Lower-dimensional Gauss-Bonnet gravity: Theory and solutions

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$$\boxed{\mathcal{G} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2}$$

String theory, gravity and cosmology (SGC2020)
APCTP
18-20 November 2020

Lovelock's Theorem

D. Lovelock, *The Einstein Tensor and Its Generalizations“, Journal of Mathematical Physics.* ***12*** *(3): 498–501 (1971).*

In **four dimensions**, the Einstein-Hilbert action is the only local action (apart from the cosmological constant and topological terms) that leads to the **second order** differential equations for the metric.

Einstein's theory is the unique theory!

Gauss-Bonnet gravity in 4 dimensions

D. Glavan and C. Lin, Einstein-Gauss-Bonnet gravity in 4-dimensional space-time, Phys. Rev. Lett. 124 (2020) 081301, [1905.03601].

Plan of the talk

I. What is Gauss-Bonnet gravity?

II. Is there 4-dimensional GB gravity?
   a) Glavan-Lin’s proposal
   b) Problems with the “naïve” D->4 limit

III. Horndeski type Gauss-Bonnet gravity in 4D
   a) Conformal trick derivation
   b) Kaluza-Klein-type derivation
   c) Solutions and basic properties

IV. Lower-dimensional theory and solutions

V. Summary

Based on

Paper 1: R.A. Hennigar, D. Kubiznak, R.B. Mann and C. Pollack, *On Taking the D->4 limit of Gauss-Bonnet Gravity: Theory and Solutions*, ArXiv:2004.09472.

Paper 2: R.A. Hennigar, D. Kubiznak, R.B. Mann and C. Pollack, *Lower-dimensional Gauss-bonnet gravity and BTZ black holes*, ArXiv:2004.12995.

Paper 3: R. A. Hennigar, D. Kubiznak, R. B. Mann, *Rotating Gauss-Bonnet BTZ black holes*, ArXiv:2005.13732.

Paper 1 has some overlap with

P. G. Fernandes, P. Carrilho, T. Clifton and D. J. Mulryne, *Derivation of Regularized Field Equations for the Einstein-Gauss-Bonnet Theory in Four Dimensions*, 2004.08362.

I) What is Gauss-Bonnet gravity?

$$\boxed{\mathcal{G} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2}$$

Einstein's Theory

To write the **Gravitational Action** we want

- **Scalar Lagrangian** -- diffeomorphism invariance
- **Second-order (E-L)** equations for the metric
- Due to the **Equivalence principle** there is no scalar with only first derivatives: $$I = I(g, \partial g)$$
- Best one can do is to write

$$S_{\mathrm{EH}}[g] = \frac{1}{16\pi G}\int \sqrt{-g}\,R(g, \partial g, \partial^2 g)$$

How come the Einstein equations are 2nd order?

Einstein's Theory

$$\delta S_{\mathrm{EH}} = \frac{1}{16\pi G}\int \delta(\sqrt{-g}R_{\alpha\beta}g^{\alpha\beta})$$

$$= \frac{1}{16\pi G}\int (\underbrace{R\delta\sqrt{-g}+\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}}_{\text{easy to calculate: } \sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu}}+\underbrace{\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}}_{\text{difficult: throw away}})$$

So we recovered $G_{\mu\nu}(g, \partial g, \partial^2 g) = 0$

Luckily we were (almost) right throwing away the last term:

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu V^\mu\,, \quad V_\mu = \nabla^\beta(\delta g_{\mu\beta}) - g^{\alpha\beta}\nabla_\mu(\delta g_{\alpha\beta})$$

$$\boxed{\sqrt{-g}R(g, \partial g, \partial^2 g) = \sqrt{-g}\tilde{R}(g, \partial g) + \partial_\mu \hat{R}^\mu(g, \partial g)}$$

Einstein's Theory

$$\mathcal{L} = R$$

- Is this just the simplest choice or can we add other scalars?

$$R^2\ ,\ R_{\mu\nu}R^{\mu\nu}\ ,\ R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}$$

$$R^3\ , \nabla_\mu R \nabla^\mu R\ , \ldots$$

- **Lovelock Theorem (1971):** In **4D**, the Einstein-Hilbert action is the only local action, apart from the cosmological constant and **topological terms (total derivatives)**, that leads to the **second-order** PDEs for the metric.

Example of topological term (in 4D)

$$\mathcal{G} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$$

Gauss-Bonnet gravity

$$\mathcal{G} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$$

- Interestingly, the **Gauss-Bonnet term** is **topological** (total derivative) **only in 4D.**

- In **D<4** it **identically vanishes**!

- In **D=5 and higher** dimensions it yields non-trivial EOMs:

$$\begin{aligned} H_{\alpha\beta} &= -\frac{1}{2}g_{\alpha\beta}\mathcal{G} + 2RR_{\alpha\beta} - 4R_{\alpha\gamma}R_\beta{}^\gamma \\ &\quad +4R_{\gamma\alpha\beta\delta}R^{\gamma\delta} + 2R_\alpha{}^{\gamma\delta\kappa}R_{\beta\gamma\delta\kappa} \end{aligned}$$

These are 2^nd^-order PDEs !!!

$$\sqrt{-g}\mathcal{G}(g, \partial g, \partial^2 g) = \sqrt{-g}\tilde{\mathcal{G}}(g, \partial g) + \partial_\mu \hat{\mathcal{G}}^\mu(g, \partial g)$$

Lovelock gravity

= Unique higher-curvature (with local action) gravity that yields **2nd-order PDEs** for the metric

$$\mathcal{L} = \frac{1}{16\pi G_N} \sum_{k=0}^{K} \alpha_k \mathcal{L}^{(k)} \qquad K = \left\lfloor \frac{d-1}{2} \right\rfloor$$

where $\mathcal{L}^{(k)}$ are the 2k-dimensional Euler densities

$$\mathcal{L}^{(k)} = \frac{1}{2^k} \delta^{a_1 b_1 \ldots a_k b_k}_{c_1 d_1 \ldots c_k d_k} R_{a_1 b_1}{}^{c_1 d_1} \ldots R_{a_k b_k}{}^{c_k d_k}$$

- k=0: cosmological term $\Lambda = -\alpha_0/2$
- k=1: Einstein-Hilbert term R **(topological in 2D)**
- k=2: Gauss-Bonnet term $\mathcal{G}$ **(topological in 4D)**
- k=3: 3rd-order Lovelock **(topological in 6D)**

