Geometric secret sharing in a model of Hawking radiation

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Based on 2003.05448
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Plan

1. Quantum extremal surfaces and the Page curve (review)

2. A simple geometric model for black holes
   entangled with radiation

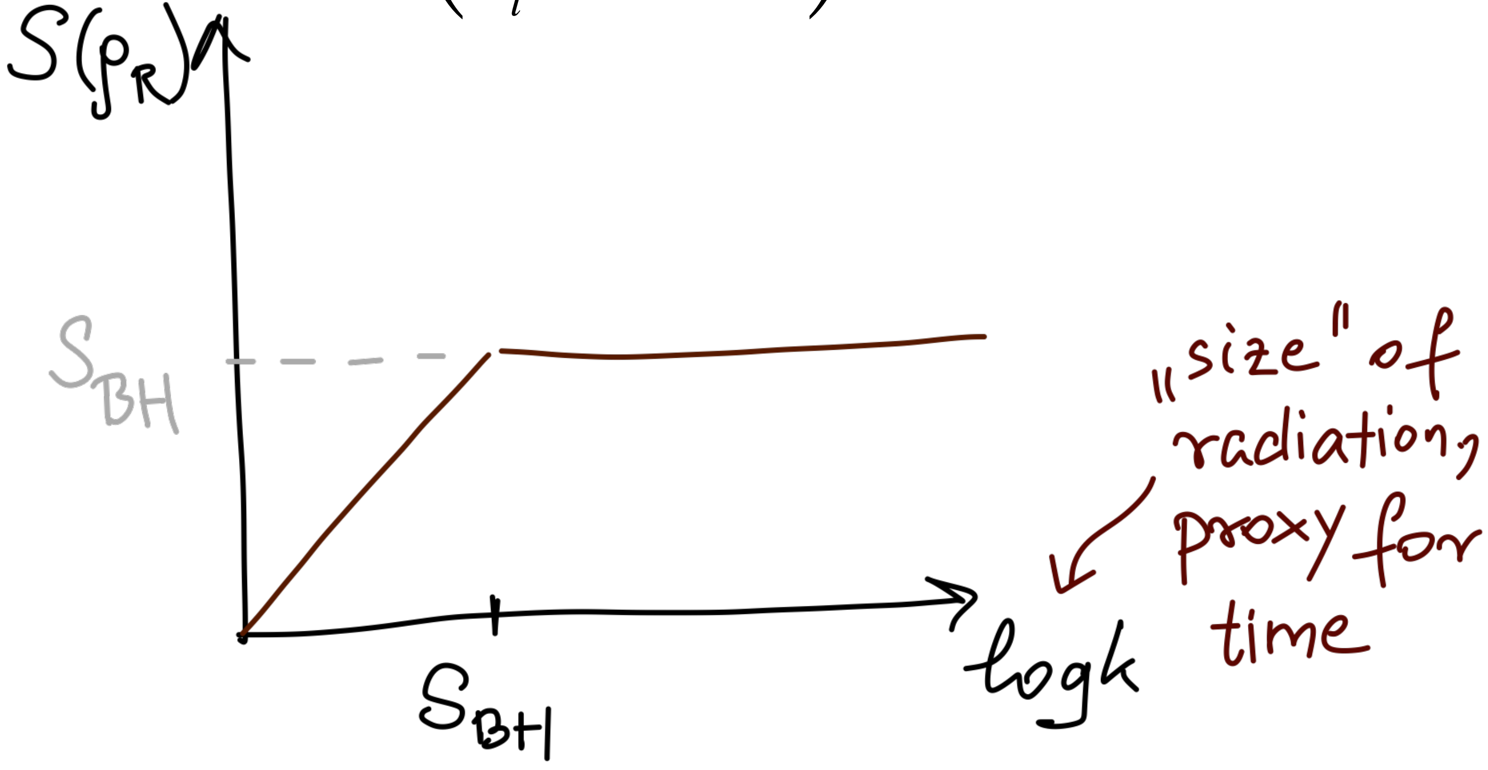
3. Shared secrets as subregions in the interior

Page curve

Simplified context: Black hole entangled with radiation

$$\sum_{i=1}^{k} \underbrace{|i\rangle_R}_{\text{states of radiation}} \underbrace{|\psi_i\rangle_A}_{\text{random black hole microstates from microcan. window}}$$

$$S(\rho_R) \overset{\text{purity of total state}}{=} S\left(\sum_i |\psi_i\rangle\langle\psi_i|_A\right) \le S_{BH}$$

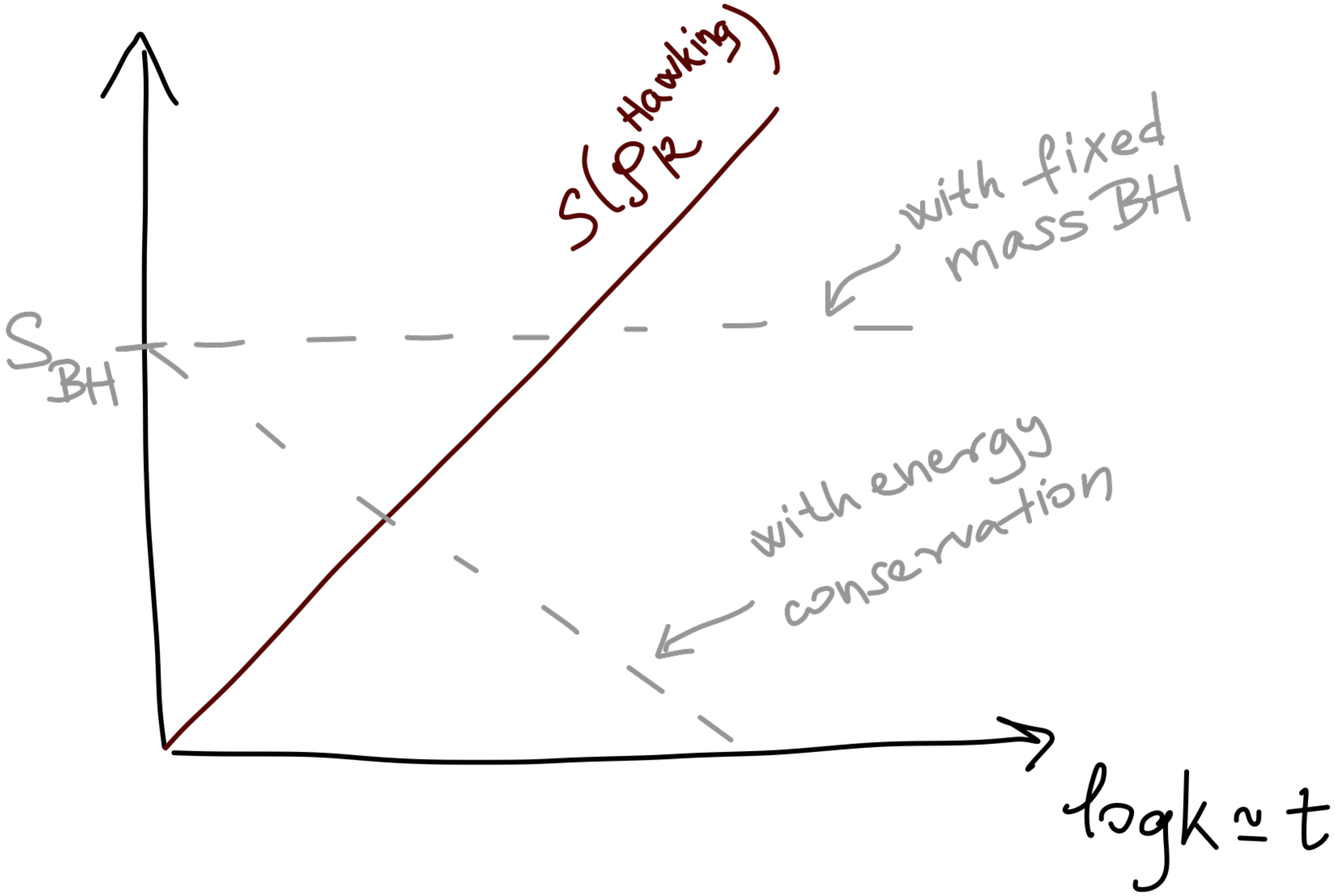


Page curve

BH information problem:

Radiation is kept being produced in the state

$$\rho_R^{\text{Hawking}} = \sum_i^k |i\rangle\langle i|_R$$



Page curve

Naively, it seems to understand the saturation/turnaround, one needs to understand black hole microstates.

However, recent work suggests it is enough to use Ryu-Takayanagi on steroids.

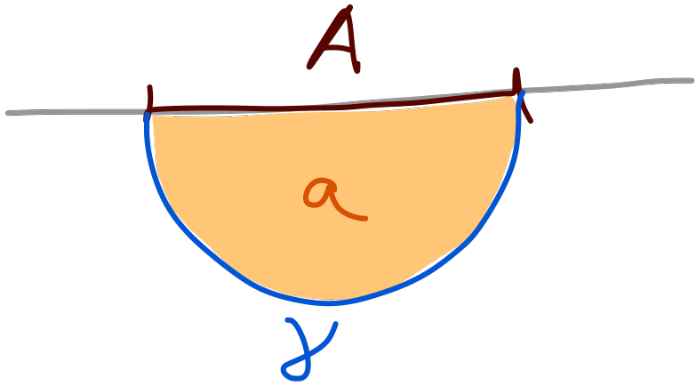
[Penington,Almheiri-Engelhardt-Marolf-Maxfield, Almheiri-Mahajan-Maldacena-Zhao, etc]

Ryu-Takayanagi:

$$S(\rho_A) = \textbf{min ext} \left[\frac{\mathscr{A}(\gamma)}{4G_N} \right]$$

Faulkner-Lewkowycz-Maldacena:

$$S(\rho_A) = \textbf{min ext} \left[\frac{\mathscr{A}(\gamma)}{4G_N} \right] + S(\rho_a^{\text{bulk}})$$



Engelhardt-Wall:

$$S(\rho_A) = \textbf{min ext} \left[\frac{\mathscr{A}(\gamma)}{4G_N} + S(\rho_a^{\text{bulk}}) \right]$$

What saves the day are surfaces that are not extremal, but this combination is extremal: Quantum extremal surfaces

Reminder: replica trick derivation

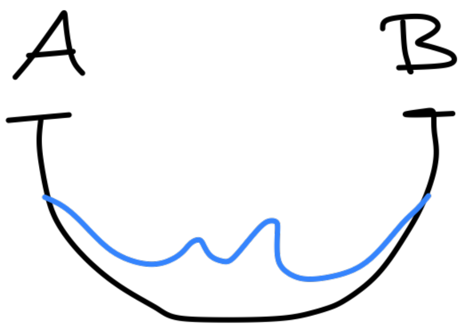
[Lewkowycz-Maldacena;Faulkner-Lewkowycz-Maldacena;Dong-Lewkowycz]

Replica trick:

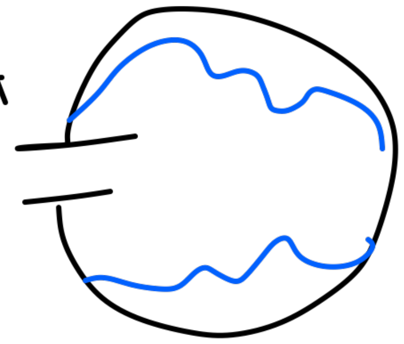
$$S(\rho) = -\,\mathbf{Tr}\rho\log\rho = \lim_{n\to 1}\partial_n \frac{\log[\mathbf{Tr}R^n]}{n} \text{ for any } R \text{ such that } \rho = \frac{R}{\mathbf{Tr}R}$$

1) Create a CFT state by Euclidean path integral with sources:

