Random Geometry Approach to $T\bar{T}$ Deformation

Shinji Hirano

University of the Witwatersrand (Wits) Mandelstam Institute for Theoretical Physics & Center for Gravitational Physics, YITP

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Plan of this talk

- 1. Motivations
- 2. Review of random geometry approach Hubbard-Stratonovich
- 3. Gravity dual "ensemble AdS₃" with statistical average over boundary metric
 - 1. Translating FT to AdS in "word by word"
 - 2. Relation to cutoff AdS two views of $T\bar{T}$ deformed CFT
- 4. Correlators from random geometry approach logarithmic corrections
 - 1. Matter correlators
 - 2. Deformed Liouville theory & Stress tensor correlators
- 5. To-do list

Motivation

- The $T\bar{T}$ deformation is remarkably tractable (and integrable) and marginally wellbehaved despite being an irrelevant deformation and power-counting nonrenormalizable
- It is a universal deformation that exits in any QFTs with stress tensor, promising a wide range of applicability and providing a new perspective on short-distance physics
- From the viewpoint of AdS/CFT, $T\bar{T}$ being irrelevant, it begs the question, what happens to the AdS boundary? McGough-Mezei-Verlinde, Guica-Monten
- The random geometry approach is a language easy-to-implement in the AdS/CFT framework
- Initial (failed) motivation: Moving away from near-horizon limit by an irrelevant $\Delta = 2d$ operator (like $T\bar{T}$) in AdS_{d+1}/CFT_d speculated by **Gubser-Hashimoto-Klebanov-Krasnitz** (1998) & Intrilligator (1999): "successful" for single-trace $T\bar{T}$ by **Giveon-Itzhaki-Kutasov** and **Hashimoto**... (and **Caetano's talk**)

Random geometry approach

A review of Cardy's method

see also Tateo's talk

Definition of $T\overline{T}$ deformation

$$\underbrace{S[\mu + \delta\mu]}_{\mathcal{T}[\mu + \delta\mu]} = \underbrace{S[\mu]}_{\mathcal{T}[\mu] \text{ theory}} + \underbrace{\delta\mu \int d^2x \left(T\bar{T} - \Theta^2\right)}_{T\bar{T} \text{ deformation} = \delta S_{T\bar{T}}}$$

✦ Best defined as an infinitesimal deformation

 $: T_{ij}$ = stress tensor of $\mathcal{T}[\mu]$ theory rather than that of $\mathcal{T}[0]$ theory

•
$$T\bar{T} = -\frac{1}{8}\epsilon_{ik}\epsilon_{jl}T^{ij}T^{kl} = -\frac{1}{4}\det T_{ij}$$

Random geometry approach — cont'd

Hubbard-Stratonovich transform — linearizing $Tar{T}$

$$\exp\left(-\delta S_{T\bar{T}}\right) \propto \int [dh] \exp\left[-\frac{1}{8\delta\mu} \int d^2x \epsilon^{ik} \epsilon^{jl} h_{ij} h_{kl} + \int d^2x h_{ij} T^{ij}\right]$$

+ *h*-integrals dominated by a saddle point $: \delta \mu$ infinitesimal

reinterpretation

- ◆ Since T_{ij} by definition is a response to a small change of the metric, $\int d^2x h_{ij} T^{ij}$ reinterpreted as $g_{ij} \mapsto g_{ij} + h_{ij}$ (change of background metric).
- *h_{ij}* is not completely "random" but constrained to diffeomorphism
 h_{ij} = ∂_iα_j + ∂_jα_i

Random geometry approach — cont'd

\blacksquare Saddle point of *h* integrals

$$\frac{1}{4\delta\mu}h_{22}^* = T_{11}, \quad \frac{1}{4\delta\mu}h_{11}^* = T_{22}, \quad \frac{1}{4\delta\mu}h_{12}^* = -T_{12}, \quad \frac{1}{4\delta\mu}h_{21}^* = -T_{21}$$

Subject to EM conservation (w/o local operator singularities)

$$\partial_i T^{ij} = 0$$
 $h_{ij} = \partial_i \alpha_j + \partial_j \alpha_i$ with $\epsilon^{ij} \partial_i \alpha_j = 0$

