Supersymmetric RG flows in 4D & Integrability

João Caetano
Simons Center for Geometry & Physics
and
Yang Institute for Theoretical Physics

$T\bar{T}$ deformation and Integrability

Based on 2006.04792 with Wolfger Peelaers & Leonardo Rastelli

Climbing up the RG

Goal: Study a special irrelevant deformation of $\mathcal{N} = 4$ SYM

$$\mathcal{L}_{\mathrm{SYM}} + h\,\mathcal{O}_8 + \ldots$$

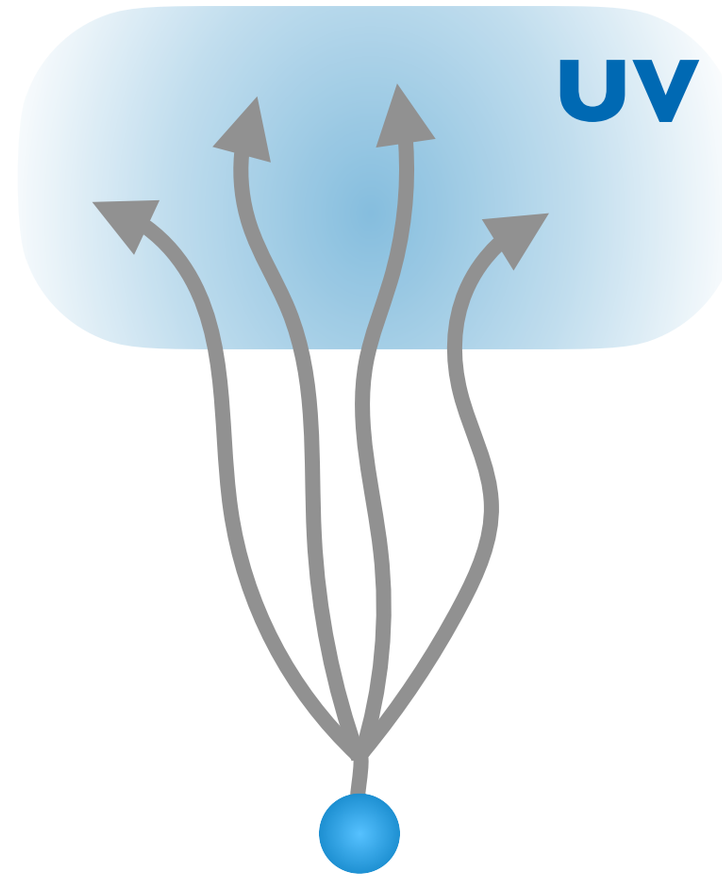
Leading **irrelevant** (single trace) operator ($\Delta = 8$) that preserves all 16 Poincaré supercharges Qs in $\mathbb{R}^4$.

Any maximally SUSY RG flow ending on $\mathcal{N} = 4$ SYM generically looks like this in the IR (unique).

Hope from 2D

$$\mathcal{L}_{\mathrm{SYM}} + h\,\mathcal{O}_8 + \ldots$$

Universal Generally ambiguous



$\mathcal{N} = 4$ SYM

How to fix the ambiguity?
What's the UV theory?

Recently:

Climbing up the RG in 2D: $T\bar{T}$ deformation seems to be well-defined. [Smirnov, Zamolodchikov,…]

New UV behaviour?

Could $\mathcal{O}_8$ be somehow a $D = 4, \mathcal{N} = 4$ version of $T\bar{T}$?

Similar story for $\mathcal{N} = 4$ SYM?