$|\psi\rangle \sim$



$\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi| \sim$



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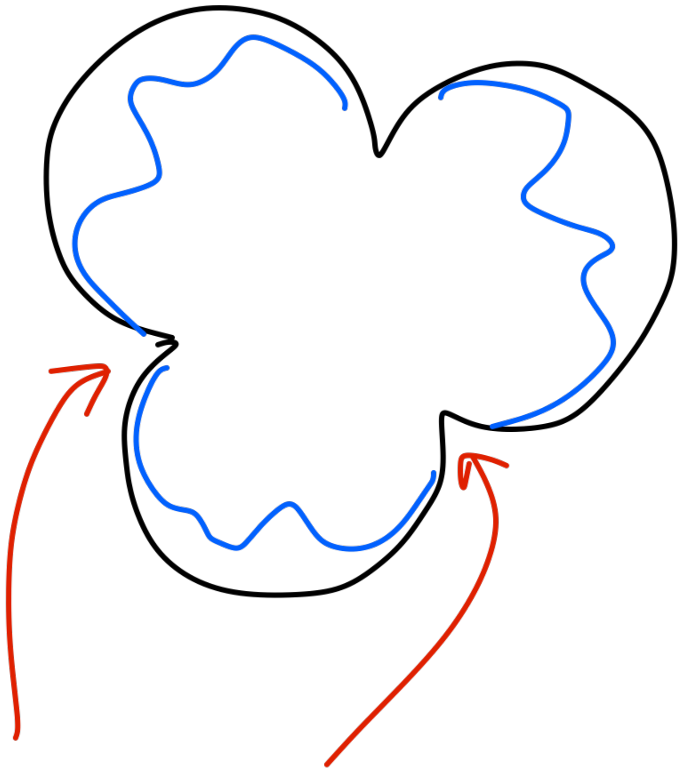
2) Calculate Renyi entropy as a partition function

$$\mathrm{Tr}\rho_A^3 \sim$$



on entangling surface: conical singularity $\Delta\phi = 2\pi(n-1)$

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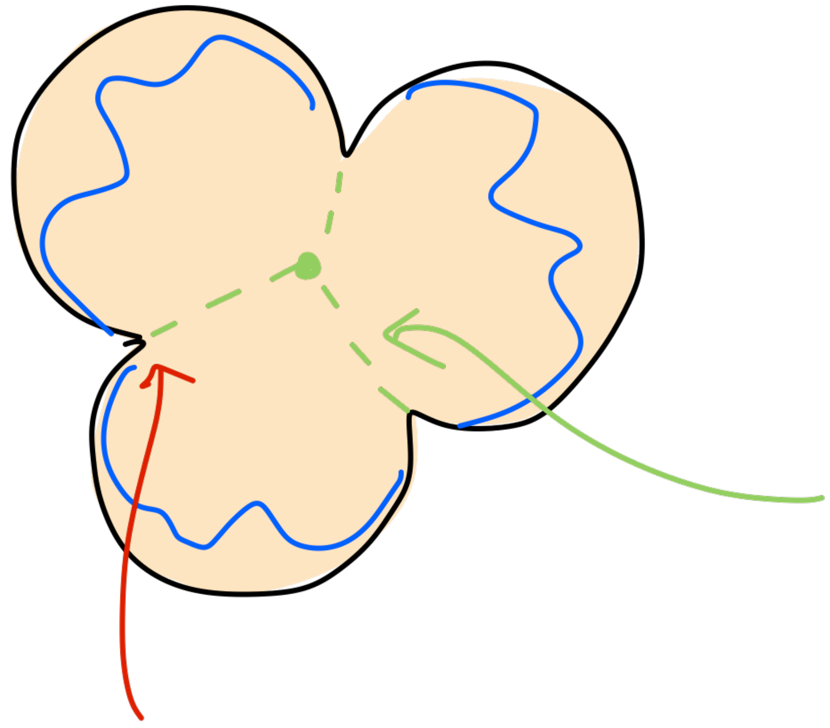
3) Fill in the bulk

$$\mathrm{Tr}\rho_A^3 \sim$$



We assume solution doesn't break $\mathbb{Z}_3$

in the bulk, conical singularity smoothed out

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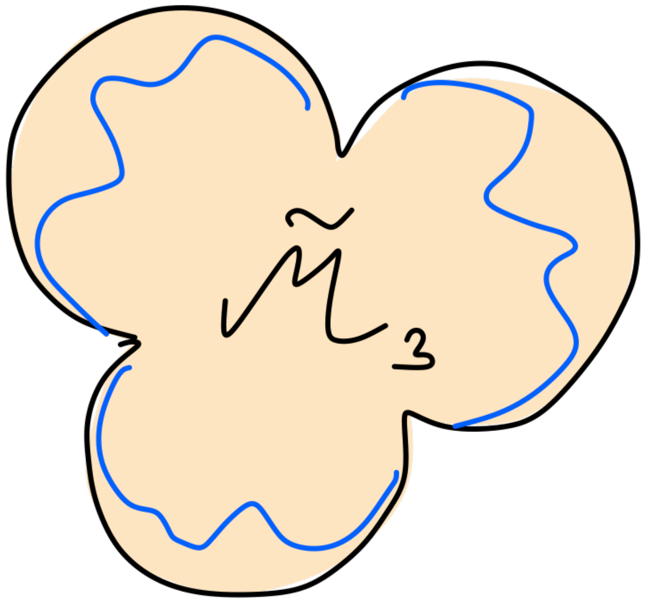

$$\log \mathbf{Tr}\rho^n \sim -\,I_{\text{grav}}(\tilde{\mathcal{M}}_n) + \log Z_{\text{matter}}(\tilde{\mathcal{M}}_n)$$

Reminder: replica trick derivation

[Lewkowycz-Maldacena;Faulkner-Lewkowycz-Maldacena;Dong-Lewkowycz]

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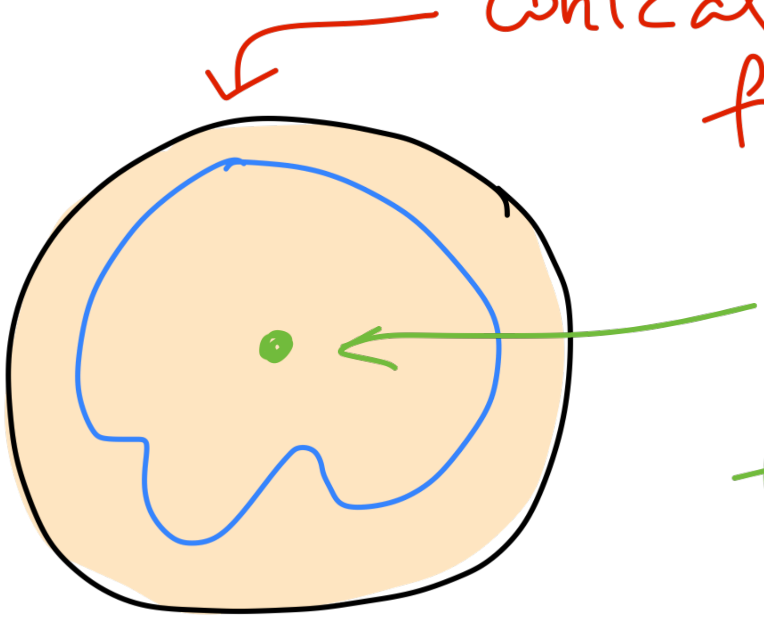
3) Orbifold the bulk

$$\mathcal{M}_n = \frac{\tilde{\mathcal{M}}_n}{\mathbb{Z}_n}$$



conical excess removed from bndy

conical deficit introduced at fixed p.t. of $\mathbb{Z}_n$

$$\Delta\phi = 2\pi\left(1-\frac{1}{n}\right)$$

$$I_{\text{grav}}(\tilde{\mathscr{M}}_n) = nI_{\text{grav}}^{\text{sing.omitted}}(\mathscr{M}_n)$$

$$= nI_{\text{grav}}^{\text{full}}(\mathscr{M}_n) + \frac{1}{4G_N}(n-1)\int_{\text{sing}}\sqrt{h}$$

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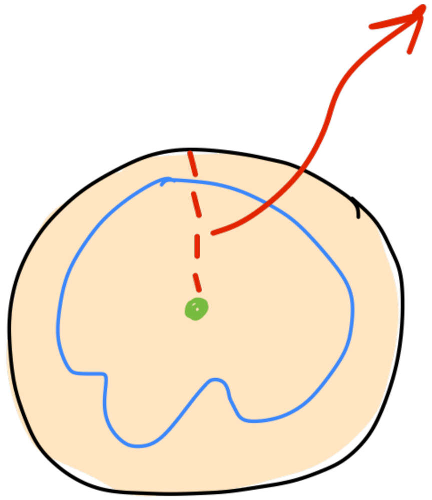
4) Analyse around n=1

$$\frac{1}{n}\log\mathbf{Tr}\rho^n|_{n\approx1} \sim -\,I_{\text{grav}}(\mathcal{M}_1) - \cancel{\delta_n I_{\text{grav}}(\mathcal{M}_1)} - \frac{1}{4G_N}(n-1)\int_{\text{sing}}\sqrt{h}(\mathcal{M}_1) + (1-n)S_{\text{matter}}$$

- **Off shell metric variations of order $(n-1)$ automatically pushed to higher order**

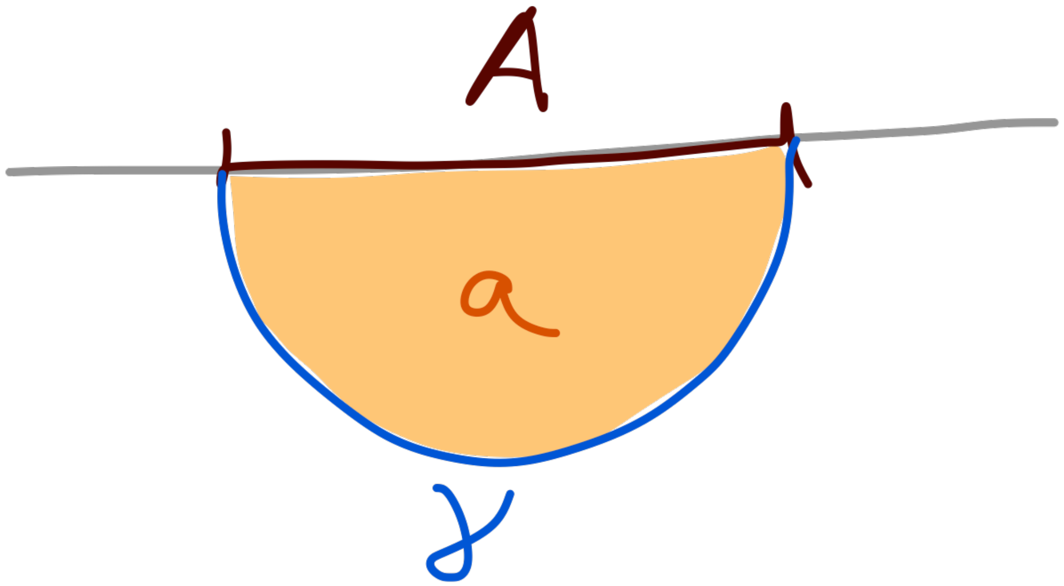
- **Position of brane is dynamical! Above is extremized over it**



$$S(\rho) = -\,\mathbf{Tr}\rho\log\rho = \textbf{ext}\left(\frac{\mathscr{A}}{4G_N} + S_{\text{matter}}\right)$$

Entanglement wedge reconstruction

(Subregion-subregion duality)



