## Solutions to Kinetic models in statistical physics

Min Jae Jo

P1) Show the following: Consider a harmonic motion: $m \frac{d^{2} x_{i}}{d t^{2}}=-k \sum_{j \in \text { n.n. of } i}\left(x_{i}-x_{j}\right)$, $\lambda x_{i}=\sum_{j} V_{i, j} x_{j}$. Then the eigenvalue i) $\lambda \sim \omega^{2}$, and ii) the spectral density is $\rho_{s}(\lambda) \sim \lambda^{\frac{d}{2}-1}$.
$\Rightarrow$ i) We use the definition of a Laplacian operator $V$ by

$$
\sum_{j} V_{i, j} x_{j}=z x_{i}-\sum_{j \in \text { n.n. of } i} x_{j}=\sum_{j \in \text { n.n. of } i}\left(x_{i}-x_{j}\right),
$$

where $z$ is the coordination number of the lattice. Then the original equation can be written as

$$
m \frac{d^{2} x_{i}}{d t^{2}}=-k \sum_{j} V_{i, j} x_{j}
$$

When $V_{i, j}$ is independent of time, this equation can be solved by the method of separation of variables. Inserting the corresponding solution of the form $x_{\ell}(t)=e^{i(\vec{q} \cdot \vec{\ell}-\omega t)}$

$$
\sum_{j} V_{i, j} x_{j}=\frac{m \omega^{2}}{k} x_{i}
$$

The oscillation frequency is simply related to the eigenvalue $\lambda=\frac{m \omega^{2}}{k}$.
ii) The spectral dimension is defined as $\rho_{s}(\omega) \sim \omega^{d_{s}-1}$. Combining the relation $\rho_{s}(\omega) d \omega=$ $\rho_{s}(\lambda) d \lambda$ and the result from i), $\rho_{s}(\lambda) \sim \lambda^{d_{s} / 2-1}$.
P2) Show the following: $P_{0}(n) \sim n^{-d / 2} \Rightarrow \mathcal{P}_{0}(s) \sim(1-s)^{d / 2-1} \rightarrow$ singular when $d<2$ as $s \rightarrow 1$.
$\Rightarrow$ Let us consider the $P_{0}(n)$ of the spectral density we obtained in P1:

$$
\begin{aligned}
P_{0}(n) & =\frac{S_{d}}{(2 \pi)^{d}} \int_{0}^{\infty} e^{-n w \lambda} \lambda^{d / 2-1} d \lambda=\frac{S_{d}}{(2 \pi)^{d}} \int_{0}^{\infty} \frac{e^{-x}}{n w}\left(\frac{x}{n w}\right)^{d / 2-1} d x \\
& =\frac{S_{d}}{(2 \pi)^{d}} \frac{\Gamma(d / 2)}{(n w)^{d / 2}} \sim n^{-d / 2},
\end{aligned}
$$

where we have used the substitution $x \equiv n w \lambda$. Then the calculation of the generating function becomes

$$
\begin{aligned}
\mathcal{P}_{0}(s) & =\frac{S_{d}}{(2 \pi)^{d}} \int_{0}^{\infty} d \lambda \frac{\lambda^{d / 2-1}}{1-s+\lambda w s}=\frac{S_{d}}{(2 \pi)^{d}(1-s)} \int_{0}^{\infty} d \lambda \frac{\lambda^{d / 2-1}}{1+\frac{\lambda w s}{1-s}} \\
& =\frac{S_{d}}{(2 \pi)^{d}} \frac{(1-s)^{d / 2-1}}{(w s)^{d / 2}} \int_{0}^{\infty} d x \frac{x^{d / 2-1}}{1+x} \\
& =\frac{S_{d}}{(2 \pi)^{d}} \frac{(1-s)^{d / 2-1}}{(w s)^{d / 2}} \pi \csc \left(\frac{d \pi}{2}\right), \quad \text { when } d<2 .
\end{aligned}
$$

Thus $\mathcal{P}_{0}(s) \sim(1-s)^{d / 2-1}$ and singular as $s \rightarrow 1$.