Defining $\mathcal{O}_8$

- Top component of a multiplet (to preserve SUSY)

- Leading irrelevant deformation is $\mathcal{Q}^4\bar{\mathcal{Q}}^4\mathbf{105}$: 1/2 BPS multiplet in the four-index symmetric traceless irrep of $SO(6)_R$

Single-trace: $\mathcal{Q}^4\bar{\mathcal{Q}}^4\,\mathrm{tr}\,\phi^{\{I}\phi^K\phi^L\phi^{M\}}$

$$= \mathrm{tr}\left[\boxed{F^4 - \frac{1}{4}(F^2)^2} + 2D_\mu\phi^I D^\mu\phi^I D_\nu\phi^J D^\nu\phi^J - D_\mu\phi^I D^\nu\phi^I D_\mu\phi^J D^\nu\phi^J + \ldots\right]$$

Same structure in the expansion of DBI action:

$$\sqrt{\det(\eta_{\mu\nu} + F_{\mu\nu})} \simeq 1 + \frac{1}{4}F^2 - \frac{1}{8}\left(F^4 - \frac{1}{4}(F^2)^2\right) + \ldots$$

Double-trace:

$$\mathcal{Q}^4 \bar{\mathcal{Q}}^4 \operatorname{tr} \phi^{\{I} \phi^K \operatorname{tr} \phi^L \phi^{M\}}$$
$$= T_{\mu\nu} T^{\mu\nu} + \mathcal{O}_\tau \mathcal{O}_{\bar{\tau}} + \ldots$$

Looks like $T\bar{T}$ indeed!

However, one does not expect the special 'topological' 2D properties:

$$\langle T\bar{T} \rangle \neq \langle T \rangle \langle \bar{T} \rangle - \frac{1}{4} \langle T^\alpha_\alpha \rangle^2$$

In this talk: **single trace** version.

The goal will not be to formulate an evolution equation for finite h.

Instead: use conformal perturbation theory.

Relation to AdS/CFT

Near horizon D3-brane geometry $\longleftrightarrow$ Low energy D3 brane effective action

Full D3-brane geometry (asymptotic to flat space) $\overset{?}{\longleftrightarrow}$ Full D3 brane effective action

Closed strings in

$$ds^2 = H^{-1/2} dx_\mu dx^\mu + H^{1/2} dx_k dx^k\,, \quad H(r) = h + \frac{R^4}{r^4}\,, \quad R^4 \equiv 4\pi g_s N (\alpha')^4$$

is dual to

$$\mathscr{L}_{\rm SYM} + h R^4 \mathcal{O}_8\,.$$

[Intriligator'99]

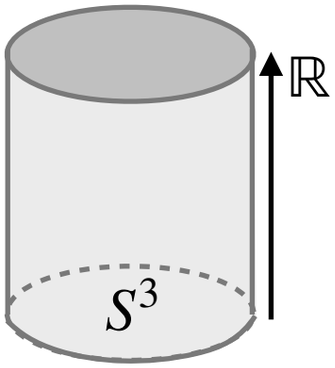
[also Gubser, Hashimoto, Klebanov…]

How to make sense of the field theory side?

Outline

- The deformation on $S^3 \times \mathbb{R}$. Off-shell classical action
- All-loop consequences of the su(2|2) x su(2|2) symmetry
- Constraints from symmetry on perturbation theory
- Integrability
- Holographic interpretation
- Conclusions

The deformation on $S^3 \times \mathbb{R}$



- For $h = 0$, the map $\mathbb{R}^4 \to S^3 \times \mathbb{R}$ is simply a Weyl transformation: full $psu(2,2|4)$ superalgebra is preserved.
- For $h \neq 0$, we can preserve a part of these symmetries (rigid superisometries):

$$psu(2|2) \times psu(2|2) \ltimes \mathbb{R}^2$$

- Bosonic part contains

Isometries of $S^3 \times R$

$$SO(4) \times \mathbb{R}_\tau \simeq SU(2)_\alpha \times SU(2)_{\dot\alpha} \times \mathbb{R}_\tau$$

R-symmetries

$$SU(2)_a \times SU(2)_{\dot a} \times U(1)_J \subset SU(4)_R$$

- Time translations are a still a good symmetry: generated by H

2 copies of $su(2|2)$, with common central extension $H - J$

- To preserve SUSY on $S^3 \times \mathbb{R}$, $\mathcal{O}_8$ has curvature corrections $\mathcal{O}(1/\ell^k)$, where ℓ is the sphere radius.