The geometry and the bulk quantum state on a

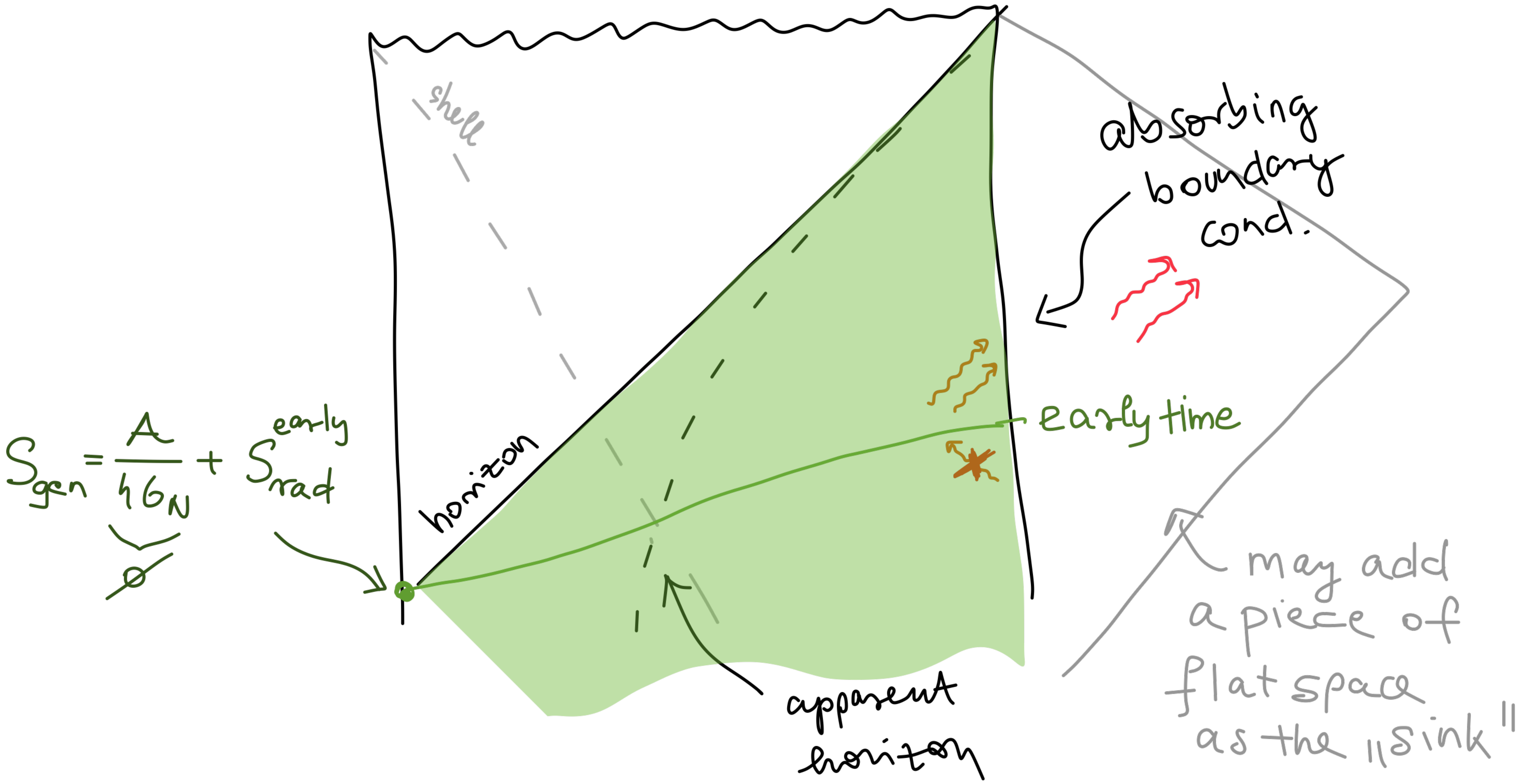
is reconstructible from the boundary state on A

where γ is the quantum extremal surface

[Jafferis-Lewkowycz-Maldacena-Suh;Dong-Harlow-Wall]

Quantum extremal surfaces and the Page curve

Sketch of the argument: [Penington]



Quantum extremal surfaces and the Page curve

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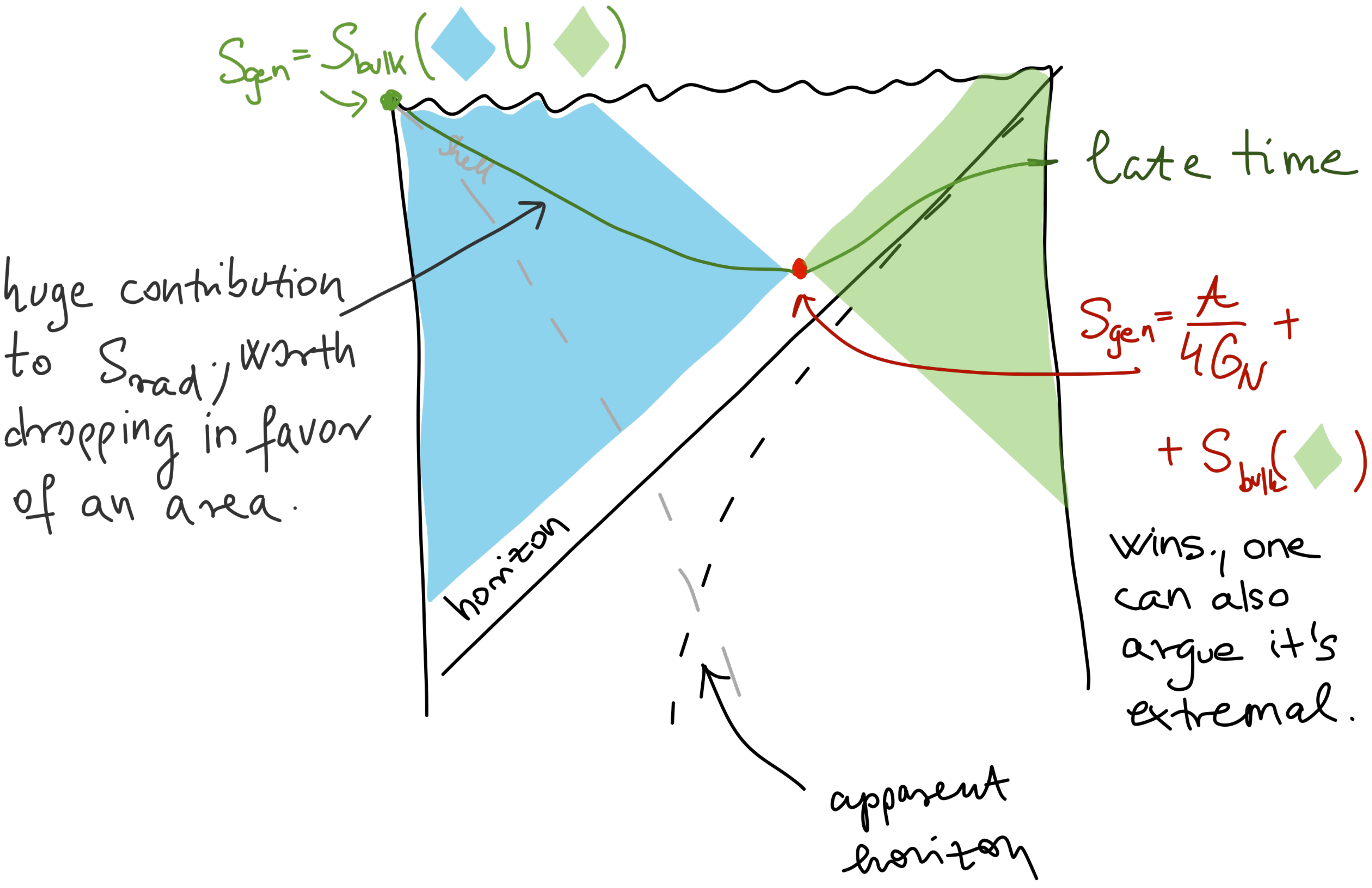


$S_{gen} = S_{bulk}($◆ $\cup$ ◆$)$

huge contribution to S_{rad}, worth dropping in favor of an area.

$S_{gen} = \frac{A}{4G_N} + S_{bulk}($◆$)$ wins, one can also argue it's extremal.

Page curve

Somehow (upgraded) RT formula knows when too many microstates would become entangled with the radiation

Various other works with more precise setups involving JT gravity, replica derivation, e.t.c
[Almheiri-Hartman-Maldacena-Shaghoulian-Tajdini;Penington-Shenker-Stanford-Yang]

Today: simple toy model in 3d showing how the RT formula knows that in

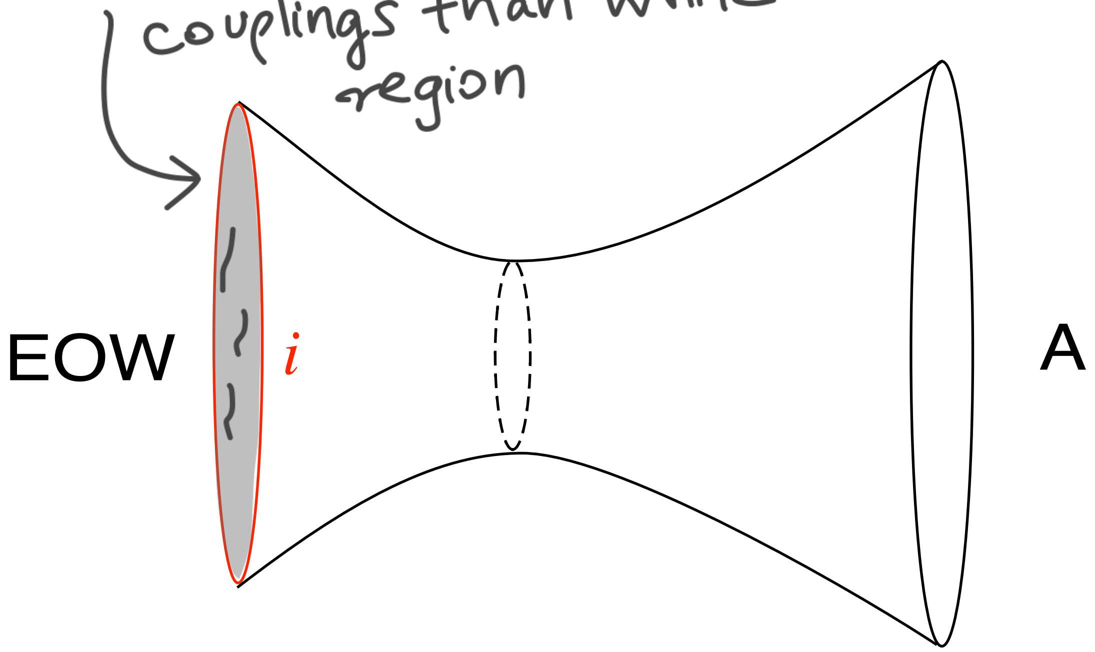
$$\sum_{i=1}^{k} |i\rangle_R |\psi_i\rangle_A$$

the states $|\psi_i\rangle_A$ cannot be all orthogonal once $k > e^{S_{BH}}$

A model of microstates

"inception disk", can have different couplings than white region

$$|\psi_i\rangle_A \sim$$



Put an EOW behind the horizon [Maldacena-Kourkoulou;...]

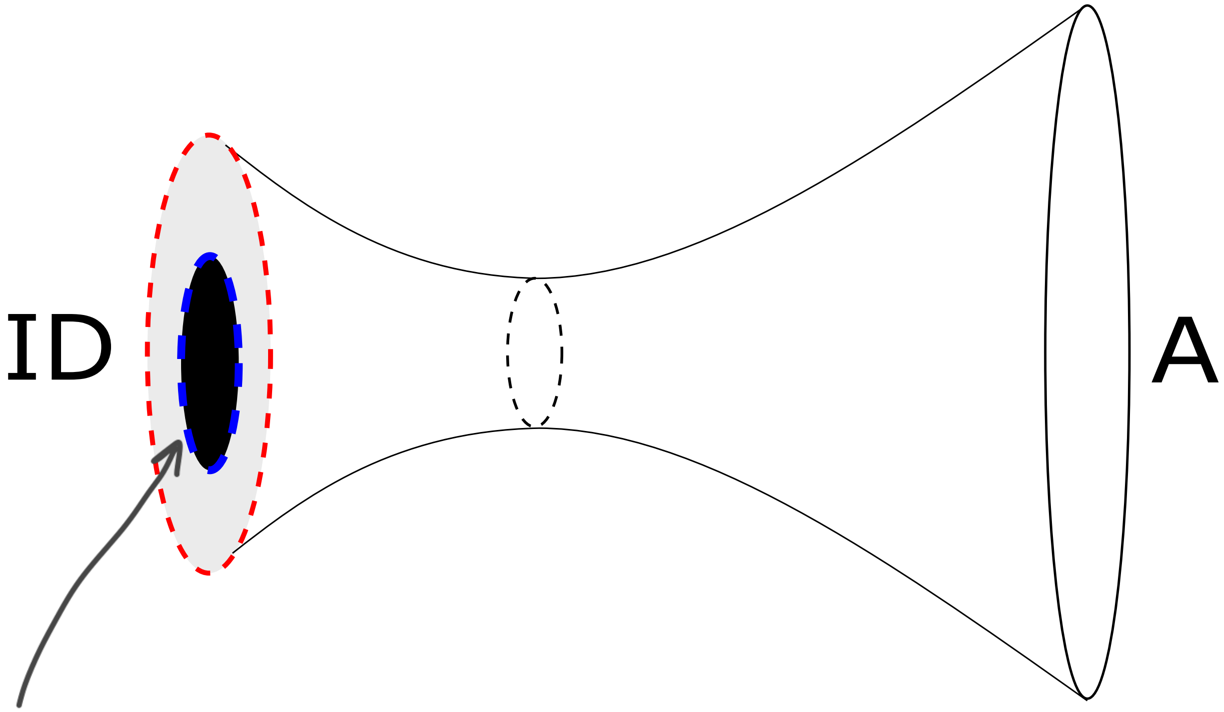
Label i runs over the states of the theory on the brane