"Natural generalization of Einstein's theory in higher dimensions"

Side remark: Beyond Lovelock

Less restricted theories with **higher-order EOM** and some nice properties – "*desperate times require desperate measures*"

- **Quasi-topological gravity** (EOM on SSS are 2nd-order)

  R. C. Myers and B. Robinson, Black Holes in Quasi-topological Gravity, JHEP 1008 (2010) 067, [1003.5357].

- **Einsteinian Cubic Gravity** (active already in 4D)

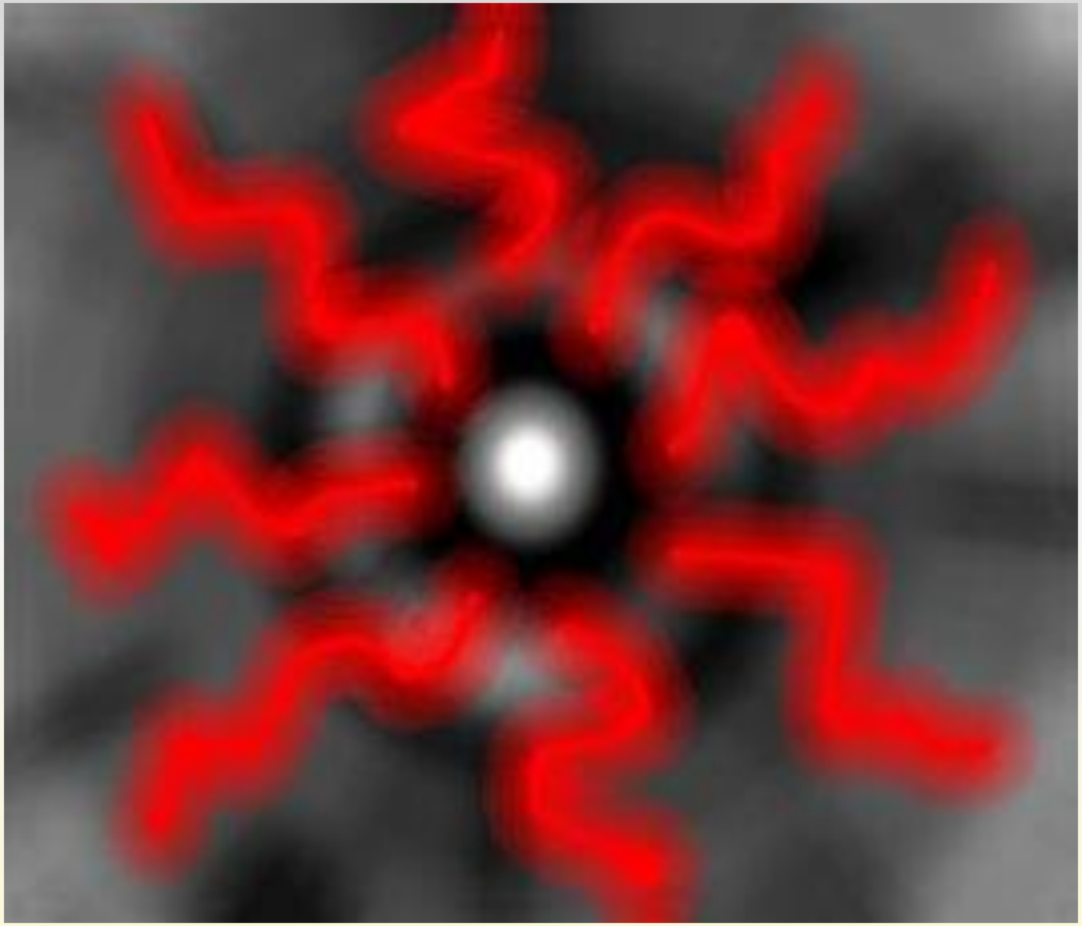
  P. Bueno and P. A. Cano, Einsteinian cubic gravity, Phys. Rev. D94 (2016) 104005,

- **SMF VSSS Gravity** (coincides with Einsteinian in 4D)

  R. A. Hennigar, D. Kubiznak and R. B. Mann,Generalized quasitopological gravity, Phys.Rev. D95(2017) 104042, [1703.01631]

In what follows we are not interested in these theories and focus on 1st-order and 2nd-order Lovelock gravity!

II) 4-dimensional Gauss-Bonnet gravity?



Glavan Lin's proposal 1

PHYSICAL REVIEW LETTERS **124,** 081301 (2020)

Einstein-Gauss-Bonnet Gravity in Four-Dimensional Spacetime

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In this Letter we present a general covariant modified theory of gravity in $D = 4$ spacetime dimensions which propagates only the massless graviton and bypasses Lovelock's theorem. The theory we present is formulated in $D > 4$ dimensions and its action consists of the Einstein-Hilbert term with a cosmological constant, and the Gauss-Bonnet term multiplied by a factor $1/(D-4)$. The four-dimensional theory is defined as the limit $D \to 4$. In this singular limit the Gauss-Bonnet invariant gives rise to nontrivial contributions to gravitational dynamics, while preserving the number of graviton degrees of freedom and being free from Ostrogradsky instability. We report several appealing new predictions of this theory, including the corrections to the dispersion relation of cosmological tensor and scalar modes, singularity resolution for spherically symmetric solutions, and others.

Glavan-Lin's proposal 2

Start with: EHGB action in D dimensions

$$S_D = \int d^D x \sqrt{-g}\left(R - 2\Lambda + \alpha \mathcal{G}\right)$$

Consider: Enhanced symmetry solutions (maximally symmetric spaces, spherical black holes, FLRW, …) by looking at a **few relevant equations.**

Observation: Remains interesting upon the **rescaling** of the coupling constant

$$\boxed{\alpha \to \alpha/(D-4)}$$

"dimensional regularization."