- **Hard in general:** typically this leads to an infinite expansion.

- We could find the deformation in closed form by using an off-shell formalism by [Berkovits] [Evans] [Pestun]

$$\begin{aligned}
\delta A_M &= \overline{\Psi}\Gamma_M \varepsilon \ , \\
\delta \Psi &= -\frac{1}{2} F_{MN}\Gamma^{MN}\varepsilon - \frac{1}{2}\Phi_I \Gamma^{\mu I} D_\mu \epsilon - K^\ell \nu_\ell \ , \\
\delta K_\ell &= -\overline{\nu}_\ell \Gamma^M D_M \Psi \ .
\end{aligned}$$

- 8 of the supersymmetries linearly realized (off-shell) provided spinors ν_l are constrained:

$$\bar{\epsilon}\Gamma^M \nu_l = 0 \, , \qquad \bar{\nu}_l \Gamma^M \nu_m = \delta_{lm} \bar{\epsilon}\Gamma^M \epsilon$$

- Bosonic symmetries in $S^3 \times \mathbb{R}$ are fully realized off-shell.

- The remaining 8 supersymmetries are realized on-shell.

- Rename the six scalars:

$$\underbrace{Z\,, \qquad \bar{Z}\,,}_{U(1)_J} \qquad \underbrace{\phi_{a\dot{a}}}_{SU(2)\times SU(2)\ \ (J=0)}$$

- Define a superfield:

$$\bar{Z}(\theta_{a\alpha}, \tilde{\theta}^{\dot{a}\dot{\alpha}}) = \bar{Z} - 2i\epsilon^{ab}\epsilon^{\alpha\beta}\,\Psi_{a\alpha}\,\theta_{b\beta} - 2i\epsilon_{\dot{a}\dot{b}}\epsilon_{\dot{\alpha}\dot{\beta}}\,\Psi^{\dot{a}\dot{\alpha}}\,\tilde{\theta}^{\dot{b}\dot{\beta}} + \ldots$$

with $\qquad \delta_{\epsilon_} Z = 0 \qquad$ (off-shell)

- Full classical action:

$$S(g_{YM}, h) = S_{SYM} + h\int_{S^3\times\mathbb{R}} \sqrt{g}\,d^4x \int d^4\theta d^4\,\tilde{\theta}\,\mathrm{tr}\,\bar{\mathrm{Z}}(\theta,\tilde{\theta})^4 + \mathrm{h.c.}$$

- Full classical action:

$$S(g_{YM}, h) = S_{SYM} + h \int_{S^3 \times \mathbb{R}} \sqrt{g} d^4x \int d^4\theta d^4\tilde{\theta}\, \mathrm{tr}\, \bar{\mathrm{Z}}(\theta, \tilde{\theta})^4 + \mathrm{h.c.}$$

- 8 off-shell supercharges manifest, on-shell invariance for the other 8

- Integrating out auxiliary fields generates an infinite power expansion in h.