The $T\bar{T}$ deformation merely amounts to curl-free diffeomorphisms: $x_i \mapsto x_i + \alpha_i$

Saddle point action

$$-\delta S_{T\bar{T}}^* = \frac{1}{8\delta\mu} \int_{\mathscr{M}} d^2 x \epsilon^{ik} \epsilon^{jl} h_{ij}^* h_{kl}^* = \frac{1}{2\delta\mu} \int_{\partial\mathscr{M}} ds^k \epsilon^{jl} \alpha_j \partial_k \alpha_l$$

Contributions only from boundaries or nontrivial cycles (or local operator singularities)

Energy spectrum

Smirnov-Zamolodchikov, Cavaglia-Negro-Szecsenyi-Tateo

As an illustration, let's calculate "SZ" formula for $\mathcal{T}[\mu]$ theory on $S^1 \times I$ from random geometry approach

Saddle point action for $S^1 \times I$ (w/o local operator insertions)

$$-\delta S^* = \frac{1}{2\delta\mu} \int_{\partial\mathcal{M}} ds^k \epsilon^{jl} \alpha_j \partial_k \alpha_l = \frac{2\pi RL}{4\delta\mu} h_{11} h_{22} \quad \text{with constant } h_{11}, h_{22}$$

Saddle point action for $S^1 \times I$ (w/o local operator insertions) $\eta_{ij} \mapsto \eta_{ij} + h_{ij}$

$$e^{-iE_n(R;\mu+\delta\mu)t} = \mathcal{N}^{-1} \int [dh] \exp\left[-\frac{2\pi iRt}{4\delta\mu}h_{11}h_{22}\right] e^{-iE_n\left(\sqrt{1+h_{22}}R;\mu\right)\sqrt{1+h_{11}}t}$$

Energy spectrum — cont'd

EXAMPLE Deformed energy spectrum at 1st order in $\delta\mu$:

 $T\bar{T}$ as deformed CFT on undeformed space (cutoff AdS)

$$E_n(R;\mu+\delta\mu) = \frac{\pi R}{2\delta\mu}h_{11}^*h_{22}^* + \left(1+\frac{1}{2}h_{11}^*\right)E_n(R;\mu) + \frac{1}{2}h_{22}^*R\partial_R E_n(R;\mu) + \mathcal{O}(\delta\mu^2)$$

whose saddle point is at

$$h_{11}^* = -\frac{\delta\mu}{\pi}\partial_R E_n(R;\mu)$$
, $h_{22}^* = -\frac{\delta\mu}{\pi R}E_n(R;\mu)$

The saddle point energy obeys inviscid Burgers' equation

$$\partial_{\mu}E_{n}(R;\mu) = -\frac{1}{2\pi}E_{n}(R;\mu)\partial_{R}E_{n}(R;\mu)$$
whose solution is the "SZ" formula
$$E_{n}(R;\mu) = \frac{\pi R}{\mu} \left[1 - \sqrt{1 - \frac{2\mu C_{n}}{\pi R^{2}}}\right]$$
with Oth order

Energy spectrum — cont'd

By iteration order by order in μ (as opposed to PDE)

TT-deformed CFT can be thought of as undeformed CFT on deformed geometry

 $E^{(i)}(\mathbf{P}, u) = C_n + O(u^{i+1})$

$$E_n^{(i)}(R;\mu) = \frac{C_n}{R^{(i)}} + \mathcal{O}(\mu^{i+1}) \qquad \text{undeformed CFT energy}$$

with
$$R^{(i)} = R\left(1 + \frac{1}{2}h_{22}^{*(i)}\right) = R\left(1 - \frac{\mu}{2\pi R}E_n^{(i-1)}\left(R;\mu\right)\right) \qquad \text{deformed geometry}$$