(Does it effect dof?)

$$\boxed{D \to 4?}$$

Glavan-Lin's proposal 3

Black hole solutions:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{D-2}^2\,,$$

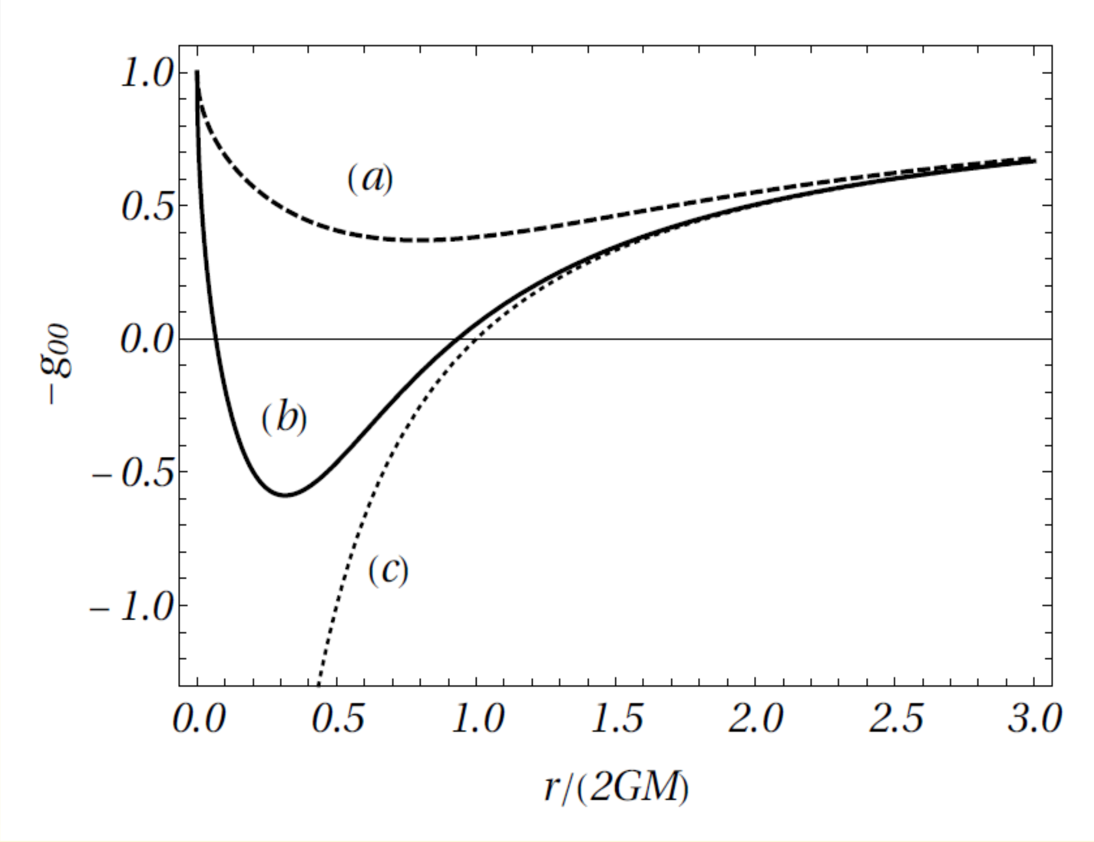
$$f_\pm(r) = 1 + \frac{r^2}{2(D-3)(D-4)\alpha}\left[1 \pm \sqrt{1 + \frac{64\pi G_D(D-3)(D-4)\alpha M}{(D-2)\Omega_{D-2}r^{D-1}} + \frac{8(D-3)(D-4)\alpha\Lambda_0}{(D-1)(D-2)}}\right]$$

- D->4 limit straightforward
- Minus branch is Schwarzschild-like

$$-g_{00} \overset{r\to\infty}{\sim} 1 - \frac{2GM}{r}$$

- "Singularity free"

$$R \propto r^{-3/2}$$



Problems

1) Is the D->4 limit well defined?

- M. Gurses, T. C. Sisman and B. Tekin, *Is there a novel Einstein-Gauss-Bonnet theory in four dimensions?*, 2004.03390.

Abstract: **No!**

- W.-Y. Ai, *A note on the novel 4D Einstein-Gauss-Bonnet gravity*, 2004.02858.

Moral: Theory **always remains higher-dimensional** (extra equations in higher dimensions)

Explicitly: Can be seen on Taub-NUT spacetimes:

- R. A. Hennigar, D. Kubiznak, R. B. Mann and C. Pollack, *On Taking the D->4 limit of Gauss-Bonnet Gravity: Theory and Solutions*, 2004.09472.

Different higher-dimensional bases (CP2, S2, T2) yield **different 4D Taub-NUTs** in the limit D->4.

Problems

2) Graviton scattering amplitudes:

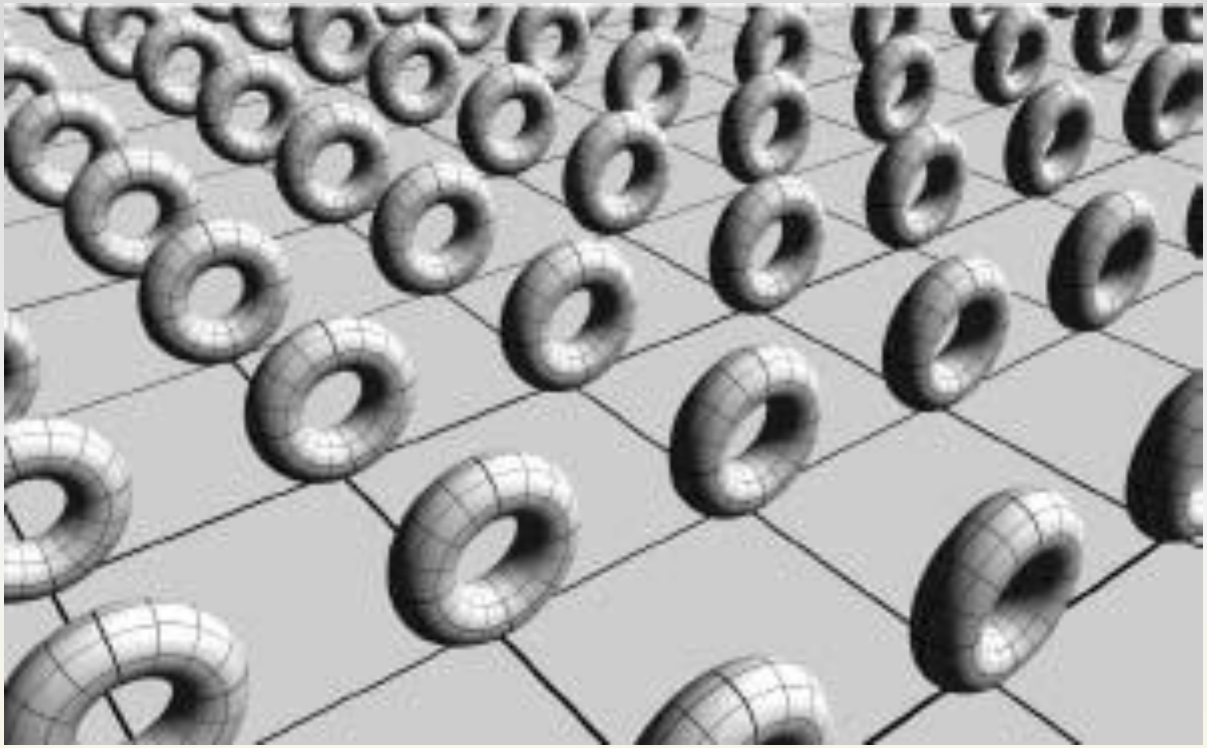
J. Bonifacio, K. Hinterbichler and L. A. Johnson, *Amplitudes and 4D Gauss-Bonnet Theory*, 2004.10716.