$$\begin{aligned}
\bar{Z}(\theta_{-I\alpha}, \theta_-^{\hat{I}\dot{\alpha}}) = {} & \bar{Z} - 2i\epsilon^{IJ}\epsilon^{\alpha\beta}\Psi_{-I\alpha}\,\theta_{-J\beta} - 2i\epsilon_{\hat{I}\hat{J}}\epsilon_{\dot{\alpha}\dot{\beta}}\Psi_-^{\hat{I}\dot{\alpha}}\,\theta_-^{\hat{J}\dot{\beta}} \\
& + \frac{1}{2}2i\epsilon^{IJ}\epsilon^{\alpha\beta}\left(+\frac{1}{2}F^{mn}(\sigma_{mn})_\alpha{}^\gamma\delta_I^K + \frac{1}{2}[\Phi^a,\Phi^b](\sigma_{a-4,b-4})_I{}^K\delta_\alpha^\gamma + K^{\hat{m}}(\sigma_{\hat{m}4})_\alpha{}^\gamma\delta_I^K\right)\theta_{-K\gamma}\theta_{-J\beta} \\
& + \frac{1}{2}2i\epsilon_{\hat{I}\hat{J}}\epsilon_{\dot{\alpha}\dot{\beta}}\left(\frac{1}{2}F^{mn}(\bar{\sigma}_{mn})^{\dot{\alpha}}{}_{\dot{\gamma}}\delta_{\hat{K}}^{\hat{I}} + \frac{1}{2}[\Phi^a,\Phi^b](\bar{\sigma}_{a-4,b-4})^{\hat{I}}{}_{\hat{K}}\delta_{\dot{\gamma}}^{\dot{\alpha}} + K^{\hat{m}}(\bar{\sigma}_{\hat{m}4})^{\dot{\alpha}}{}_{\dot{\gamma}}\delta_{\hat{K}}^{\hat{I}}\right)\theta_-^{\hat{K}\dot{\gamma}}\theta_-^{\hat{J}\dot{\beta}} \\
& + 2i\epsilon_{\hat{K}\hat{J}}\epsilon_{\dot{\zeta}\dot{\beta}}\left(D_m\Phi_a - \delta_m^4\left(K^a + \frac{1}{\ell}\Phi^a\right)\right)(\bar{\sigma}^{a-4})^{\hat{K}K}(\bar{\sigma}^m)^{\dot{\zeta}\zeta}\theta_{-K\zeta}\theta_-^{\hat{J}\dot{\beta}} \\
& - \frac{1}{2}2i\epsilon_{\hat{K}\hat{J}}\epsilon_{\dot{\zeta}\dot{\beta}}\Big(\delta_{\hat{L}}^{\hat{K}}\Big(\big(-(\bar{\sigma}^m)^{\dot{\gamma}\zeta}D_m\Psi_{+I}^{\dot{\alpha}} + \frac{4}{2\ell}\Psi_{+I}^{\dot{\alpha}}(\bar{\sigma}_4)^{\dot{\gamma}\zeta}\big)\epsilon^{IK}(\epsilon_{\dot{\alpha}\dot{\eta}}\delta_{\dot{\gamma}}^{\dot{\zeta}} + \epsilon_{\dot{\gamma}\dot{\eta}}\delta_{\dot{\alpha}}^{\dot{\zeta}}) \\
& \qquad + \big(D_m\Psi_{+I}^{\dot{\alpha}}(\sigma_m)_{\alpha\dot{\alpha}}\epsilon^{IK} - i[Z,\Psi_{-I\alpha}]\epsilon^{IK} - [\Phi_b,\Psi_{+\alpha}^{\hat{I}}]\epsilon_{\hat{I}\hat{M}}(\bar{\sigma}_{b-4})^{\hat{M}K}\big)(\bar{\sigma}^4)^{\dot{\alpha}\zeta}(\bar{\sigma}_4)^{\dot{\delta}\alpha}(\epsilon_{\dot{\alpha}\dot{\eta}}\delta_{\dot{\delta}}^{\dot{\zeta}} + \epsilon_{\dot{\delta}\dot{\eta}}\delta_{\dot{\alpha}}^{\dot{\zeta}})\Big) \\
& \qquad + [\Psi_{+\alpha}^{\hat{I}}\epsilon^{\alpha\zeta}\delta_{\dot{\eta}}^{\dot{\zeta}}, (\bar{\sigma}^{a-4})^{\hat{M}K}\Phi_a](\epsilon_{\hat{I}\hat{L}}\delta_{\hat{M}}^{\hat{K}} + \epsilon_{\hat{M}\hat{L}}\delta_{\hat{I}}^{\hat{K}})\Big)\theta_-^{\hat{L}\dot{\eta}}\theta_{-K\zeta}\theta_-^{\hat{J}\dot{\beta}} \\
& - \frac{1}{2}2i\epsilon_{\hat{K}\hat{J}}\epsilon_{\dot{\zeta}\dot{\beta}}\Big(\epsilon^{LK}\Big(\big(D_m\Psi_{+\alpha}^{\hat{K}}(\bar{\sigma}^m)^{\dot{\zeta}\theta} - \frac{4}{2\ell}\Psi_{+\alpha}^{\hat{K}}(\bar{\sigma}_4)^{\dot{\zeta}\theta}\big)(\epsilon^{\alpha\zeta}\delta_\theta^\eta + \epsilon^{\alpha\eta}\delta_\theta^\zeta) \\
& \qquad + \big(-D_m\Psi_{+\delta}^{\hat{K}}(\bar{\sigma}_m)^{\dot{\alpha}\delta} + i[Z,\Psi_-^{\hat{K}\dot{\alpha}}] + [(\bar{\sigma}_{b-4})^{\hat{K}I}\Phi_b, \Psi_{+I}^{\dot{\alpha}}]\big)(\bar{\sigma}^4)^{\dot{\zeta}\theta}(\sigma_4)_{\alpha\dot{\alpha}}(\epsilon^{\alpha\zeta}\delta_\theta^\eta + \epsilon^{\alpha\eta}\delta_\theta^\zeta)\Big) \\
& \qquad + [-\Psi_{+I}^{\dot{\zeta}}\epsilon^{\zeta\eta}, (\bar{\sigma}^{a-4})^{\hat{K}M}\Phi_a](\epsilon^{IL}\delta_M^K + \epsilon^{IK}\delta_M^L)\Big)\theta_{-L\eta}\theta_{-K\zeta}\theta_-^{\hat{J}\dot{\beta}}
\end{aligned}$$