P3) Show that i) $F(t) \simeq \frac{1}{\sqrt{\pi}} \frac{1}{t^{3 / 2}}$ in $1 d$; ii) $F(t) \simeq \frac{4 \pi}{t(\ln t)^{2}} \frac{1}{t^{3 / 2}}$ in $2 d$; and iii) $F(t) \simeq \frac{(1-\mathcal{R})^{2}}{8 \pi^{3 / 2}} \frac{1}{t^{3 / 2}}$ in $1 d$. Find $\mathcal{R}$ in $3 d$.
$\Rightarrow$ i) From the Lecture note, the Laplace transform of the occupation probability at the origin is

$$
\begin{equation*}
P(s) \simeq \int_{0}^{\infty}(4 \pi t)^{-d / 2} e^{-s t} d t=\frac{\Gamma(1-d / 2) s^{d / 2-1}}{(4 \pi)^{d / 2}}, \quad \text { when } d / 2<1 \tag{1}
\end{equation*}
$$

Thus, $P(s) \simeq(4 s)^{-1 / 2}$ in $1 d$. Then the Laplace transform of the first-passage probability to the origin,

$$
\begin{equation*}
F(s)=1-\frac{1}{P(s)} \simeq 1-\sqrt{4 s} \tag{2}
\end{equation*}
$$

and the eventual return probability $R=F(s=0)=1$. Because $F(s)$ does not diverge for small $s$, it is not convenient to invert this Laplace transform directly. Instead, we consider the derivative of $F(s)$ :

$$
-F^{\prime}(s)=\mathcal{L}(t F(t))=s^{-1 / 2}
$$

where $\mathcal{L}$ is a notation for the Laplace transform. We can then directly apply Eq. (1) to find that $t F(t)=(\pi t)^{-1 / 2}$ and thus $F(t) \simeq \frac{1}{\sqrt{\pi}} \frac{1}{t^{3 / 2}}$.
ii) Firstly, we must cut off the integral for $t<1$ to eliminate the spurious singularity that arises from using the long-time form of $P(t)$ for short times. Then we have

$$
\begin{aligned}
P(s) & \simeq \int_{0}^{\infty}(4 \pi t)^{-1} e^{-s t} d t=\frac{\Gamma(0, x)}{4 \pi} \\
& \simeq-\frac{\ln s}{4 \pi}, \quad \text { when } s \rightarrow 0
\end{aligned}
$$

Then the Laplace transform of the first-passage probability to the origin,

$$
F(s) \simeq 1+\frac{4 \pi}{\ln s}, \quad \text { when } s \rightarrow 0
$$

Again, the eventual return probability $R=F(s=0)=1$ and we consider the derivative of $F(s)$ :

$$
-F^{\prime}(s)=\mathcal{L}(t F(t))=\frac{4 \pi}{s(\ln s)^{2}}, \quad \text { when } s \rightarrow 0
$$

Since the Laplace transform of a constant function equals $1 / s$, the function $t F(t)$ must vary weakly in time to give the extra factor of $(\ln s)^{-2}$ and thus $F(t) \simeq \frac{4 \pi}{t(\ln t)^{2}}$.
iii) For $d=3$,

$$
P(s) \simeq \int_{0}^{\infty}(4 \pi t)^{-3 / 2} e^{-s t} d t
$$

and it is useful to differentiate to both sides of the above equation:

$$
P^{\prime}(s) \simeq \frac{-1}{(4 \pi)^{3 / 2}} \int_{0}^{\infty} t^{-1 / 2} e^{-s t} d t=\frac{-1}{8 \pi \sqrt{s}}
$$

where we have used Eq. (1). Then $P(s)=P(0)-\frac{\sqrt{s}}{4 \pi}$ and $F(s)$ is given by

$$
\begin{aligned}
F(s) & =1-\frac{1}{P(s)} \simeq 1-\frac{1}{P(0)-\frac{\sqrt{s}}{4 \pi}} \simeq 1-\frac{1}{P(0)}\left(1+\frac{\sqrt{s}}{4 \pi P(0)}\right) \\
& =1-\frac{1}{P(0)}-\frac{\sqrt{s}}{4 \pi P(0)^{2}}
\end{aligned}
$$