- There are **no** other than GR tree level graviton scattering amplitudes in 4D

- By taking the limit of higher-dimensional Gauss-Bonnet scattering amplitudes, one obtains amplitudes of a certain **scalar-tensor theory**

So, is there Gauss-Bonnet gravity in 4D?

III) Horndeski type Gauss-Bonnet gravity in 4D



https://publicism.info/science/elegant/9.html

Conformal trick: D->2 limit of GR

R. B. Mann and S. F. Ross, The D->2 limit of general relativity, Class. Quant. Grav. 10 (1993) 1405{1408, [gr-qc/9208004].

- Start with EH $$S_D^{\mathrm{EH}} = \kappa \int d^D x \sqrt{-g} R$$ (topological in D=2)
- Evaluate it for the **conformally rescaled** metric $\tilde{g} = e^{\psi} g$
$$S_D^{\mathrm{EH}} = \kappa \int d^D x \sqrt{-\tilde{g}} \tilde{R}$$
$$\tilde{R} = e^{-\psi} \left(R - \frac{(D-2)(D-1)}{4} (\partial \psi)^2 - (D-1) \Box \psi \right)$$
- **Expand** around $\epsilon = D - 2$, **rescale the coupling** according to
$$\boxed{\frac{1}{2}(D-2)\kappa \to \kappa}$$
and take the limit
$$D \to 2$$

Conformal trick: D->2 limit of GR

- The limit D->2 diverges, so we introduce a (in D=2 topological) **counterterm**:

$$S_D^{\mathrm{EH}} = \kappa \left(\int d^D x \sqrt{-\tilde{g}}\tilde{R} - \int d^D x \sqrt{-g} R \right)$$
$$= \kappa \int d^D x \sqrt{-g}\, e^{\epsilon\psi/2} \left[\left(R - (\epsilon+1)\Box\psi \right) - \frac{1}{4}\epsilon(\epsilon+1)(\partial\psi)^2 \right] - R \Big]$$

- Take the limit D->2, and throw away boundary terms, to obtain the following **scalar-tensor theory (JT gravity)**

$$S_2^{\mathrm{EH}} \equiv \lim_{\epsilon\to 0} S_D^{\mathrm{EH}} = \kappa \int d^2 x \sqrt{-g} \left[\psi R + \frac{1}{2}(\partial\psi)^2 \right]$$

- This theory has interesting **BH solutions**, whose metric is in some sense D->2 limit of D-dimensional Kerr solution, e.g.

A. M. Frassino, R. B. Mann and J. R. Mureika, Lower-Dimensional Black Hole Chemistry, Phys. Rev. D 92 (2015) 124069, [1509.05481].

Conformal trick: D->4 limit of GB

P. G. Fernandes, P. Carrilho, T. Clifton and D. J. Mulryne, *Derivation of Regularized Field Equations for the Einstein-Gauss-Bonnet Theory in Four Dimensions*, 2004.08362 & Paper 1

- Replace EH part with the GB part

$$S_D^{GB} = \alpha\left(\int d^D x\sqrt{-\tilde{g}}\tilde{\mathcal{G}} - \int d^D x\sqrt{-g}\mathcal{G}\right)$$

- Expand around $\epsilon = (D-4)$, rescale the coupling according to $(D-4)\alpha \to \alpha$, and take the limit D->4.
- After field redefinition: $\psi \to -2\phi\,, \quad g_{ab} \to -\frac{1}{2}g_{ab}\,, \quad \alpha \to \alpha/2$

$$S = \int d^4x\sqrt{-g}\Big[R - 2\Lambda + \alpha\Big(\phi\mathcal{G} + 4G^{ab}\partial_a\phi\partial_b\phi - 4(\partial\phi)^2\Box\phi + 2((\nabla\phi)^2)^2\Big)\Big]\,,$$

The new theory

$$S = \int d^p x \sqrt{-g}\Big[R - 2\Lambda + \alpha\Big(\phi\mathcal{G} + 4G^{ab}\partial_a\phi\partial_b\phi - 4(\partial\phi)^2\Box\phi + 2((\nabla\phi)^2)^2\Big)\Big],$$

- Is a **scalar-tensor** theory of **Horndeski-type**

  G. W. Horndeski, *Second-order scalar-tensor field equations in a four-dimensional space*, International Journal of Theoretical Physics 10 (1974) 363-384.

- It can also be derived by **Kaluza-Klein compactification** in the limit of vanishing extra dimensions:

$$ds_D^2 = ds_4^2 + e^{2\phi} d\Sigma_{D-4}^2$$

H. Lu and Y. Pang, *Horndeski Gravity as D->4 Limit of Gauss-Bonnet*, 2003.11552. (see also T. Kobayashi, 2003.12771)

Kaluza-Klein approach

H. Lu and Y. Pang, *Horndeski Gravity as D->4 Limit of Gauss-Bonnet*, 2003.11552. (see also T. Kobayashi, 2003.12771)

- Start with EHGB $$S_D = \int d^D x \sqrt{-g}\left(R - 2\Lambda + \hat{\alpha}\mathcal{G}\right)$$
- **Compactify** on

$$ds_D^2 = ds_p^2 + e^{2\phi} d\Sigma^2_{D-p,\,\lambda} \qquad R_{abcd} = \lambda(g_{ac}g_{bd} - g_{ad}g_{bc})$$