Classical Lagrangian

- At leading order in h, we have

$$\mathscr{L} = \mathscr{L}_{\text{SYM}} + h\left(\mathcal{O}_8 + \frac{\mathcal{O}_7}{\ell} + \ldots + \frac{\mathcal{O}_4}{\ell^4}\right)$$

Components of **105** supermultiplet

e,g. $$\mathcal{O}_4 = \text{sTr}\left(3\,Z^2\bar{Z}^2 - 6\,Z\bar{Z}\phi_p\phi_p + \phi_p\phi_p\phi_q\phi_q\right)$$

- Quantum level, need counterterms: **ambiguous!**

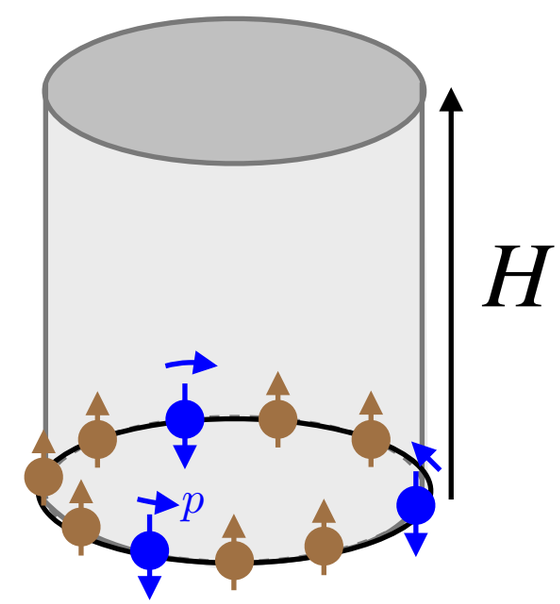
Spin chains and Integrability

- Inspired by the 2D $T\bar{T}$ deformation:

Fix ambiguity by Integrability?

- Is there a choice of RG trajectory which preserves integrability of $\mathcal{N} = 4$ SYM?
- No dilatation operator.
- Instead, ask about energy spectrum of states in the cylinder.

- **Idea: regard the generator of time translations $H(g^2, h)$ as a deformation of the spin chain Hamiltonian of $H(g^2)$ of $\mathcal{N} = 4$ SYM**



- In $\mathcal{N} = 4$ SYM, time translation generator $H(g^2)$ is an integrable spin chain Hamiltonian acting on the space of single trace states.