Take this brane theory to be holographic
and fill in the brane with the dual
(results in a fuzzball-like geometry)

Entangled microstates

$$\sum_{i=1}^{k} |i\rangle_R |\psi_i\rangle_A$$

$\rho_A \sim$



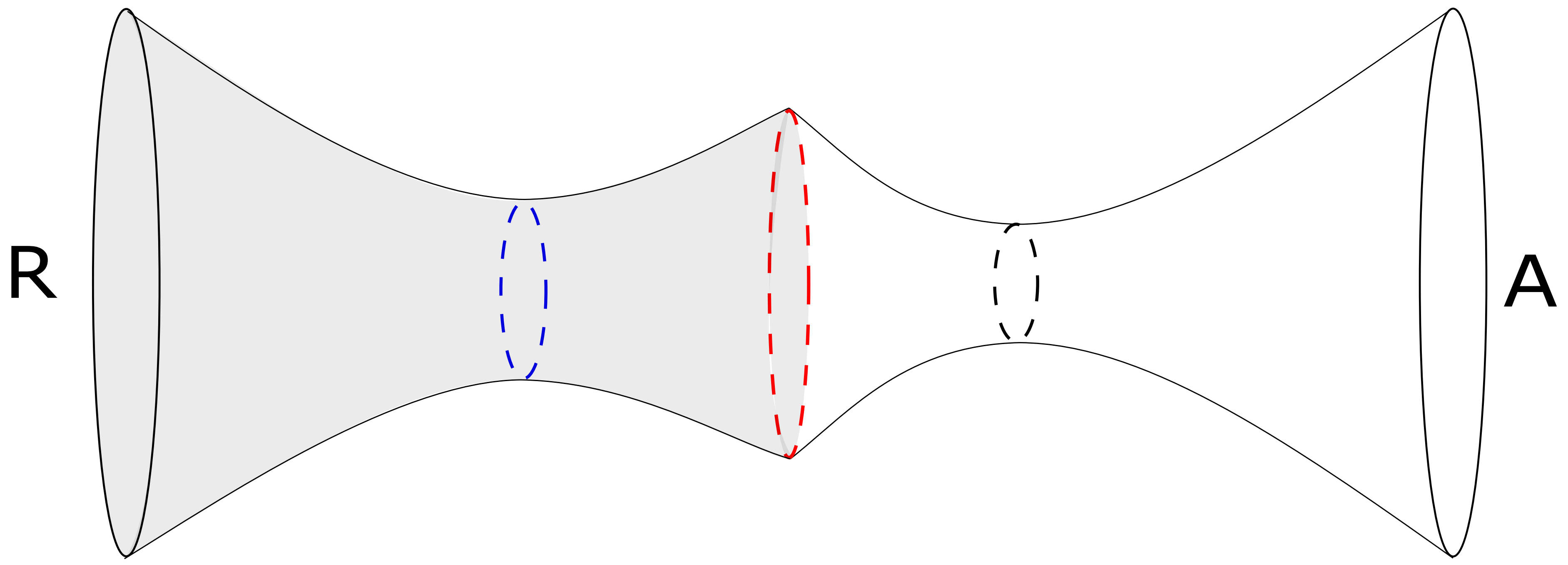
now there is
a black hole
inside the inception
geometry

Entangled microstates

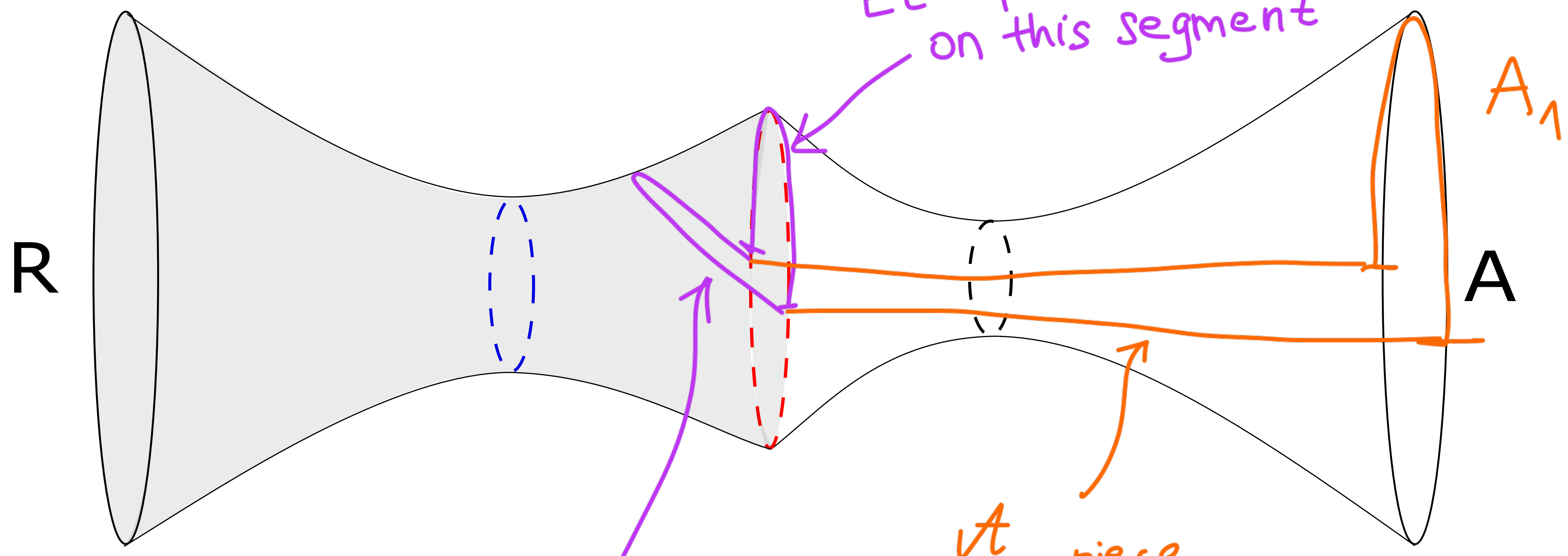
$$\sum_{i=1}^{k} |\, i\rangle_R \,|\, \psi_i\rangle_A$$

A possible choice of purification:

Take R to be the UV CFT of the brane theory and $|\, i\rangle_R$ to be from a microcanonical window of eigenstates



Quantum extremal surfaces



EE of brane d.o.f. on this segment

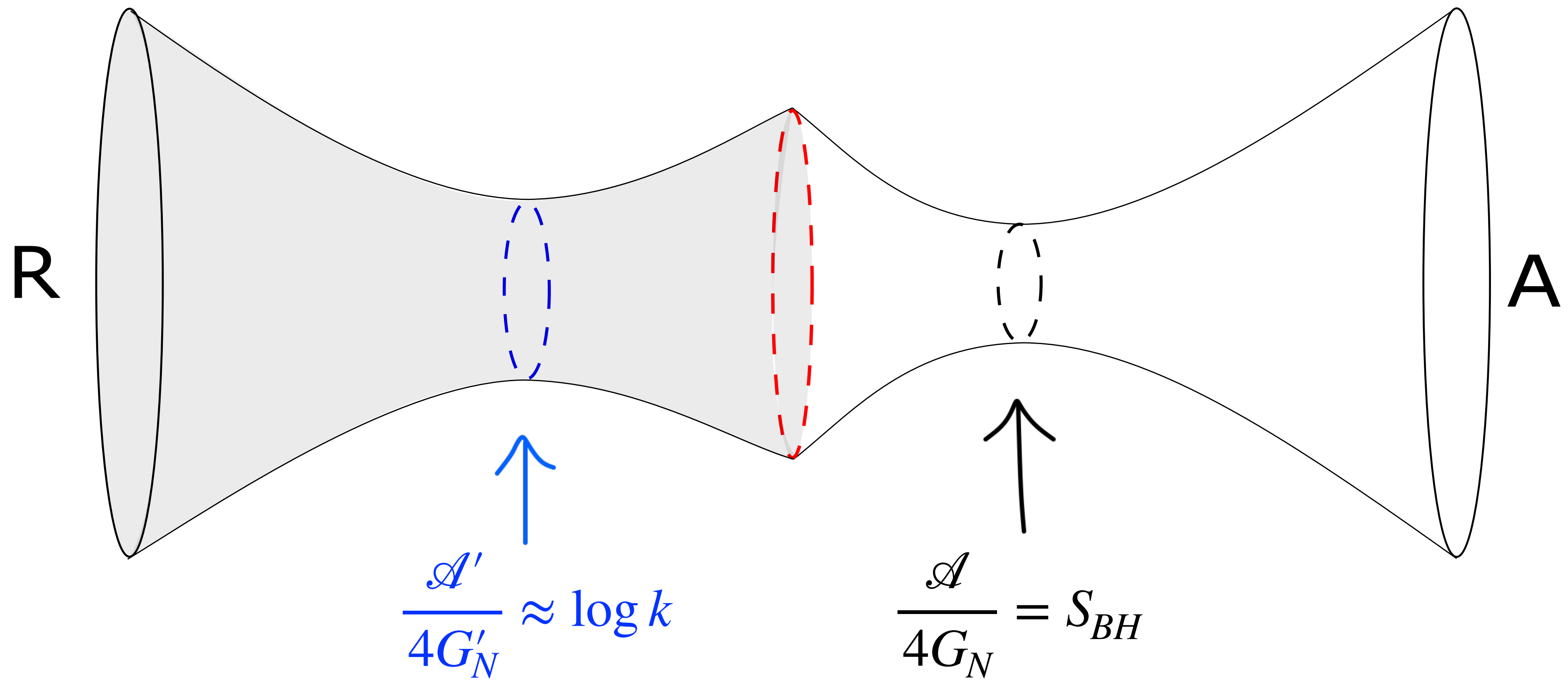
this EE is calculated via RT in inception side

$\frac{\mathscr{A}}{4G_N}$ piece

$$S(A_1) = \textbf{min ext}\left[\frac{\mathscr{A}}{4G_N} + S(\textbf{brane segment})\right]$$

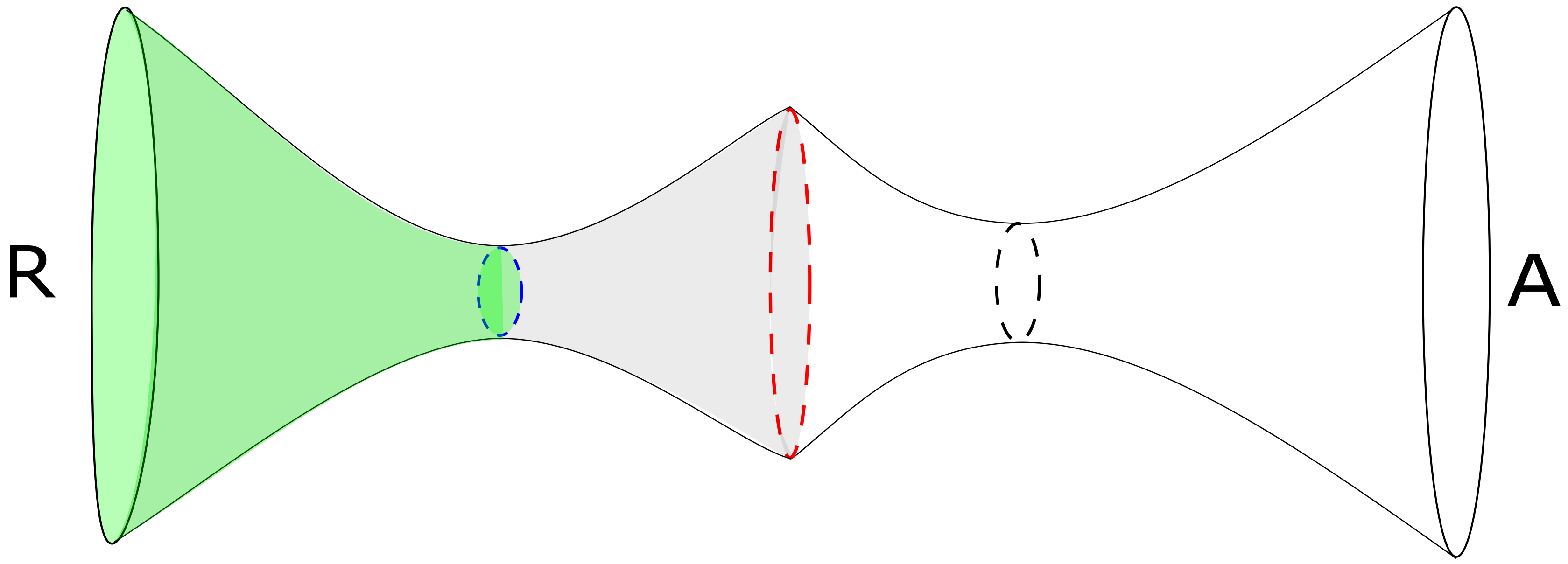
$$= \textbf{min ext}\left[\frac{\mathscr{A}}{4G_N} + \frac{\mathscr{A}'}{4G'_N}\right]$$