+ The *i*-th order energy spectrum $\sim \mathcal{O}(\mu^{i})$

$$E_n^{(i)}(R;\mu) = E_n^{(i-1)}(R) + \frac{\pi R}{2\mu} h_{11}^{*(i)} h_{22}^{*(i)} + \frac{1}{2} \left(h_{11}^{*(i)} - h_{22}^{*(i)} \right) E_n^{(i-1)} \left(R;\mu \right) + \mathcal{O}(\mu^{i+1})$$

whose saddle point energy in the continuum limit $i \to \infty$ obeys

$$E_n(R;\mu) = E_n(R) + \frac{\mu}{2\pi R} \left(E_n(R;\mu) \right)^2 \quad \Longleftrightarrow \quad E_n(R;\mu) = \frac{C_n}{R \left(1 - \frac{\mu}{2\pi R} E_n(R;\mu) \right)}$$

This is solved by the SZ formula

for a related point, see Sfondrini, Tateo, Jiang and Guica's talks

Gravity dual

An ensemble of AdS₃ with "Gaussian" average over boundary metric (diffeomorphisms)

$$\underbrace{\mathcal{O}_{AdS[\delta\mu]}\left[g_{ij}^{bdy}(x),\phi^{bdy}(x)\right]}_{observable in AdS[\delta\mu]} = \mathcal{N}^{-1} \int [dh] \exp\left[-\frac{1}{8\delta\mu} \int d^2x e^{ik} e^{il} h_{ij} h_{kl}\right] \underbrace{\mathcal{O}_{AdS}\left[g_{ij}^{bdy}(x+\alpha),\phi^{bdy}(x+\alpha)\right]}_{observable in AdS}$$

$$(dh] e^{\frac{1}{\delta\mu} \int d^2x h^2} g^{bdy} + h$$

$$e^{\frac{1}{\delta\mu} \int d^2x h^2} g^{bdy} + h$$

$$g^{bd} + h$$

Gravity dual — cont'd

Translation from field theory to gravity

- (1) Start with AdS/CFT w/o deformation AdS / $\mathcal{T}[0](=CFT)$ correspondence
- (2) Add new DOF, i.e. boundary metric deformations h_{ij} restricted to diffeomorphisms which "randomly" fluctuate over scale $\delta \mu$ via Hubbard-Stratonovich transform
- * This bodes well with the fact that the $T\bar{T}$ operator is geometric and irrelevant, driving an RG flow to UV over scale $\delta\mu$ and thus integrating-in new DOF
- (3) Perform coordinate transformations $x_i \mapsto x_i + \alpha_i$

∴ In GKP-W dictionary $\exp\left(\int d^2x h_{ij}T^{ij}\right)$ corresponds to a non-normalisable deformation of the metric, $g_{ij} \rightarrow g_{ij} + h_{ij}$, which in this case is restricted to curl-free diffeomorphisms

(4) Integrate over h_{ij} or more precisely curl-free α_i – AdS[$\delta\mu$] / $\mathcal{T}[\delta\mu]$ correspondence

Gravity dual — cont'd

Translation from energy spectrum in $\mathcal{T}[\mu]$ theory to BTZ[μ]

$$ds_{\rm BTZ}^2 = -\left(\frac{\rho^2}{4} - \frac{M}{2} + \frac{M^2}{4\rho^2}\right)dt^2 + \left(\frac{\rho^2}{4} + \frac{M}{2} + \frac{M^2}{4\rho^2}\right)dy^2 + \frac{d\rho^2}{\rho^2}$$

with $y \sim y + 2\pi R$

CFT energy identified with ADM mass

in contrast to quasi-local energy in cutoff AdS

$$E_n(R) = M_{ADM} = RM \implies M = \frac{C_n}{R^2}$$

+ Diffeomorphism induced by $T\bar{T}$ deformation translates to

$$g_{tt} \mapsto (1+h_{11}) g_{tt}$$
, $g_{yy} \mapsto (1+h_{22}) g_{yy}$, $M \mapsto M/(1+h_{22})$

which yields

$$M_{ADM} \left[g_{ij}^{bdy}(x+\alpha) \right] = \sqrt{\frac{1+h_{11}}{1+h_{22}}} M_{ADM}$$