- e.g. at one-loop, $H(g^2) = g^2 H^{(1)} + \ldots$

$$g^2 H^{(1)} |ZZ\ldots XZZX\ldots\rangle = \boxed{g^2(\mathbb{I} - \mathbb{P})} |ZZ\ldots XZZX\ldots\rangle$$

Scalars

XXX spin chain

- $H(g^2, h)$ is a deformation of this spin chain Hamiltonian. **Integrable?**

Symmetry constraints: single magnon

- Dynamics of a single magnon contained by triply centrally extended $\mathfrak{su}(2|2)$ symmetry [Beisert]

$$\{S^{\alpha}{}_{a}, Q^{b}{}_{\beta}\} = \delta^{a}_{b}\, L^{\alpha}{}_{\beta} + \delta^{\alpha}_{\beta}\, R^{b}{}_{a} + \frac{1}{2}\,\delta^{a}_{b}\,\delta^{\alpha}_{\beta}\,(H - J)$$

$$\{Q^{a}{}_{\alpha}, Q^{b}{}_{\beta}\} = \epsilon^{ab}\,\epsilon_{\alpha\beta}\,P \qquad \{S^{\alpha}{}_{a}, S^{\beta}{}_{b}\} = \epsilon_{ab}\epsilon^{\alpha\beta}\,K$$

- Energy of a **single magnon** fixed to all-orders

$$\mathrm{tr}\left(Z\ldots Z\chi Z\ldots Z\right)$$

$$H - J = \sqrt{1 + 16\,\alpha(g^2, h)\,\sin^2\left(\frac{p}{2}\right)}$$

Unfixed function of the coupling constants

Symmetry constraints: two magnons

$$\mathrm{tr}\left(Z\ldots\ \chi_1\ \ldots\ \chi_2\ \ldots Z\right)$$

- The matrix structure of two body S-matrix completely fixed

$$S(p_1,p_2) = e^{i\theta(p_1,p_2)}\ \hat{\mathbb{S}}(p_1,p_2)$$

Dressing phase Matrix part

- Only freedom sits in θ!

- Automatically satisfies Yang-Baxter equation: strong hint of integrability

- $n \to n$ factorization and stronger evidence for integrability? Need explicit computations.

$SU(2|2) \ltimes U(1)$ sector

- Impractical to do perturbation theory from our action.
- Symmetry is however very powerful.
- Subsector $SU(2|2) \ltimes U(1)$ (analogous to the $SU(2|3)$ of $\mathcal{N} = 4$ SYM)

$$\boxed{\phi_a \equiv \phi_{a1}\,, \qquad \psi_\alpha \equiv \psi_{\alpha 1}\,, \qquad Z} \qquad a = 1,2\,, \quad \alpha = \pm$$

- $H(g^2, h)$ admits a double expansion in g^2 and h, but from an abstract symmetry point of view they are on same footing.
- **A term of order $\mathcal{O}(g^{2i}h^j) \leftrightarrow 2i + j$ loop order.**

Constraining H from symmetry

- Symmetry constrains the action of generators of $SU(2|2) \ltimes U(1)$ on spin chain states.

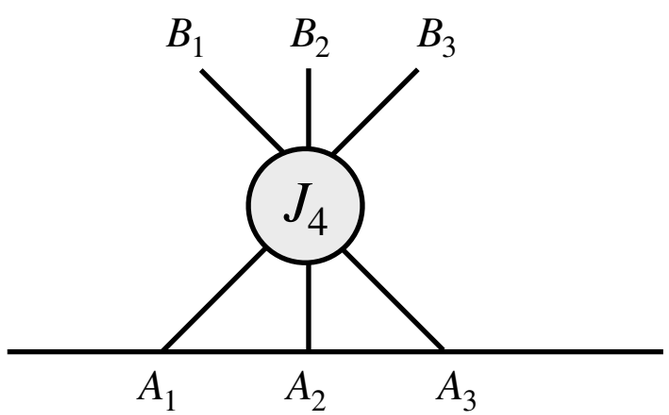
- The invariant tensor structures can be parametrized by the symbols

$$\left\{ \begin{matrix} A_1 \ldots A_n \\ B_1 \ldots B_m \end{matrix} \right\}$$

e.g. $\left\{ \begin{matrix} 1234 \\ 4321 \end{matrix} \right\} |1341234134\rangle = |1344321134\rangle$