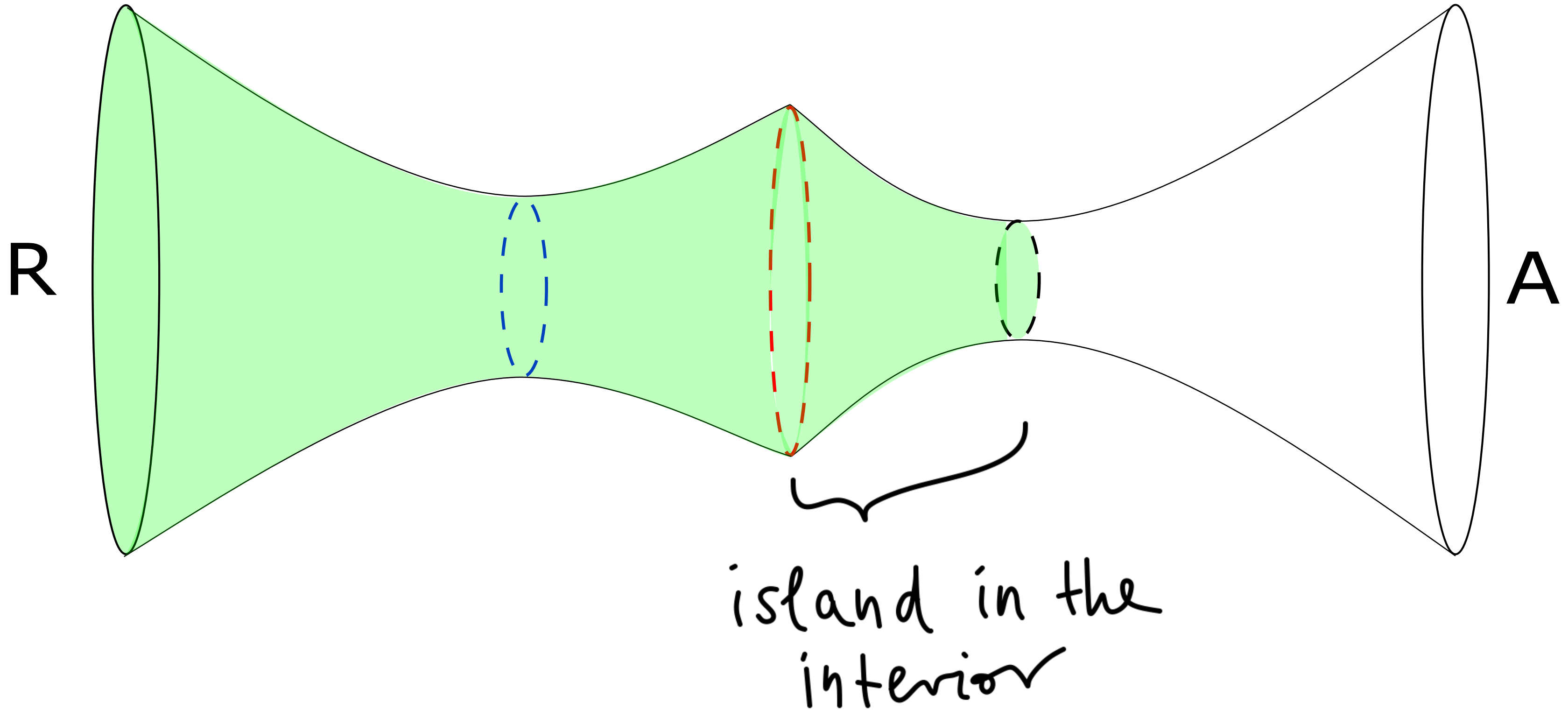
Page transition



$$S(A) = S(R) = \textbf{min}\{\log k, S_{BH}\}$$

Entanglement wedge of the radiation





Gluing the inception geometry

So far we talked about states dual to geometries on a Cauchy slice

To motivate the RT rule, one also needs a Euclidean preparation

There is no top-down way of doing this, but also the details don't seem to matter as long as there is a long wormhole with two extremal surfaces

The way we think about the glued geometry motivates the junction conditions:

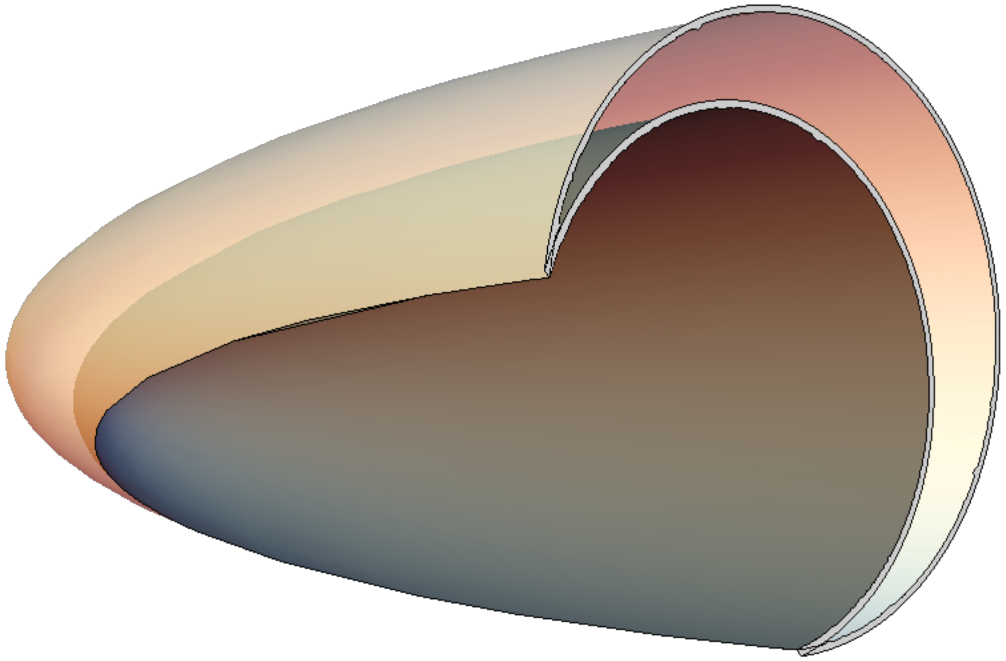
$$h_{ab} = h'_{ab} \qquad \underbrace{\frac{1}{G_N}K_{ab}}_{\text{Neumann b.c.: } = \frac{\delta S_{\text{brane}}}{\delta h_{ab}}} = \frac{1}{G'_N}K'_{ab}$$

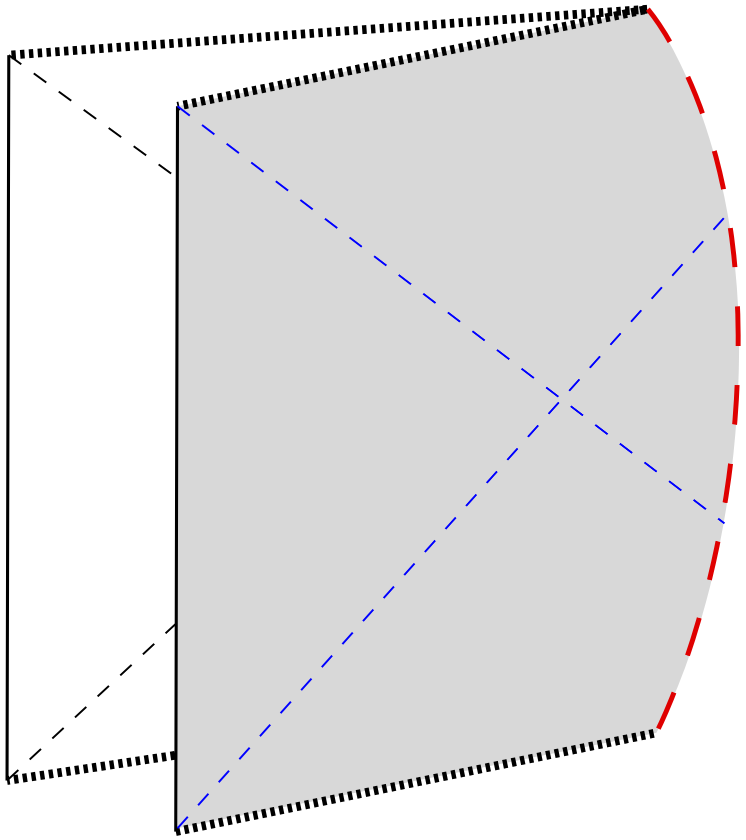
make it entirely holographic.

Gluing the inception geometry

$$h_{ab} = h'_{ab} \qquad \frac{1}{G_N} K_{ab} = \frac{1}{G'_N} K'_{ab}$$

- K_{ab} **jumps when** $G_N \neq G'_N$
- **We choose to glue convex to convex, to obtain long wormholes**





Gluing the inception geometry

With BTZ metric on both sides

$$ds^2 = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\varphi \qquad f(r) = \frac{r^2 - r_h^2}{\ell^2}, \qquad \beta = \frac{2\pi\ell^2}{r_h},$$

The solution is

$$\tau(r) = \ell^2 \sqrt{\frac{r_t^2 - r_h^2}{r_h^2 - r_b^2}} \int_{r_t}^{r} d\tilde{r} \frac{\sqrt{\tilde{r}^2 - r_b^2}}{(\tilde{r}^2 - r_h^2)\sqrt{\tilde{r}^2 - r_t^2}} \qquad \tau'(r) = \ell'^2 \sqrt{\frac{r_t^2 - r_h'^2}{r_h'^2 - r_b^2}} \int_{r_t}^{r} d\tilde{r} \frac{\sqrt{\tilde{r}^2 - r_b^2}}{(\tilde{r}^2 - r_h'^2)\sqrt{\tilde{r}^2 - r_t^2}}$$

$$r_t = \sqrt{\frac{\ell^2 G_N^2 r_h'^2 - \ell'^2 G_N'^2 r_h^2}{\ell^2 G_N^2 - \ell'^2 G_N'^2}}, \qquad r_b = \sqrt{\frac{\ell^2 r_h'^2 - \ell'^2 r_h^2}{\ell^2 - \ell'^2}}$$