- Resultant **effective p-dimensional** action is

$$\begin{aligned} S_p = \frac{1}{16\pi G_p}\int d^p x\sqrt{-g}e^{(D-p)\phi}\Big\{ & R - 2\Lambda_0 + (D-p)(D-p-1)\big((\partial\phi)^2 + \lambda e^{-2\phi}\big) \\ & +\alpha\Big(\mathrm{GB} - 2(D-p)(D-p-1)\left[2G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \lambda R e^{-2\phi}\right] \\ & -(D-p)(D-p-1)(D-p-2)\left[2(\partial\phi)^2\Box\phi + (D-p-1)((\partial\phi)^2)^2\right] \\ & +(D-p)(D-p-1)(D-p-2)(D-p-3)\left[2\lambda(\partial\phi)^2 e^{-2\phi} + \lambda^2 e^{-4\phi}\right]\Big)\Big\}, \end{aligned}$$

Kaluza-Klein approach

- In $p \leq 4$ one can substract topological (zero) term

$$-\frac{\alpha}{16\pi G_p}\int d^p x\sqrt{-g}\,\mathrm{GB}$$

- Rescale the coupling alpha and take the limit:

$$\alpha \to \frac{\alpha}{D-p} \qquad D \to p$$

(Limit of **0-dim.** internal space)

Gauss-Bonnet in $p \leq 4$

$$S = \int d^p x\sqrt{-g}\Big[R - 2\Lambda + \alpha\Big(\phi\mathcal{G} + 4G^{ab}\partial_a\phi\partial_b\phi - 4(\partial\phi)^2\Box\phi + 2((\nabla\phi)^2)^2\Big)\Big],$$

$$S_\lambda = \int d^p x\sqrt{-g}\left(-2\lambda R e^{-2\phi} - 12\lambda(\partial\phi)^2 e^{-2\phi} - 6\lambda^2 e^{-4\phi}\right)$$

Solutions: GB Black Hole

H. Lu and Y. Pang, *Horndeski Gravity as D->4 Limit of Gauss-Bonnet*, 2003.11552.

- Constructed the SSS solutions of the **Horndeski-GB theory**

$$ds^2 = -f dt^2 + \frac{dr^2}{fh} + r^2 d\Omega^2$$

- **Special class** of solutions has $h = 1$

$$f_\pm = 1 + \frac{r^2}{2\alpha}\left(1 \pm \sqrt{1 + \frac{4}{3}\alpha\Lambda + \frac{8\alpha M}{r^3}}\right)$$

$$\phi_\pm = \int \frac{1 \pm \sqrt{f}}{\sqrt{f} r} dr$$

Metric **coincides** with the naïve D->4 limit. True also for non-trivial internal space curvature lambda.

Solutions: GB Black Hole

- The same spacetime considered as "**quantum gravity** corrected metric"
  - Y. Tomozawa, Quantum corrections to gravity, 1107.1424.
  - G. Cognola, R. Myrzakulov, L. Sebastiani and S. Zerbini, Einstein gravity with Gauss-Bonnet entropic corrections, Phys. Rev. D 88 024006, [1304.1878].
- **Have a theory** – so can use Wald's formalism to calculate **entropy**, which picks up *logarithmic corrections*:

$$S = \frac{1}{G_4}\left(\pi r_+^2 + 4\alpha\pi \log \frac{r_+}{L}\right)$$

- Many papers have studied observational features
  - Light bending
  - Black hole shadow
  - Thin accretion disc

Solutions: GB Taub-NUT?

R. A. Hennigar, D. Kubiznak, R. B. Mann and C. Pollack, *On Taking the D->4 limit of Gauss-Bonnet Gravity: Theory and Solutions*, 2004.09472.

$$ds^2 = -f(dt + 2n\cos\theta d\varphi)^2 + \frac{dr^2}{fh} + (r^2+n^2)d\Omega^2$$

- Concentrating on $h = 1$ (works for higher-dim GB)

δf $$\phi_\pm = \int \frac{\sqrt{f(3fn^2+n^2+r^2)} \pm rf}{f(r^2+n^2)}$$

δh Gives hope for D->4 limit of higher-dimensional GB Taub-NUTs, but!

$\delta\phi$ $$f = -\frac{1}{4}\frac{r^2+n^2}{l^2} \qquad \Lambda = \frac{3}{16}\frac{4n^2+\alpha}{n^4}\,, \quad l = \pm n$$

There is NO Lorentzian Taub-NUT solution! (with h=1)

Rotating Black Strings

R. A. Hennigar, D. Kubiznak, R. B. Mann, *Rotating and Charged Gauss-Bonnet BTZ black holes*, ArXiv:2005.13732.

$$ds^2 = -f_{GB}dt^2 + \frac{dr^2}{f_{GB}} + r^2 d\varphi^2 + r^2 dx^2\,, \quad \phi = \ln(r/l)\,,$$

$$f_{GB}^{\pm} = \frac{r^2}{2\alpha}\left(1 \pm \sqrt{1 - \frac{4\alpha}{r^2} f_E}\right), \quad f_E = \frac{r^2}{\ell^2} - \frac{2M}{r}\,, \qquad (29)$$

(see also Lin, Yang, Wei, Wang, Liu - ArXiv:2006.07913)

Performing the "boost"

$$t \to \Xi_{\rm eff} t - a\varphi\,,\ \varphi \to \frac{at}{\ell_{\rm eff}^2} - \Xi_{\rm eff}\varphi\,,\ \Xi_{\rm eff}^2 = 1 + \frac{a^2}{\ell_{\rm eff}^2}$$