- By compatibility with **planar** Feynman diagrams: at a given loop order k, the generators take the form

$$J_k \sim \left\{ \begin{matrix} A_1 \ldots A_n \\ B_1 \ldots B_m \end{matrix} \right\}, \text{ with } n + m = k + 2$$



One-loop results

- By imposing closure of the algebra (and parity) we find at one-loop:

$$H_{1-\text{loop}} = \alpha_1^2 \Big(\left\{ {ab \atop ab} \right\} + \left\{ {a\beta \atop a\beta} \right\} + \left\{ {\alpha b \atop \alpha b} \right\} + \left\{ {\alpha\beta \atop \alpha\beta} \right\} - \left\{ {ab \atop ba} \right\} - \left\{ {a\beta \atop \beta a} \right\} - \left\{ {\alpha b \atop b\alpha} \right\} + \left\{ {\alpha\beta \atop \beta\alpha} \right\}$$
$$+ \left\{ {Z\beta \atop Z\beta} \right\} + \left\{ {\beta Z \atop \beta Z} \right\} + \left\{ {Zb \atop Zb} \right\} + \left\{ {bZ \atop bZ} \right\} - \left\{ {Z\beta \atop \beta Z} \right\} - \left\{ {Zb \atop bZ} \right\} - \left\{ {bZ \atop Zb} \right\} - \left\{ {\beta Z \atop Z\beta} \right\} \Big)$$

- One overall unfixed constant (coupling definition).

- It matches the familiar one-loop result of $SU(2|3)$ chain of $\mathcal{N} = 4$ SYM: **symmetry enhancement**

- So this implies one-loop integrability (same as $\mathcal{N} = 4$ SYM)

Two-loop

- Calculations more involved but doable.

- Closure of $SU(2|2) \ltimes U(1)$ algebra + hermiticity + parity: fixes $H_{2-\text{loop}}$ up to **10** real parameters. **7** of them correspond to change of basis/vanish on cyclic states. **Finally, only 3 are left.**

- Imposing dispersion relation (or using central extensions), fix **1** of them.

- **Only 2 parameters left:** they are related to the freedom of redefining the coupling.

- Same enhancement and get back the $SU(2|3)$ of $\mathcal{N} = 4$ SYM, so integrability guaranteed again.

- Three-loop technically much more involved.

Long-range integrable spin chains

- Restricting to $SU(2|2) \ltimes U(1)$ subsector, is there a long-range integrable spin chain **different** from the $\mathcal{N} = 4$ case?

- For the simplest case of $U(1)$ spin chain (XXZ spin chain), Beisert, Fievét, de Leeuw, Loebbert found a large class of models:

$$\exp(ip(u_k)L) = \prod_{j \neq k} \exp(-2i\theta(u_k, u_j)) \frac{\sinh \hbar(u_k - u_j + i)}{\sinh \hbar(u_k - u_j - i)}$$

$$\theta(u_k, u_j) = \sum_{s>r=2}^{\infty} \beta_{r,s}(q_r(u_k)q_s(u_j) - q_s(u_k)q_r(u_j)) + \sum_{r=2}^{\infty} \eta_r(q_r(u_k) - q_r(u_j))$$

Arbitrary parameters

- XXZ spin chain is a closed subsector of $SU(2|2) \ltimes U(1)$. In our case, $\hbar = 0$. For XXZ, ample freedom is left.