Real, timelike trajectory requires $r_h, r_h' > r_b$ **and** $r_h, r_h' < r_t$

Gluing the inception geometry

The solution exists, the upshot of the constraints

is that we need $G'_N < G_N$ in order to see a Page transition

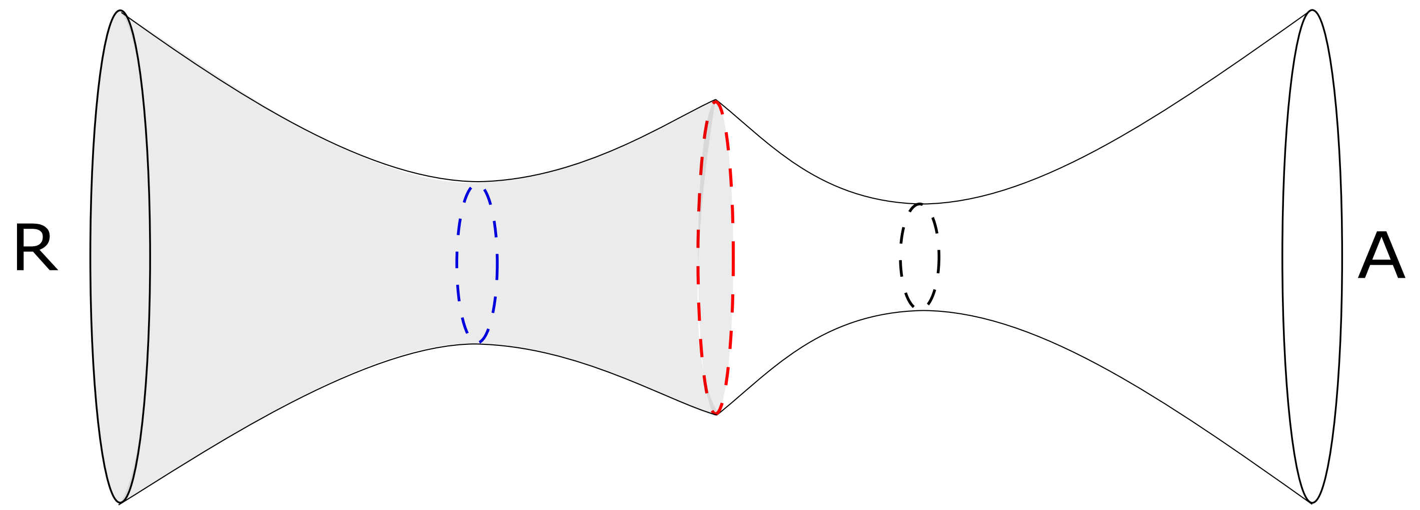
Intuitively, we therefore need more degrees of freedom
in the brane CFT than the original

This is natural, since we want to entangle

more than $e^{S_{BH}}$ qubits with the black hole

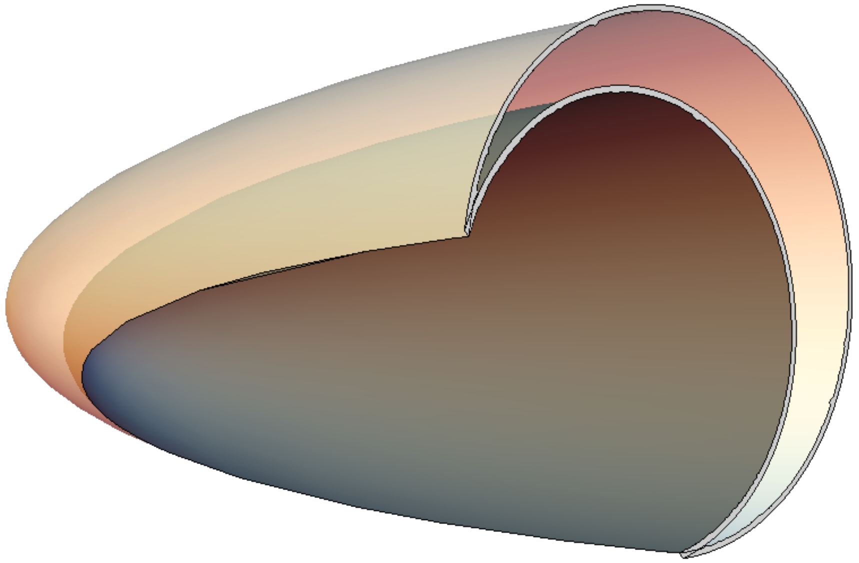
The Page transition in the RT surface tells us
that we cannot do this, without invoking
a microscopic theory for the black hole

Replica saddles

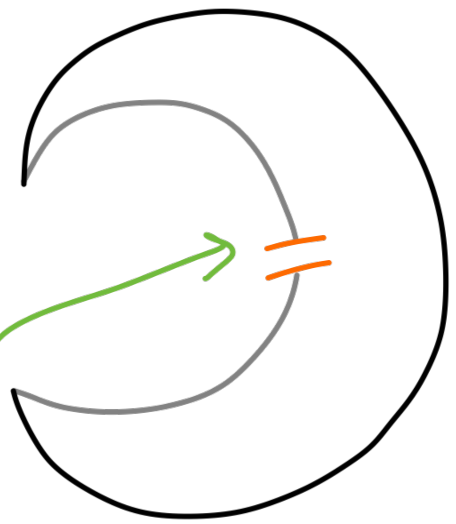


The Page transition between the two horizons may be understood now using the replica trick, along the lines of

[Almheiri-…,Penington,…]