Using $\mathcal{R}=F(s=0)=1-P(0)^{-1}$, the above equation is rewritten as follows:

$$
F(s)=\mathcal{R}-\frac{(1-\mathcal{R})^{2} \sqrt{s}}{4 \pi} .
$$

Note that $\mathcal{R}=1-P(0)^{-1}<1$ and the random walk is transient. By comparing with Eq. (2), the asymptotic form of $F(t)$ is given by

$$
F(t) \simeq \frac{(1-\mathcal{R})^{2}}{8 \pi^{3 / 2}} \frac{1}{t^{3 / 2}}
$$

P4) Show that i) $F(t) \sim \frac{y_{0}}{\sqrt{8 \pi D t^{3}}} e^{-y_{0}^{2} / 8 D t} \sim t^{-3 / 2}$ and ii) the survival probability $S(t)=$ $1-\frac{2}{\sqrt{\pi}} \int_{y_{0} / \sqrt{8 D t}}^{\infty} e^{-u^{2}} d u \rightarrow \frac{y_{0}}{\sqrt{2 \pi D t}}$.
$\Rightarrow$ i) The first-passage probability to the origin at time $t$ is just the flux to this point with the probability density $c(y, t)=\frac{1}{\sqrt{8 \pi D t}}\left[e^{-\left(y-y_{0}\right)^{2} / 8 D t}-e^{-\left(y+y_{0}\right)^{2} / 8 D t}\right]$ :

$$
F(t)=\left.2 D \frac{\partial c(y, t)}{\partial y}\right|_{y=0}=\frac{y_{0} e^{-y_{0}^{2} / 8 D t}}{\sqrt{8 \pi D t^{3 / 2}}} \sim t^{-3 / 2}, \quad \text { as } t \rightarrow \infty
$$

ii) The survival probability $S(t)$ of the two particles may be found from $S(t)=1-\int_{0}^{t} F\left(0, t^{\prime}\right) d t^{\prime}$. Using the substitution $x^{2}=y_{0}^{2} / 8 D t^{\prime}$ leads to

$$
S(t)=1-\frac{2}{\sqrt{\pi}} \int_{y_{0} / \sqrt{8 D t}}^{\infty} e^{-x^{2}} d x=\operatorname{erf}\left(y_{0} / \sqrt{8 D t}\right) \rightarrow \frac{y_{0}}{\sqrt{2 \pi D t}}, \quad \text { as } t \rightarrow \infty
$$

P5) Consider a Sierpinski gasket. i) Obtain the fractal dimension.
Consider random walks on an infinite Sierpinski gasket. ii) Obtain the spectral dimension and random walk dimension.


Figure 1: Compartment of a Sierpinski gasket.
$\Rightarrow$ i) Let us consider a Sierpinski gasket in Fig. 1. Observe that it is a triangle, and consists of three smaller triangles with a triangular space between them. These three triangles are identical copies of the entire fractal, so we decide that our value for $n$ will be 3 . Now what is the scaling factor? Observation also easily tells us that the edge of the smaller triangles is half the edge of the fractal itself. Thus our scaling factor $r$ is 2. Applying these to formula, we get:

$$
d_{f}=\frac{\ln n}{\ln r} \simeq 1.585<2
$$

ii) In Fig. 1, some of the equations for the eigenvalue problem are given by

$$
\begin{aligned}
& \lambda x_{1}=4 x_{1}-x_{2}-x_{3}-x_{5}-x_{6}, \\
& \lambda x_{2}=4 x_{2}-x_{1}-x_{3}-x_{4}-x_{5}, \\
& \lambda x_{3}=4 x_{3}-x_{1}-x_{2}-x_{4}-x_{6},
\end{aligned}
$$

or equivalently

$$
\left(\begin{array}{ccc}
\lambda-4 & 1 & 1 \\
1 & \lambda-4 & 1 \\
1 & 1 & \lambda-4
\end{array}\right)\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right)=-\left(\begin{array}{c}
x_{5}+x_{6} \\
x_{4}+x_{5} \\
x_{4}+x_{6}
\end{array}\right)
$$

Solving the equation with respect to $x_{1}, x_{2}$, and $x_{3}$

$$
\left(\begin{array}{l}
x_{1}  \tag{3}\\
x_{2} \\
x_{3}
\end{array}\right)=-\frac{1}{(\lambda-5)(\lambda-2)}\left(\begin{array}{ccc}
\lambda-3 & -1 & -1 \\
-1 & \lambda-3 & -1 \\
-1 & -1 & \lambda-3
\end{array}\right)\left(\begin{array}{l}
x_{5}+x_{6} \\
x_{4}+x_{5} \\
x_{4}+x_{6}
\end{array}\right)
$$

