

Comparison with old large N limit

⋆ Stokes phenomenon can accommodate the old computation of [Kinney, Maldacena, Minwalla, Raju 05]

- The submanifold of real fugacities sits entirely within a Stokes line
- More strongly, all contributions of order N^2 from $\mathrm{T\text{-}\textsc{transformed sol}}$'s pair up into competing terms, and potentially cancel out.

Universal black holes

Special case:
$$\begin{aligned} J_1 &= J_2 \\ R_1 = R_2 &= R_3 \end{aligned} \quad \leftrightarrow \quad \begin{aligned} \tau &= \sigma \\ \Delta_1 = \Delta_2 = \Delta_3 \equiv \Delta &= \frac{2\tau - 1}{3} \end{aligned}$$

Such black holes exist in 5d $\mathcal{N} = 1$ minimal gauged SUGRA

Uplift to any $\mathrm{AdS}_5 \times \mathrm{SE}_5$ dual to 4d $\mathcal{N} = 1$ SCFT

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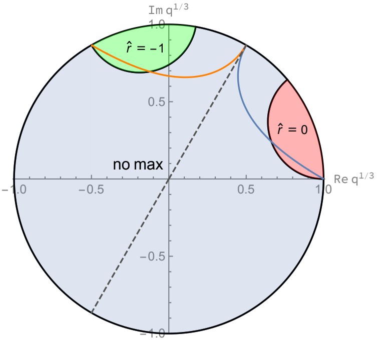

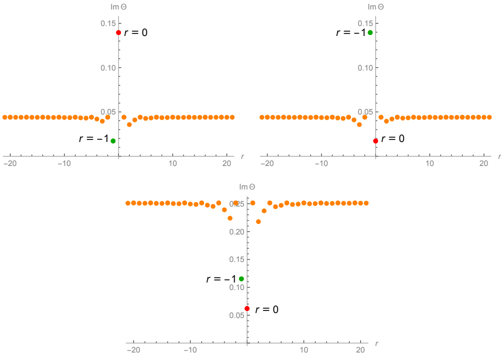

Conclusions

Summary:

- Careful analysis of superconformal index of $\mathcal{N} = 4$ SYM,
  using an alternative Bethe Ansatz formulation.
  At large N, each Bethe Ansatz solution plays the role of a saddle point.
- One solution exactly reproduces the Bekenstein-Hawking entropy of BPS black holes in AdS_5.
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Open questions:

- What do the other solutions represent?
- What is the nature of the phase transitions?
- Can we compute corrections?
- What signatures of quantum gravity emerge?