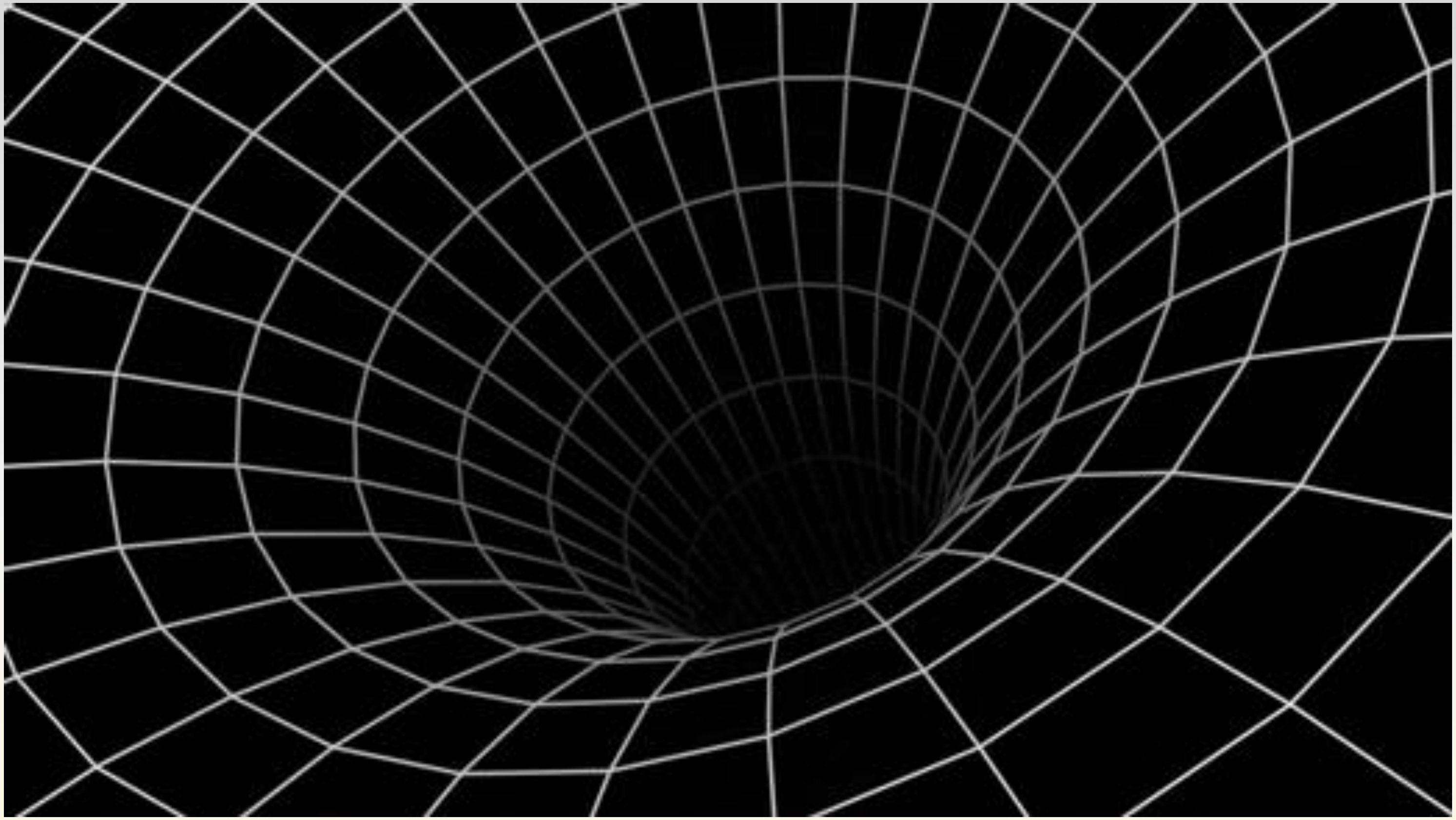
We recover:

$$ds^2 = -f_{GB}(\Xi_{\rm eff}dt - a d\varphi)^2 + \frac{r^2}{\ell_{\rm eff}^4}(a dt - \Xi_{\rm eff}\ell_{\rm eff}^2 d\varphi)^2 + \frac{dr^2}{f_{GB}} + r^2 dx^2\,,$$

Some properties of the theory

- **Is the theory well posed?** May be – probably not?
  - A.D. Kovacs, H.S. Reall, *Well-posed formulation of Lovelock and Horndeski theories*, PRD 101, 124003 (2020).
- **Asymptotic structure:** no propagating scalar dof
  - H. Lu, P. Mao, *Asymptotic structure of Einstein-Gauss-Bonnet theory in lower dimensions*, ArXiv:2004.14400.
- **Observational constraints**
  - T. Clifton, P. Carrilho, P.G.S. Fernandes, D.J. Muryne, *Observational constraints on the regularized 4D Einstein-Gauss-Bonnet theory of gravity*, ArXiv:2006.15017.
  - J-X. Feng, B-M Gu, F-W. Shu, *Theoretical and observational constraints on regularized 4D Einstein-Gauss-Bonner gravity*, ArXiv:2006.16751.

IV) Lower-dimensional Theory and Solutions



https://www.quantamagazine.org/tag/quantum-gravity/

New Horndeski 3D GB Gravity

$$S = \int d^p x \sqrt{-g} \Big[R - 2\Lambda + \alpha \Big(\phi \mathcal{G} + 4 G^{ab} \partial_a \phi \partial_b \phi - 4 (\partial\phi)^2 \Box \phi + 2((\nabla\phi)^2)^2 \Big) \Big] \,,$$

The theory is applicable in p=3 dimensions

- Obtained by D->3 **KK procedure** with 0-dimensional internal space

  H. Lu and Y. Pang, *Horndeski Gravity as D->4 Limit of Gauss-Bonnet*, 2003.11552.

- Can equally be obtained via the **conformal trick** in D=3 dimensions

  R. A. Hennigar, D. Kubiznak, R. B. Mann and C. Pollack, *Lower-dimensional Gauss-bonnet gravity and BTZ black holes*, 2004.12995.

Gauss-Bonnet BTZ Black Holes

R. A. Hennigar, D. Kubiznak, R. B. Mann and C. Pollack, *Lower-dimensional Gauss-bonnet gravity and BTZ black holes*, 2004.12995.

$$ds^2 = -f_{GB}dt^2 + \frac{dr^2}{f_{GB}} + r^2 d\varphi^2 , \quad \phi = \ln(r/l) ,$$

$$f_{GB}^{\pm} = -\frac{r^2}{2\alpha}\left(1 \pm \sqrt{1 + \frac{4\alpha}{r^2} f_E}\right) ,$$

(see also Konoplya & Zhidenko, 2003.12171)

- Minus branch describes a BH which is asymptotically BTZ (with modified Lambda)

- It has **identical thermodynamics** to standard BTZ black hole (entropy calculated through Wald formalism)

$$T = \frac{f_E'(r_+)}{4\pi} = \frac{f_{GB}'(r_+)}{4\pi}$$

- It has singularity in the origin (repulsive potential)