Likewise, $x_{7}, x_{8}$, and $x_{9}$ can be represented by $x_{5}, x_{10}$, and $x_{11}$. The equation of $x_{5}$ is given by

$$
\begin{equation*}
\lambda x_{5}=4 x_{5}-x_{1}-x_{2}-x_{7}-x_{8} \tag{4}
\end{equation*}
$$

and inserting the solutions $x_{1}, x_{2}$ in Eq. (3) and $x_{7}, x_{8}$ to the right hand side of Eq. (4)

$$
\left(5 \lambda-\lambda^{2}\right) x_{5}=4 x_{5}-\left(x_{4}+x_{6}+x_{10}+x_{11}\right)
$$

Note that the equation is represented by $x_{4}, x_{6}, x_{10}$, and $x_{11}$. Thus it reduces to the eigenvalue problem with the edge length $2^{k-1} a$, where $k$ is the iteration number and $a$ is the unit length. The renormalized eigenvalue $\lambda^{\prime}$ is given by $\lambda^{\prime}=5 \lambda-\lambda^{2}$. For $\lambda \rightarrow 0, \lambda^{\prime}=5 \lambda$. If we introduce an quantity $D_{k}(\Lambda)$, the number of eigenvalues in the interval $0<\lambda<\Lambda$ with the edge length $2^{k} a$, we obtain $D_{k}(\Lambda)=D_{k-1}(5 \Lambda)$ for $\Lambda \rightarrow 0$.
Then spectral density becomes

$$
\rho(\lambda)=\frac{d}{d \lambda} \lim _{k \rightarrow \infty} \frac{1}{n} D_{k}(\lambda),
$$

and for $\lambda \rightarrow 0$

$$
\begin{equation*}
\rho(\lambda)=\frac{5}{3} \rho(5 \lambda) . \tag{5}
\end{equation*}
$$

The spectral dimension $d_{s}$ is defined as $\rho(b \omega)=b^{d_{s}-1} \rho(\omega)$. Moreover, the eigenvalue $\lambda=\omega^{2} / \omega_{0}^{2}$ gives the relation $\rho(\omega)=\omega \rho\left(\omega^{2} / \omega_{0}^{2}\right)$. Thus we obtained the relation $\rho(b \lambda)=b^{d_{s} / 2-1} \rho(\lambda)$, which gives the spectral dimension by Eq. (5): $d_{s} \simeq 1.365$ and $d_{w}=2 d_{f} / d_{s} \simeq 2.322$.
P5') Consider a Sierpinski gasket. i) Obtain the fractal dimension.
Consider random walks on an infinite Sierpinski gasket. ii) Obtain the spectral dimension and self-avoiding walk dimension $\nu$.
$\Rightarrow \mathrm{i})$ This is the same problem with the P 5 - i .
ii) In two dimensional Sierpinski gasket, the partition function is

$$
\begin{aligned}
Z & =\operatorname{Tr} \prod_{\langle i, j\rangle}\left[1+v O_{i j}\right] \\
& =\operatorname{Tr} \prod_{\Delta}\left[1+v\left(O_{12}+O_{23}+O_{31}\right)+v^{2}\left(O_{12} O_{23}+O_{12} O_{31}+O_{23} O_{31}\right)+v^{3} O_{12} O_{23} O_{31}\right],
\end{aligned}
$$

where $\prod_{\triangle \otimes}$ is the product of the unit triangles and 1,2 , and 3 indicate the vertices of the triangle. Calculating the trace of a, b, c spins in Fig. 2(a), the equation is represented by 1, 2, 3 spins. Then we generally set the factor of $\triangle$ as

$$
T_{\triangle}=1+x\left(O_{12}+O_{23}+O_{31}\right)+y\left(O_{12} O_{23}+O_{12} O_{31}+O_{23} O_{31}\right)+z O_{12} O_{23} O_{31}+w\left(O_{12}^{2}+O_{23}^{2}+O_{31}^{2}\right),
$$



Figure 2: Renormalization of a Sierpinski gasket.
where we set $x, y, z$, and $w$ as the independent parameters. The renormalization transformation can be obtained by performing the partial trace. Using the decimation procedure, an exact real space renormalization group transformation for general $n$ can be obtained in the fourdimensional parameter space $(x, y, z, w)$. In the limit $n \rightarrow 0$, the transformation reduces as

$$
\begin{align*}
& x^{\prime}=x^{2}+x^{3}+2 x y+y^{2}+2 x^{2} y, \\
& y^{\prime}=x^{2} y+2 x y^{2}, \\
& z^{\prime}=w^{\prime}=0 \tag{6}
\end{align*}
$$

Eq. (6) has a fixed point at $x^{*}=(\sqrt{5}-1) / 2, y^{*}=z^{*}=w^{*}=0$, and the eigenvalue can be found from the linearized transformation. Thus, $\lambda_{T}=2^{1 / \nu}=(7-\sqrt{5}) / 2$ or $\nu=0.7986 \cdots$.