- Study should be repeated to the full $SU(2|2) \ltimes U(1)$

BPS spectrum

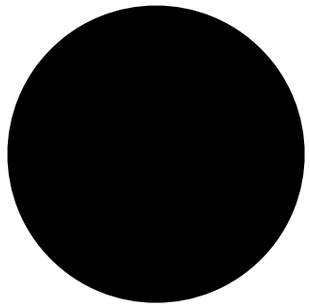
- In $\mathcal{N} = 4$ SYM, symmetry enhancement $SU(2|2) \to SU(2|3)$ generally comes from adding Bethe roots at infinity ($u = \infty$).
- We expect dressing phase to break the symmetry (at sufficient high loop order), and so adding a Bethe root at infinity is expected to produce a different solution (different energy)
- From XXZ analysis, scattering of 2 roots at $u = \infty$ is trivial.
- Prediction: old BPS states of $\mathcal{N} = 4$ SYM, must remain protected when the deformation is turned on
- Seems to be compatible with the analysis of the superconformal index

Holographic interpretation

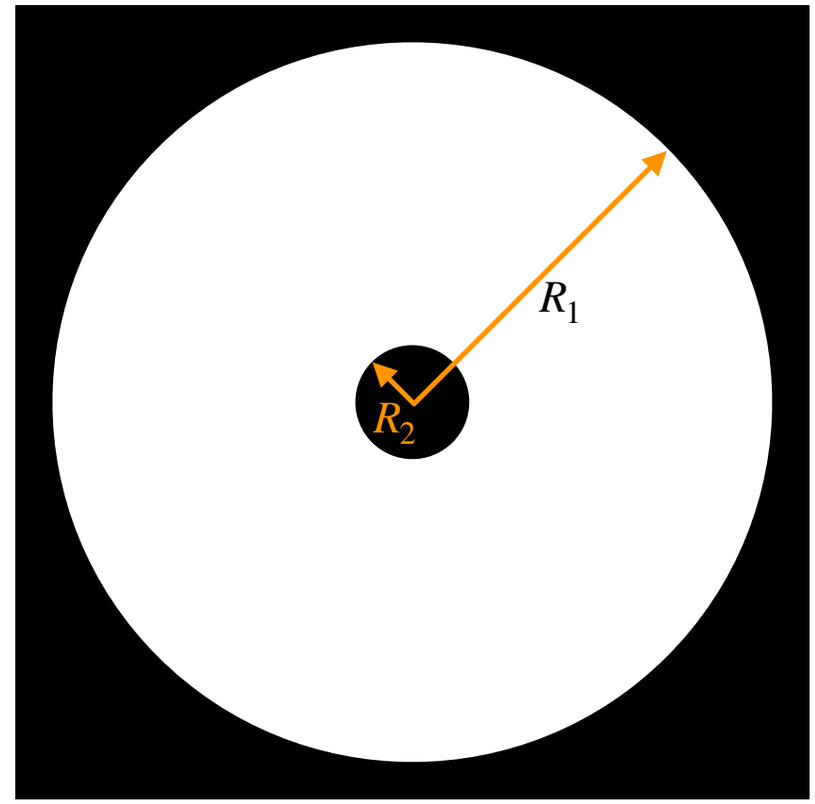
- A natural setting are the *bubbling* geometries of Lin Lunin Maldacena (LLM).

- LLM geometries describe the backreaction of additional D3 branes (giant gravitons) in global $AdS_5 \times S^5$. The extreme UV asymptotics are always $AdS_5 \times S^5$.

- LLM are fully determined bu prescribing $\pm 1/2$ boundary conditions on a 2D plane: bicoloring of the plane



LLM picture for $AdS_5 \times S^5$



Simplest non-trivial LLM geometry with $U(1)_J$ isometry

$$\frac{h}{l^4} \sim \frac{R_2^2}{R_1^2}$$

- Integrability of the sigma-model on this background? Chervonyi & Lunin's analysis shows that massless geodesic equation is not separable.

Outlook

- Higher loop and more systematic analysis of $SU(2|2) \ltimes U(1)$ spin chains

- $SU(2|2)$ integrable spin chains different from the $\mathcal{N} = 4$ case? Which RG flow would it describe in the field theory side

- Relation with LLM, regardless of integrability

- Other geometries: $\mathbb{R}^4$, S^4.

- Integrable backgrounds with 16 supercharges, other than $AdS_5 \times S^5$?

- Double-trace version.

Thank you!