Charged Gauss-Bonnet BTZ Black Holes

$$f_{GB} = -\frac{r^2}{2\alpha}\left(1 \pm \sqrt{1 + \frac{4\alpha}{r^2} f_E}\right)$$

$$f_E = \frac{r^2}{\ell^2} - m - 2e^2 \ln(r/r_0)\,, \quad A = -e \ln(r/r_0) dt$$

- The same form for NLE, for example BI theory:

$$\mathcal{L}_{BI} = -b^2\left(\sqrt{1 + \frac{2F}{b^2}} - 1\right)$$

R. A. Hennigar, D. Kubiznak, R. B. Mann, Rotating and Charged *Gauss-Bonnet BTZ black holes*, 2005.13732.

Rotating Gauss-Bonnet BTZ BH

- Constructed by **boost** on static solution

$$t \to \Xi_{\rm eff} t - a\varphi\,, \;\; \varphi \to \frac{at}{\ell_{\rm eff}^2} - \Xi_{\rm eff}\varphi\,, \;\; \Xi_{\rm eff}^2 = 1 + \frac{a^2}{\ell_{\rm eff}^2}$$

$$ds^2 = -f_{GB}(\Xi_{\rm eff}dt - a d\varphi)^2 + \frac{r^2}{\ell_{\rm eff}^4}(a dt - \Xi_{\rm eff}\ell_{\rm eff}^2 d\varphi)^2 + \frac{dr^2}{f_{GB}}$$

$$\phi = \ln(r/l)\,,$$

- Have ergoregion and outer horizon, but **no Cauchy horizon** (eliminated by singularity – cosmic censorship?)

- For $\alpha>0$, funny relation to ADM type coordinates

- **TDs no longer universal** – distinct from the Einstein case

R. A. Hennigar, D. Kubiznak, R. B. Mann, *Rotating and Charged Gauss-Bonnet BTZ black holes*, 2005.13732.

Summary

1) Recently, a lot of interest in **4D Gauss-Bonnet Gravity**.

2) The original proposal by Glavan and Lin of taking the **naïve D->4 limit** of solutions/EOMs while rescaling the coupling constant

$$\alpha \to \alpha/(D-4)$$

is interesting but **does not work** – not a 4D metric theory.

3) However, one can make sense of the limit in two alternative scenarios – by **KK compactification** and by **conformal trick**. The resultant theory is a **scalar-tensor theory of Horndeski type**.

$$S = \int d^p x \sqrt{-g}\Big[R - 2\Lambda + \alpha\Big(\phi\mathcal{G} + 4G^{ab}\partial_a\phi\partial_b\phi - 4(\partial\phi)^2\Box\phi + 2((\nabla\phi)^2)^2\Big)\Big]\,, \qquad \boxed{p=4,3,2}$$

4) **<u>Generalizations</u>**
e.g. Easson, Manton Svesko - ArXiv:2005.12292 (includes dilaton); Colleaux - ArXiv:2010.14174.

Summary

5. **The theory is "nice"** as
   a) It is derived from a more fundamental theory (c.f. EdGB, or other Horndeski theories)

   b) Admits **interesting analytic solutions**, some of which coincide with the naïve D->4 limit.

   For example: **4D BH solution**

$$f_{\pm} = 1 + \frac{r^2}{2\alpha}\left(1 \pm \sqrt{1 + \frac{4}{3}\alpha\Lambda + \frac{8\alpha M}{r^3}}\right)$$

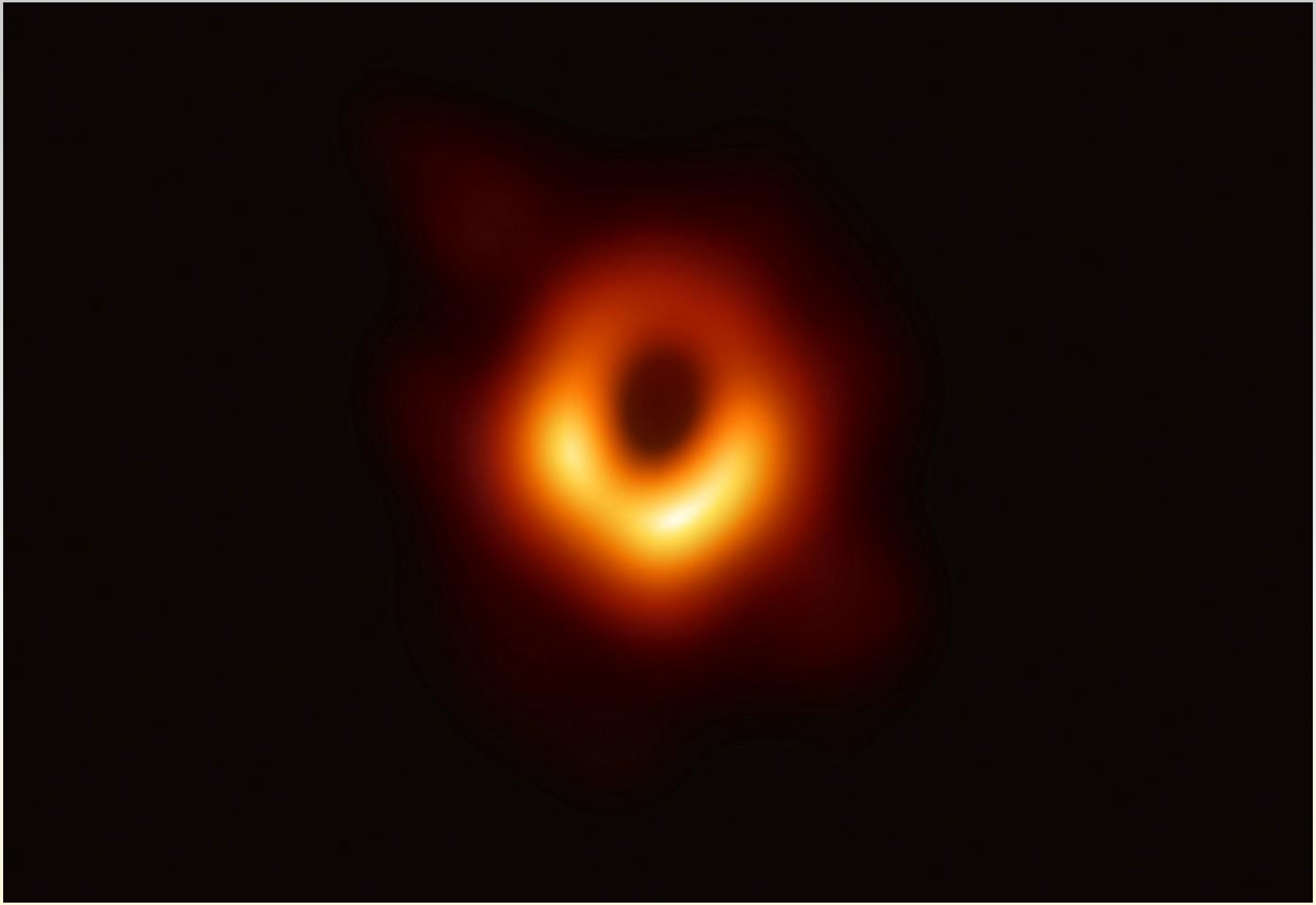
   Seem interesting from both **theoretical** and **observational** aspects (logarithmic corrections to *entropy*, "removal of *singularities*", observational predictions)

   c) Is it well posed?