P6) Show i) and ii). quoted from the Reference below
$\Rightarrow$ i) We first consider the mean-field equation appropriate to describe the reaction $A_{1}+A_{2}+$ $\cdots+A_{N} \rightarrow \phi$

$$
\frac{\partial \rho_{i}(t)}{\partial t}=-k \rho_{1}(t) \rho_{2}(t) \cdots \rho_{N}(t)
$$

where $\rho_{i}(t)$ is the density of the $i$ th species at time $t$ and $k$ is a rate constant. When the initial densities of all reactants are equal, the asymptotic form of $\rho_{i}(t)$ is

$$
\rho_{i}(t) \sim(k t)^{-1 /(N-1)}, \quad \text { for } d>d_{c}
$$

We now turn our attention for $d<d_{c}$. For simplicity, we treat the three-body reaction, but the generalization to the $N$-body case is immediate. In analogy with the two-body reaction, we write a scaling form for $\rho_{i}(t)$ by assuming that the decay will be a power-law behavior for $t$ less than the shortest time scale,

$$
\begin{equation*}
\rho_{i}(t) \simeq C_{i} t^{-\alpha} f_{i}\left(x_{23}, x_{13}, x_{12}\right), \tag{7}
\end{equation*}
$$

where $f_{i}\left(x_{23}, x_{13}, x_{12}\right)$ is a scaling function of the dimensionless variables $x_{i j}=t / t_{i j}$, and $C_{i}$ is a constant which depends on the initial conditions.
To determine $\alpha$, it is useful to consider the particular initial condition $\rho_{1}(0)<\rho_{2}(0)=\rho_{3}(0)$, so that $t_{23}$ diverges, while $t_{13}=t_{12}=\tau$. Eq. (7) can be written as

$$
\begin{equation*}
\rho_{i}(t) \simeq C_{i} t^{-\alpha} \bar{f}_{i}(t / \tau), \tag{8}
\end{equation*}
$$

where $\bar{f}(t / \tau)=f_{i}\left(0, t / t_{13}, t / t_{12}\right)$, and $\alpha$ is assumed to be independent of the initial conditions. From the conservation of the particle density difference, $\rho_{1}(t)-\rho_{2}(t)=\rho_{1}(0)-\rho_{2}(0)$, and from Eq. (8), we have

$$
\left(\rho_{1}(0)-\rho_{2}(0)\right) t^{\alpha}=C_{1} \bar{f}_{1}(t / \tau)-C_{2} \bar{f}_{2}(t / \tau),
$$

and $\rho_{1}(0)-\rho_{2}(0)$ can be rewritten as $\left(\sqrt{\rho_{1}(0)}+\sqrt{\rho_{2}(0)}\right) \tau^{-d / 4}$, from which we can immediately determine $\alpha$ and $C_{i}$ to be

$$
\alpha=d / 4, \quad C_{1}=C_{2} \simeq\left(\sqrt{\rho_{1}(0)}+\sqrt{\rho_{2}(0)}\right) .
$$

In the limit $\rho_{1}(0) \rightarrow \rho_{2}(0)$, the $t^{-\alpha}$ decay law will be recovered in Eq. (8). Therefor, we find $\rho_{i}(t) \simeq \sqrt{\rho_{i}(0)} t^{-d / 4}$.
To generalize to the $N$-body decay for equal initial densities of the reactants, we choose the initial condition $\rho_{1}(0)<\rho_{2}(0)=\rho_{3}(0)=\cdots=\rho_{N}(0)$, and follow the same steps in 3-body decay. This gives $\rho_{i}(t) \simeq \sqrt{\rho_{i}(0)} t^{-d / 4}$ for $d<d_{c}=4 /(N-1)$.
ii) We can apply scaling to the single-species $N A \rightarrow \phi$. The scaling form in Eq. (8) may be written as

$$
\rho_{A}(t) \simeq \rho_{A}(0) f(t / \tau),
$$

and we postulate that $\rho_{A}(t)$ will be independent of the initial density as $t \rightarrow \infty$. This condition immediately fixes the decay law to be $\rho_{A}(t) \sim t^{-d / 2}$ for $d<d_{c}=2 /(N-1)$, while the mean-field decay of $t^{-1 /(N-1)}$ should hold for $d>d_{c}$.