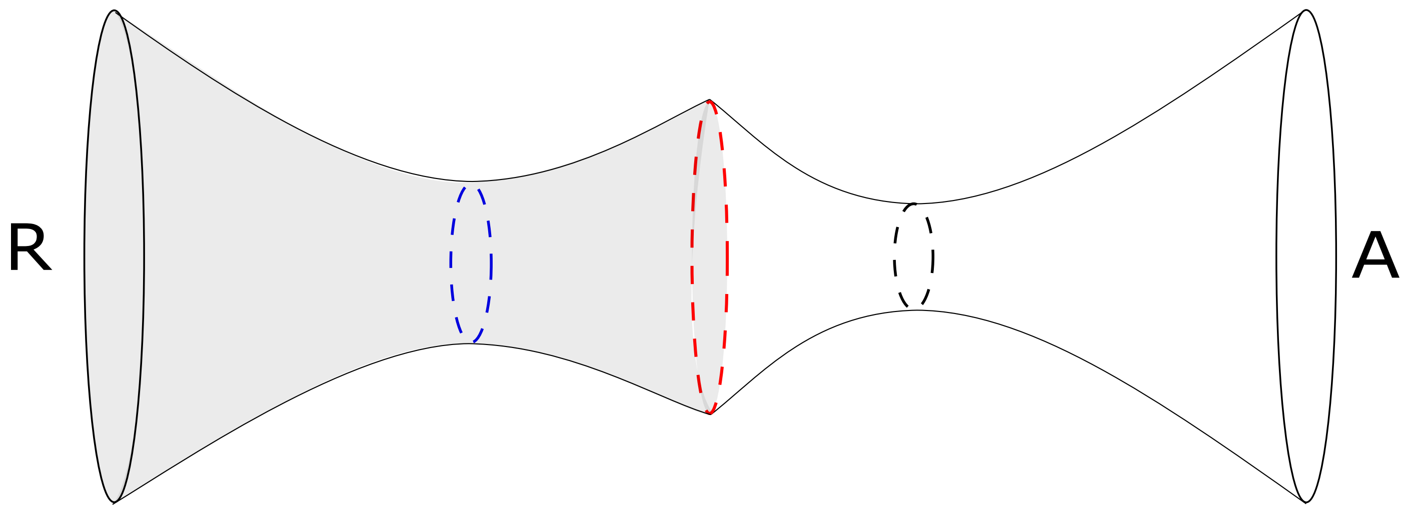


$$\longrightarrow \quad \rho_R \sim$$



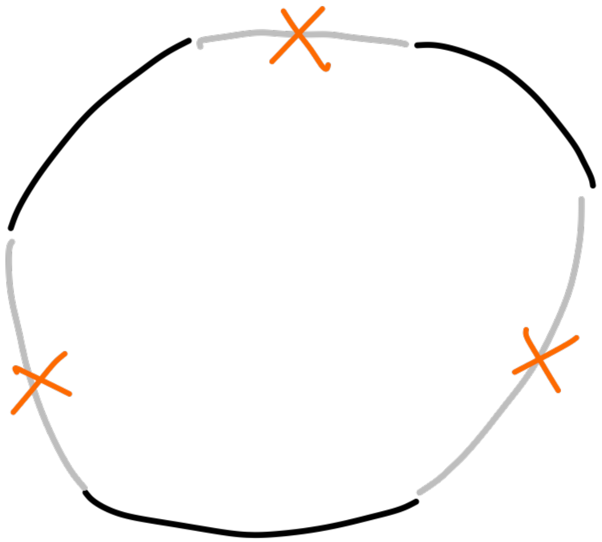
cut it open
along t=0 slice of R system

Replica saddles

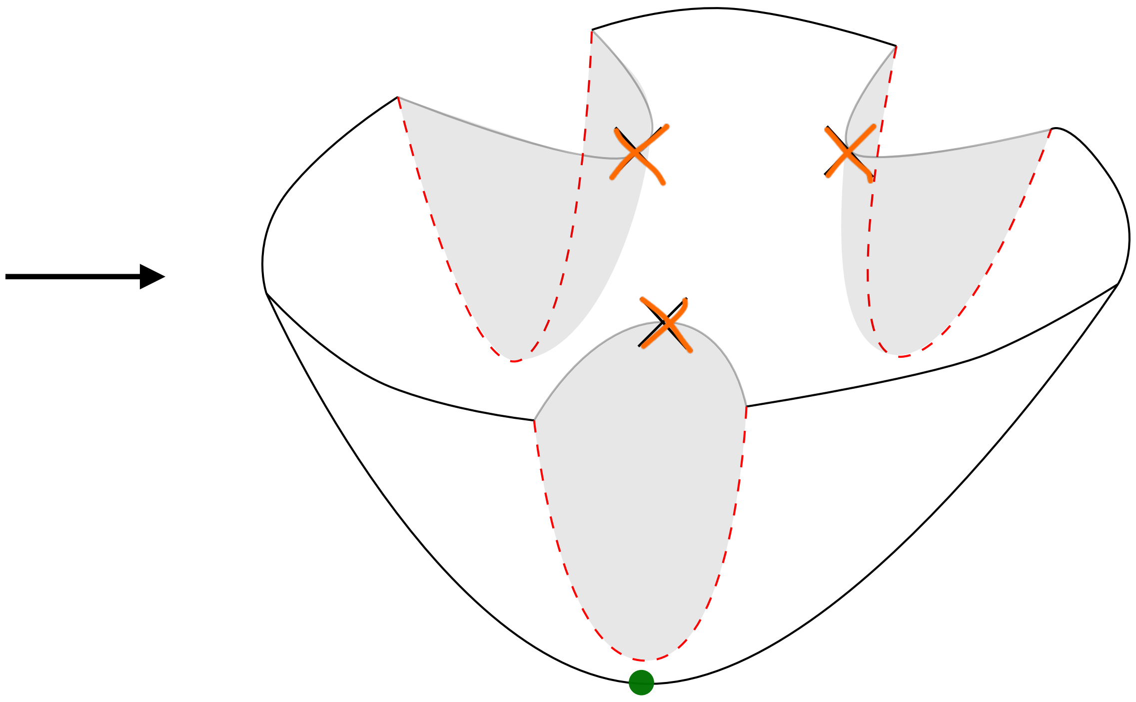


Glue cyclically to form Rényi entropies

$$\mathbf{Tr}\rho_R^3 \sim$$

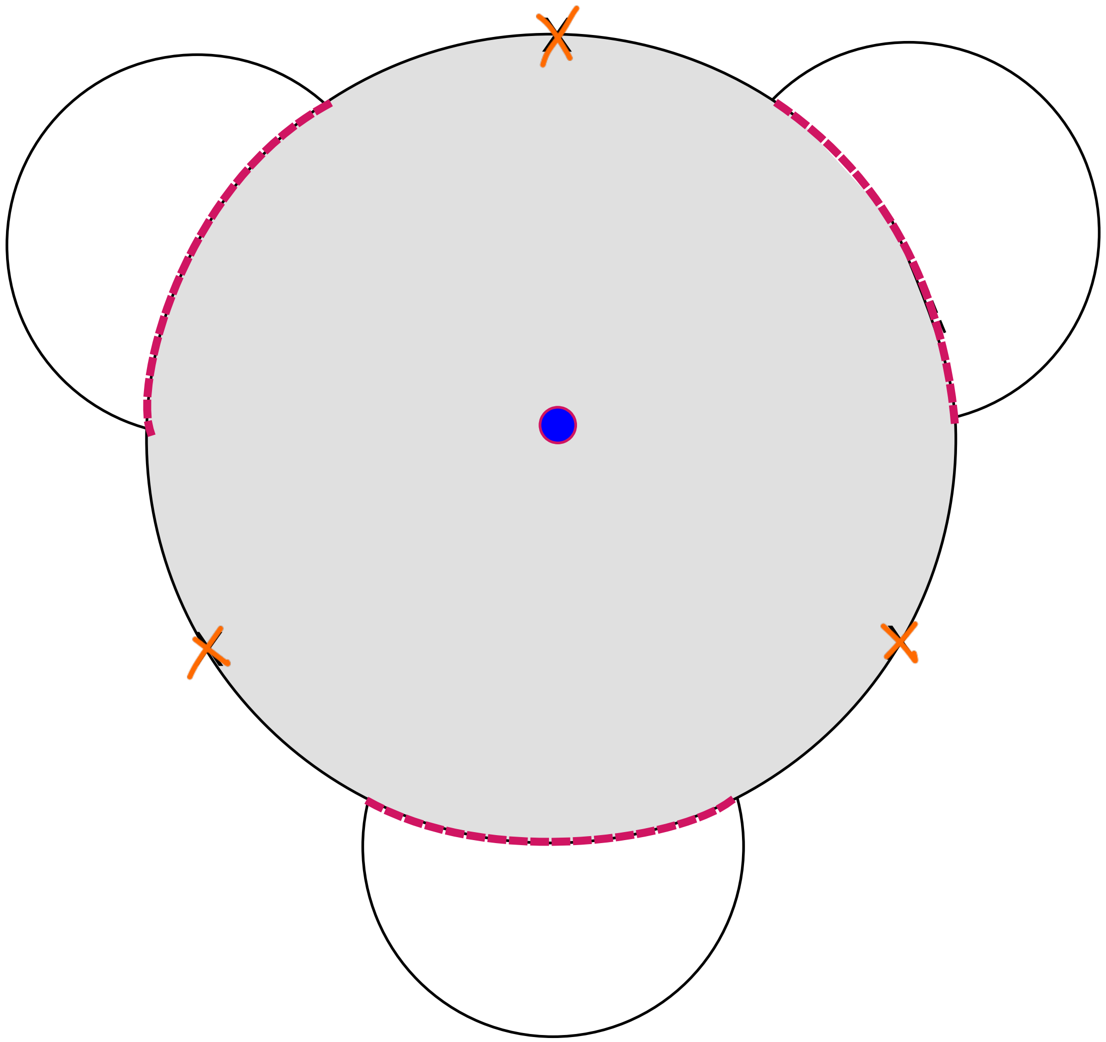


Fill in the boundary with a replica symmetric bulk saddle

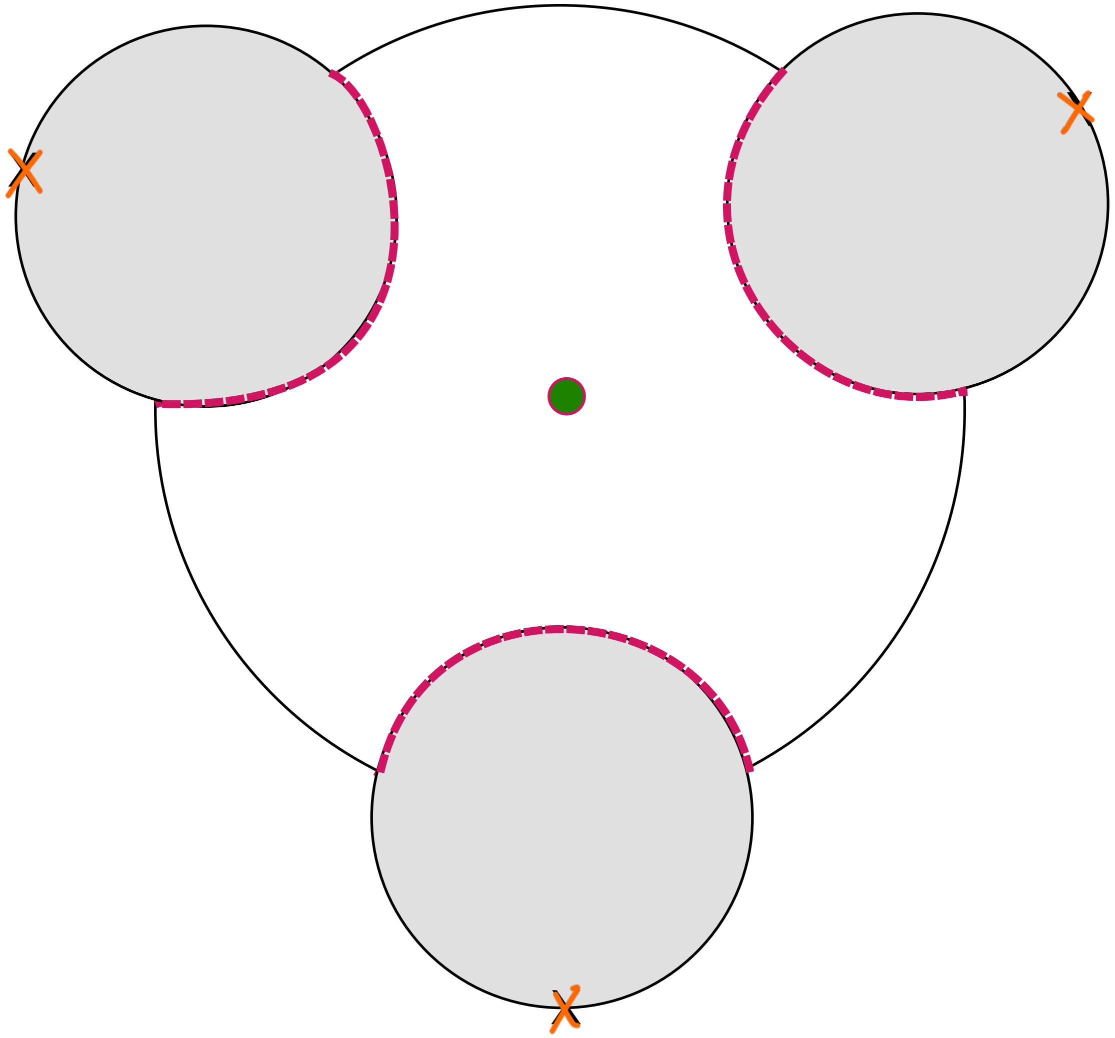


Replica saddles

There are two ways to fill it in in a replica symmetric way



Hawking phase:
Fixed point in inception side



Page phase:
Fixed point on real BH side

In a nutshell

Toy model: microstates modelled by EOWs, with their own holographic dual

Semi-classical calculation knows when we would want to entangle too many reservoir qubits with the black hole and prevents it by a “Page transition”

The transition maybe understood in terms of the replica trick, via a phase transition in the dominant saddle
[Almheiri-Hartman-Maldacena-Shaghoulian-Tajdini;Penington-Shenker-Stanford-Yang]

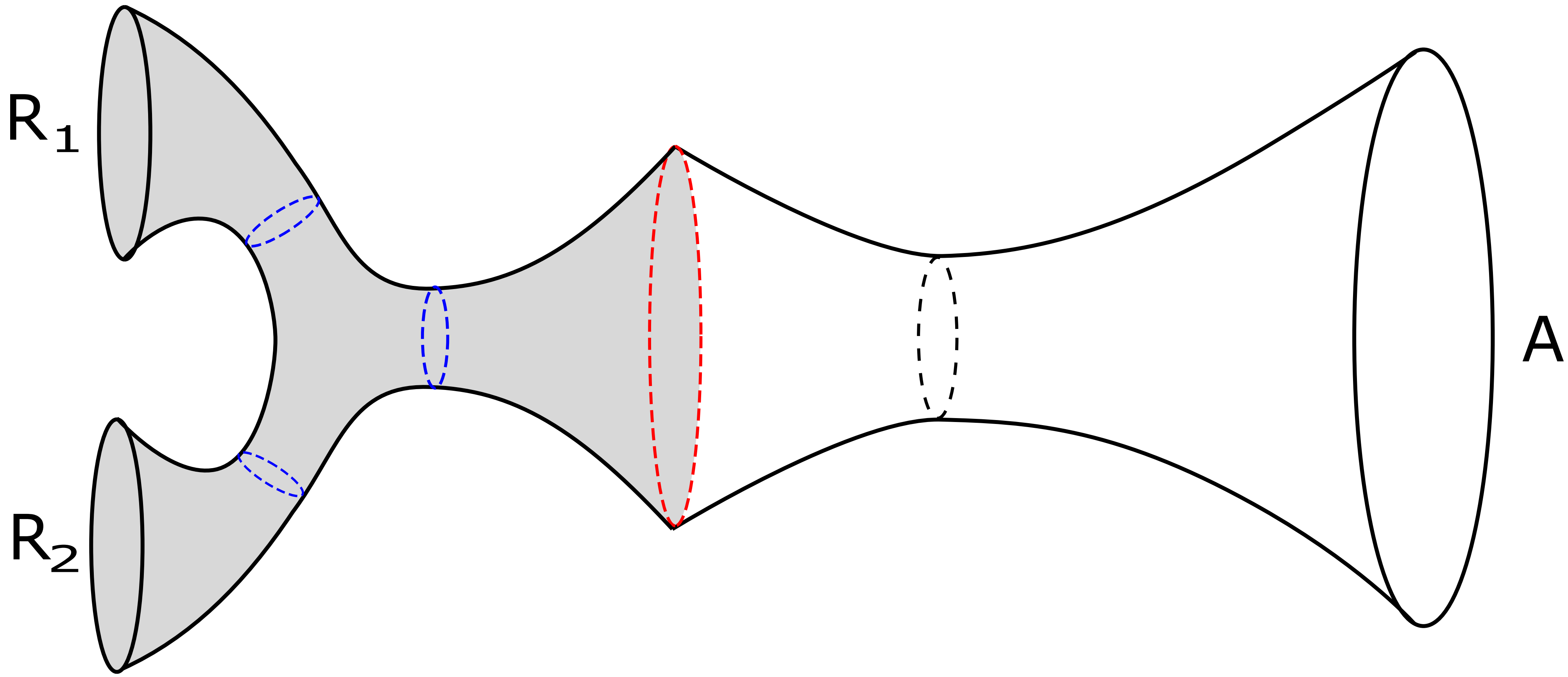
After the transition, the interior is reconstructible from the radiation reservoir.
In more realistic models, this is only true for part of the interior, which has been called an island
[Penington,Almheiri-Mahajan-Maldacena-Zhao]