6. **Alternatives**: e.g. breaking the temporal diffeomorphism invariance, see Aoki, Gorji, Mukohyama - ArXiv:2005:03859.

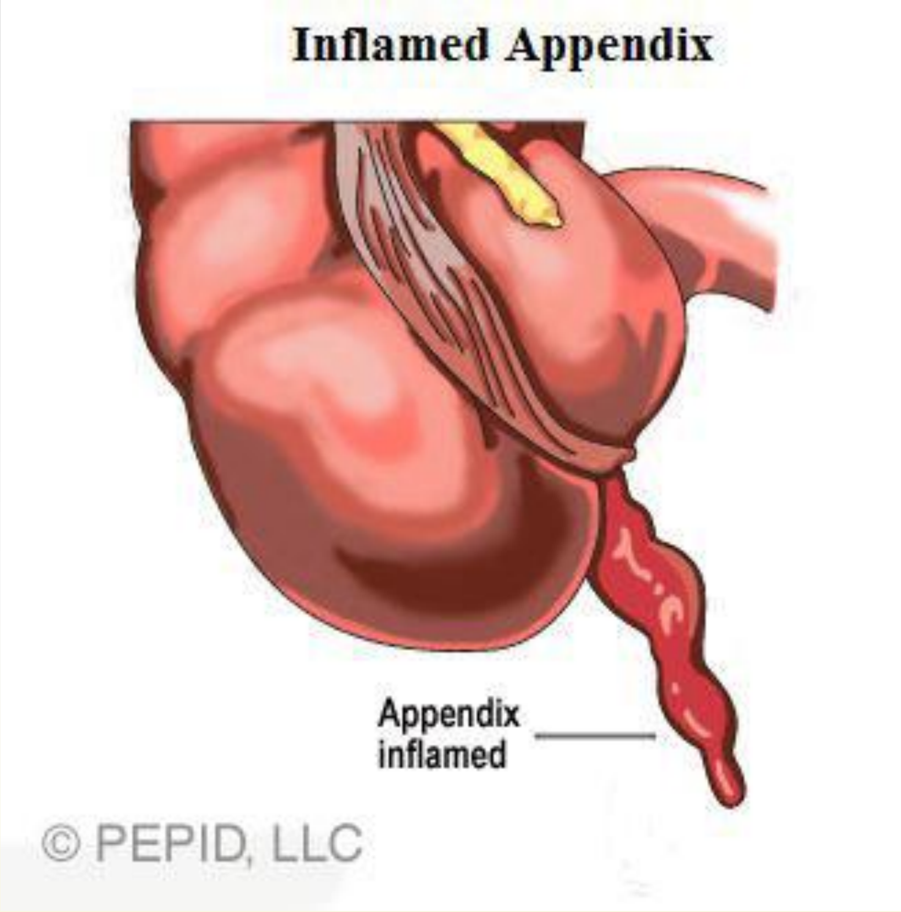
Summary

7. **Is there Gauss-Bonnet gravity in the sky?**



https://medium.com/predict/black-holes-shadow-seen-for-the-first-time-46578a7a2787

Appendices



Equations of motion

$$
\begin{aligned}
\mathcal{E}_\phi = & -\mathcal{G} + 8G^{ab}\nabla_b\nabla_a\phi + 8R^{ab}\nabla_a\phi\nabla_b\phi - 8(\Box\phi)^2 + 8(\nabla\phi)^2\Box\phi + 16\nabla^a\phi\nabla^b\phi\nabla_b\nabla_a\phi \\
& + 8\nabla_b\nabla_a\phi\nabla^b\nabla^a\phi - 24\lambda^2 e^{-4\phi} - 4\lambda R e^{-2\phi} + 24\lambda e^{-2\phi}\left[(\nabla\phi)^2 - \Box\phi\right] \\
= & \, 0\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}_{ab} = & \Lambda g_{ab} + G_{ab} + \alpha\Big[\phi H_{ab} - 2R\left[(\nabla_a\phi)(\nabla_b\phi) + \nabla_b\nabla_a\phi\right] + 8R^c_{(a}\nabla_{b)}\nabla_c\phi + 8R^c_{(a}(\nabla_{b)}\phi)(\nabla_c\phi) \\
& - 2G_{ab}\left[(\nabla\phi)^2 + 2\Box\phi\right] - 4\left[(\nabla_a\phi)(\nabla_b\phi) + \nabla_b\nabla_a\phi\right]\Box\phi - \left[g_{ab}(\nabla\phi)^2 - 4(\nabla_a\phi)(\nabla_b\phi)\right](\nabla\phi)^2 \\
& + 8(\nabla_{(a}\phi)(\nabla_{b)}\nabla_c\phi)\nabla^c\phi - 4g_{ab}R^{cd}\left[\nabla_c\nabla_d\phi + (\nabla_c\phi)(\nabla_d\phi)\right] + 2g_{ab}(\Box\phi)^2 - 2g_{ab}(\nabla_c\nabla_d\phi)(\nabla^c\nabla^d\phi) \\
& - 4g_{ab}(\nabla^c\phi)(\nabla^d\phi)(\nabla_c\nabla_d\phi) + 4(\nabla_c\nabla_b\phi)(\nabla^c\nabla_a\phi) + 4R_{acbd}\left[(\nabla^c\phi)(\nabla^d\phi) + \nabla^d\nabla^c\phi\right] \\
& + 3\lambda^2 e^{-4\phi}g_{ab} - 2\lambda e^{-2\phi}\left(G_{ab} + 2(\nabla_a\phi)(\nabla_b\phi) + 2\nabla_b\nabla_a\phi - 2g_{ab}\Box\phi + g_{ab}(\nabla\phi)^2\right)\Big] \\
= & \, 0\,,
\end{aligned}
$$

Interesting consequence:

$$
0 = g^{ab}\mathcal{E}_{ab} + \frac{\alpha}{2}\mathcal{E}_\phi = 4\Lambda - R - \frac{\alpha}{2}\mathcal{G}
$$