P7) Suppose that the cluster size dis. at $t=0$ is given by $c_{k}(0)=b k^{-\gamma}$, where $2<\gamma<3$. Solve the generating function $g(z, t)$ and $c_{k}(t)$.
$\Rightarrow$ In the limit $z \rightarrow 1$, the generating function $g(z, 0)$ is asymptotically related by

$$
\begin{equation*}
g(z, 0)=\sum_{k=1}^{\infty} c_{k}(0) z^{k}=\sum_{k=1}^{\infty} b k^{-\gamma} z^{k}=1+b \Gamma(1-\alpha)(1-z)^{\alpha-1} . \tag{9}
\end{equation*}
$$

Using Eq. (9), taking the limits $t \rightarrow \infty$ and $z \rightarrow 1$, and keeping only the leading term, the generating function becomes

$$
\begin{aligned}
g(z, t) & =(1+t)^{-2} \frac{g(z, 0)}{1-[t /(1+t)] g(z, 0)} \\
& \simeq \frac{t^{-1}}{1+C t(1-z)^{\alpha-1}},
\end{aligned}
$$

with $C=-b \Gamma(1-\alpha)$. From the grouping of variables in the denominator, we identify the scaling variable as $w=k /(C t)^{1 /(\alpha-1)}$, and

$$
c_{k}(t) \simeq t^{-1}(C t)^{-1 /(\alpha-1)} f_{\alpha}(w),
$$

as the scaling form of the mass distribution. Indeed,

$$
g(z, t)=\sum_{k=1}^{\infty} c_{k}(t) z^{k} \simeq t^{-1} \int_{0}^{\infty} d w f_{\alpha}(w) e^{-w s}=\frac{t^{-1}}{1+s^{\alpha-1}},
$$

where $s=(C t)^{1 /(\alpha-1)}(1-z)$. Hence the scaling function $f_{\alpha}(w)$ is the inverse Laplace transform of $\left(1+s^{\alpha-1}\right)^{-1}$.

P8)

$$
M_{1}(t) \xrightarrow{t \rightarrow \infty} \begin{cases}? & \text { for } d=1, \\ ? & \text { for } d=2 \\ ? & \text { for } d=3\end{cases}
$$

$\Rightarrow$ The mass distribution is simply the superposition of Gaussian propagators due to all sources from $t=0$ until the present:

$$
M_{1}(\mathbf{r}, t)=J \int_{0}^{\infty} \frac{d t^{\prime}}{\left(4 \pi D t^{\prime}\right)^{d / 2}} e^{-r^{2} / 4 D t^{\prime}}
$$

Using the substitution $x \equiv r^{2} /\left(4 D t^{\prime}\right)$,

$$
M_{1}(\mathbf{r}, t)=\frac{J r^{2-d}}{4 D \pi^{d / 2}} \Gamma\left(\frac{d}{2}-1, \frac{r^{2}}{4 D t}\right),
$$

where $\Gamma$ is the incomplete gamma function. The solution in the limits $r \ll D t$ leads to

$$
M_{1}(t) \xrightarrow{t \rightarrow \infty} \begin{cases}\sqrt{\pi} r \Gamma\left(-\frac{1}{2}, \frac{r^{2}}{4 D t}\right)=4 \sqrt{\pi D t} & \text { for } d=1 \\ \Gamma\left(0, \frac{r^{2}}{4 D t}\right)=-\gamma_{E}-\ln \left(\frac{r^{2}}{4 D t}\right)+\mathcal{O}\left(\frac{r^{2}}{4 D t}\right) & \text { for } d=2 \\ r^{-1} \pi^{-1 / 2} \Gamma\left(\frac{1}{2}, \frac{r^{2}}{4 D t}\right)=r^{-1} & \text { for } d=3\end{cases}
$$

where $\gamma_{E} \simeq 0.577215$ is Euler's constant.

## References

K. Kang, P. Meakin, J. H. Oh, and S. Redner, "Universal behavior of $N$-body decay processes," J. Phys. A, vol. 17, pp. L665-L670 (1984).