Partial islands

Divide radiation into two (or more) parts

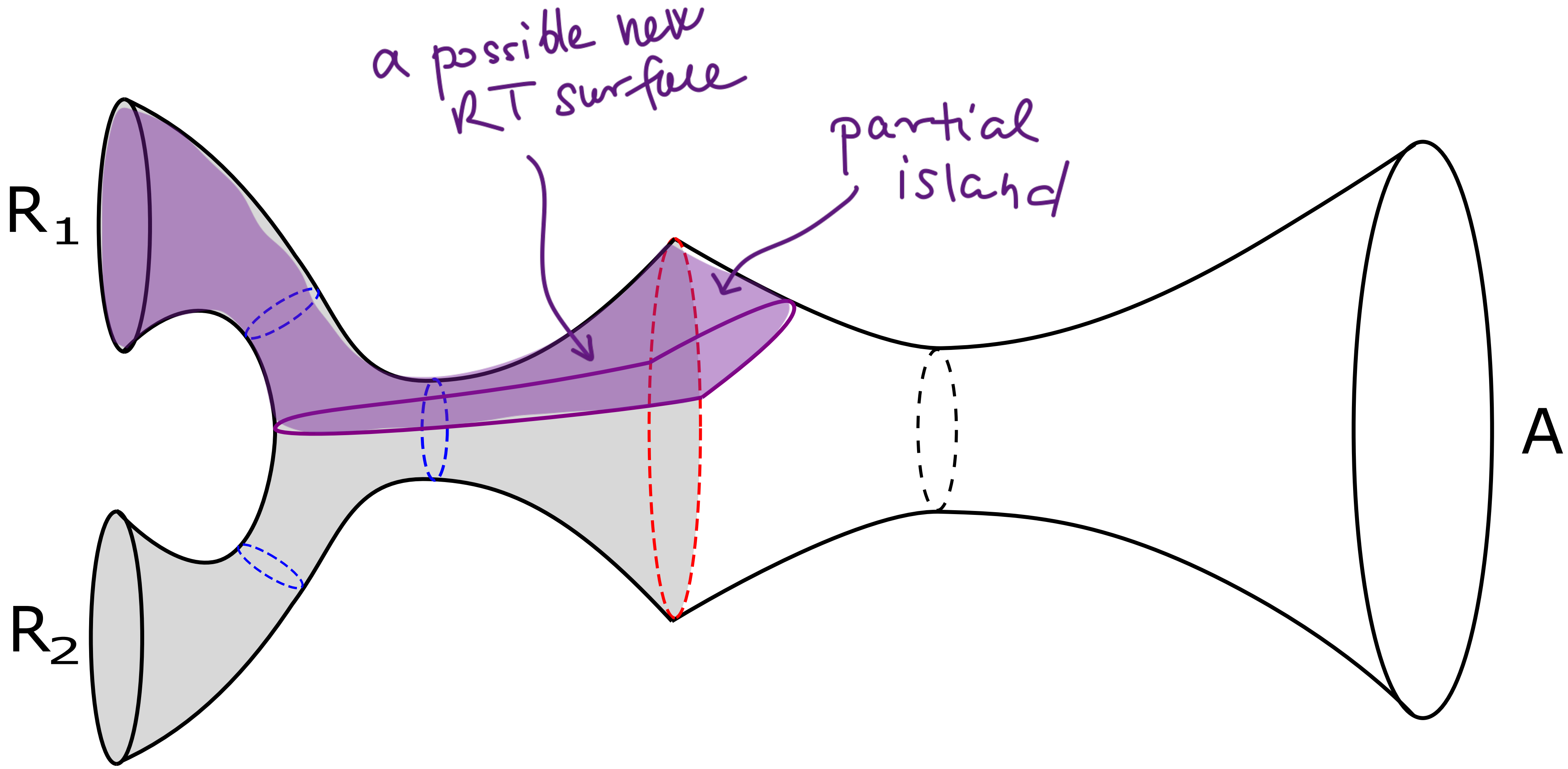
$$\sum_{ii_1i_2} \left[c^i_{i_1i_2} \,|\, i_1\rangle_{R_1} \otimes |\, i_2\rangle_{R_2}\right] \otimes |\, \psi_i\rangle_B$$

Depending on $c^i_{i_1i_2}$ we may represent the state as gluing multiboundary wormholes

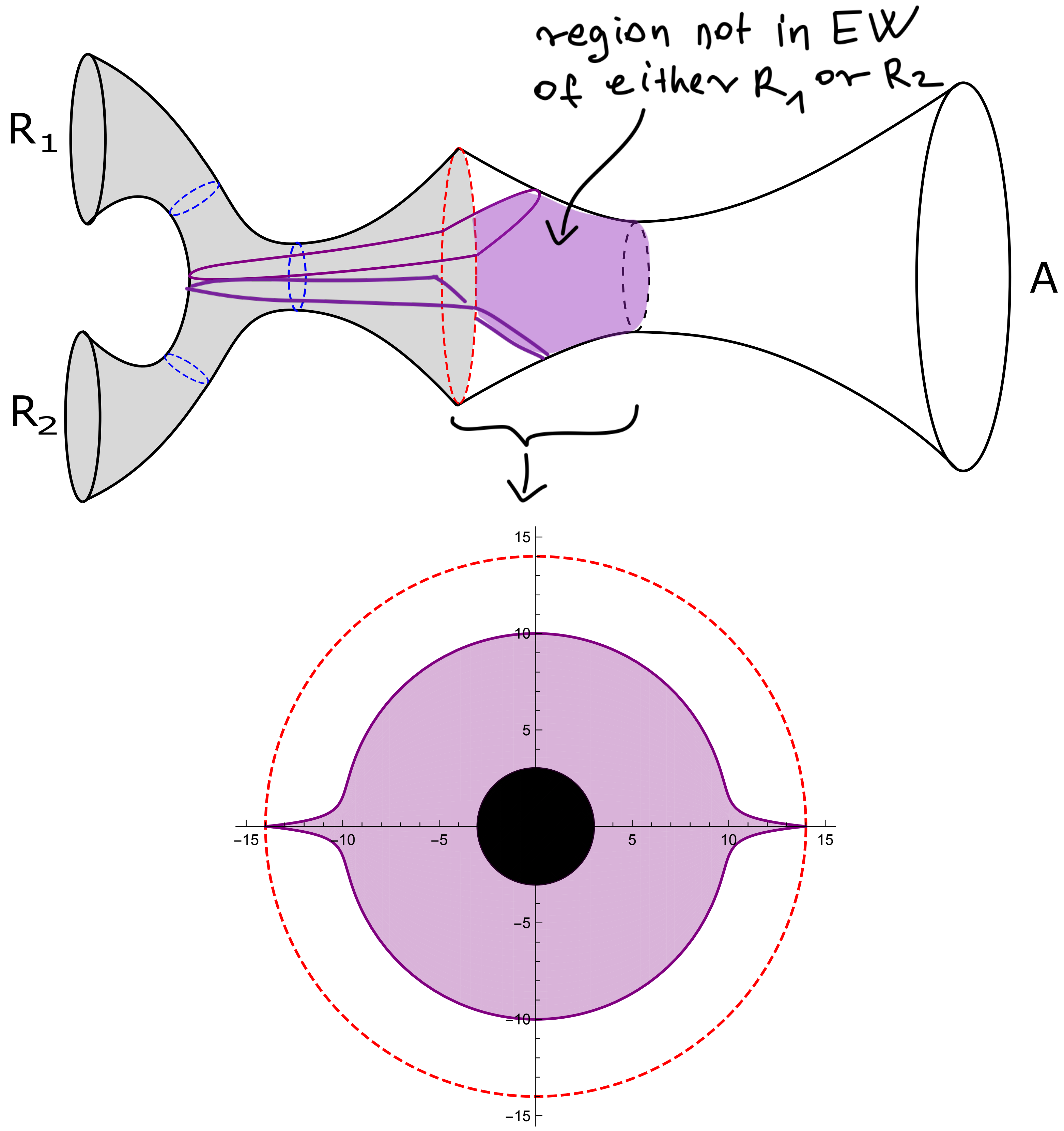


Partial islands

New extremal surfaces

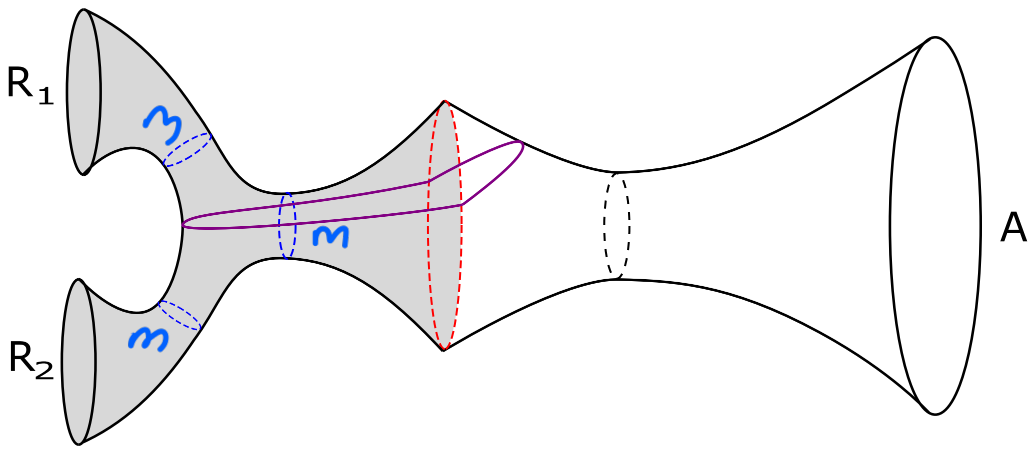


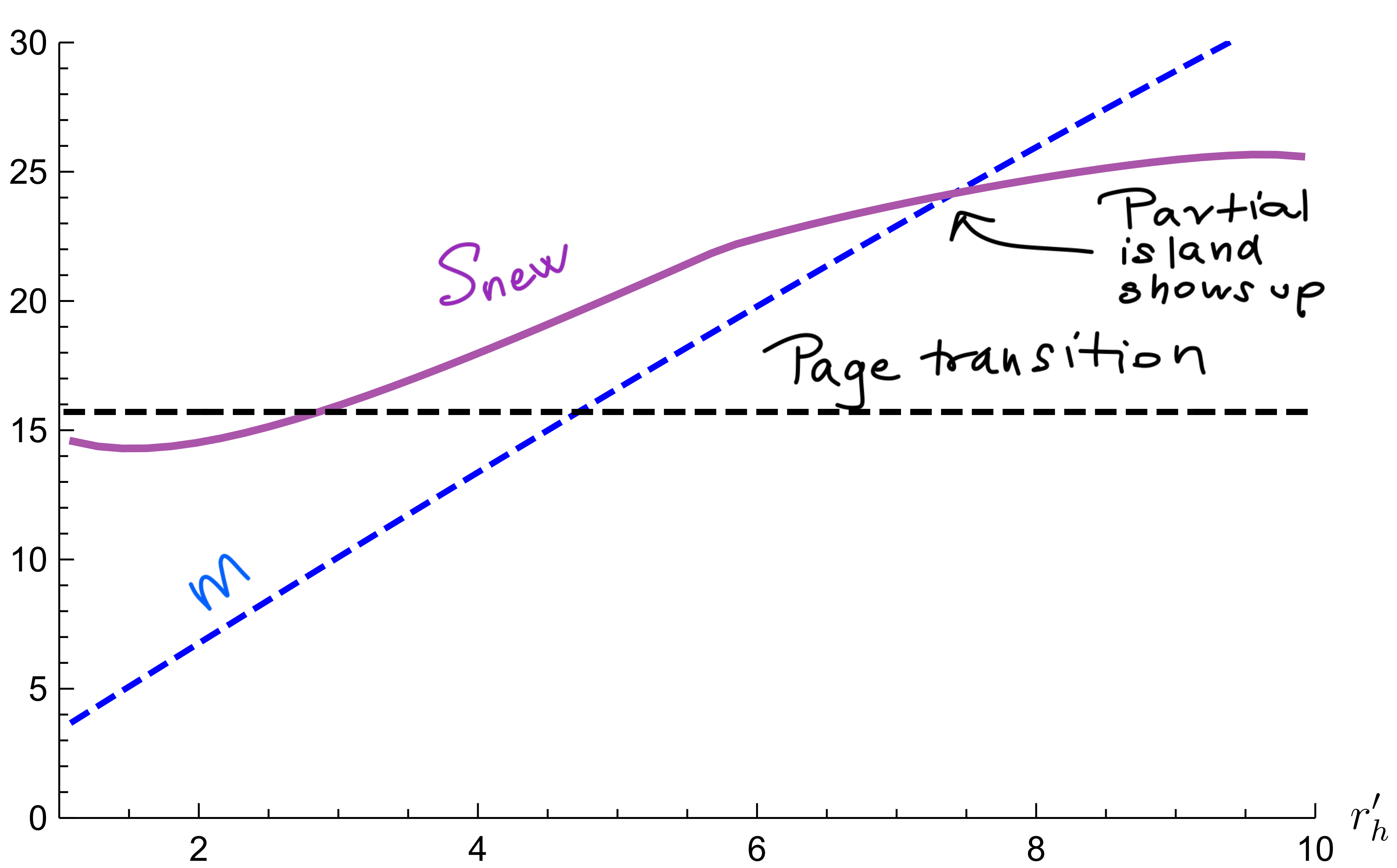
There is such an extremal surface, dominates over the causal horizon a bit after the Page transition


region not in EW
of either R1 or R2
R1
R2
A
15
10
5
−5
−10
−15
−15
−10
−5
5
10
15

“Eyeland”

The new RT surface in covering space





Summary

- **Simple geometric model for QES for black holes entangled with reference system**

- **Captures Page behaviour, and the replica saddle exchange mechanism**

- **Purifying with multi boundary wormholes gives interesting phenomena like partial islands and subregions that are shared secrets**
  (similar to usual holographic error correction story [Almheiri-Dong-Harlow])

- **Horizon is always part of the secret**