Gravity dual — cont'd

+ The gravity dual of the energy spectrum calculation

$$e^{-iH_{\text{BTZ}[\delta\mu]}\left[g^{bdy}(x)\right]t} = \mathcal{N}^{-1} \int [dh] \exp\left[-\frac{1}{8\delta\mu} \int d^2x \epsilon^{ik} \epsilon^{jl} h_{ij} h_{kl}\right] e^{-iH_{\text{BTZ}}\left[g^{bdy}(x+\alpha)\right]t}$$

This is identical to the FT calculation with translation $E_n(R;\mu) = M_{ADM}(R;\mu)$

$$\begin{array}{ll} \mbox{minimal length} & M_{ADM}(R;\mu) = R(\mu)M(\mu) = \frac{\pi R}{\mu} \left[1 - \sqrt{1 - \frac{2\mu M_{ADM}}{\pi R}} \right] & \mbox{from Burgers' eqn} \\ \mbox{global AdS} & \mbox{(ground state)} & R(\mu) = \frac{R}{2} \left(1 + \sqrt{1 - \frac{2\mu M}{\pi}} \right) & \mbox{from iteration} \\ \mbox{equations are also Jiang's talks} & M(\mu) = M \left(\frac{R}{R(\mu)} \right)^2 & \mbox{from iteration} \end{array}$$

* $T\bar{T}$ -deformed gravity dual = BTZ with deformed boundary radii & mass

see also Tateo's talk for 4D MBI theory

Relation to cutoff AdS

1. Undeformed CFT on deformed space (ensemble AdS) BTZ with deformed boundary radii & mass $\beta(\mu) = \frac{2\pi}{\sqrt{8GM(\mu)}}$ (inverse Hawking temperature)

$$R(\mu) = \frac{R}{2} \left(1 + \sqrt{1 - \frac{2\mu M}{\pi}} \right), \quad \beta(\mu) = \frac{\beta}{2} \left(1 + \sqrt{1 + \frac{\pi\mu}{G\beta^2}} \right) \quad M(\mu) = M \left(\frac{R}{R(\mu)} \right)^2$$

2. Deformed CFT on undeformed space ("cutoff" AdS) McGough-Mezei-Verlinde Alternatively, "cutoff" BTZ with undeformed "boundary" radii

$$\beta = \left(1 - \frac{\mu M(\mu)}{2\pi}\right) \beta(\mu), \qquad R = \left(1 + \frac{\mu M(\mu)}{2\pi}\right) R(\mu)$$
Similar

Similar to Guica-Monten

proper radii at cutoff surface

in the Fefferman-Graham coordinates

if the "boundary" is located at

$$ds_{\text{EBTZ}}^{2} = \frac{\rho^{2}}{4} \left(1 - \frac{8GM(\mu)}{\rho^{2}} \right)^{2} d\tau^{2} + \frac{\rho^{2}}{4} \left(1 + \frac{8GM(\mu)}{\rho^{2}} \right)^{2} dy^{2} + \frac{d\rho^{2}}{\rho^{2}} dy^{2} dy^{2} + \frac{d\rho^{2}}{\rho^{2}} dy^{2} dy^{2} + \frac{d\rho^{2}}{\rho^{2}} dy^{2} dy^{2} dy^{2} + \frac{d\rho^{2}}{\rho^{2}} dy^{2} dy$$

 $16\pi G$

μ

Evolving boundary geometry vs. Deforming theory



Matter correlators on \mathbb{R}^2

- Generalize Cardy's method to correlators, i.e. the case with local operator singularities
 - Recap: In the absence of singularities (w/o local operators)

$$\partial_i T^{ij} = 0 \quad \xrightarrow{\text{saddle}} \quad h_{ij} = \partial_i \alpha_j + \partial_j \alpha_i \quad \text{with} \quad e^{ij} \partial_i \alpha_j = 0$$

• New: In the presence of singularities with local operators

$$\partial_i T^{ij} = \sum_a J^j(x_a, \partial_{x_a}) \delta^2(x - x_a) \xrightarrow{\text{saddle}} h_{ij} = \partial_i \alpha_j + \partial_j \alpha_i + \underbrace{\delta_{ij} \left(\phi - \partial_k \alpha_k \right)}_{\text{Weyl+conf}}$$

with α_i and ϕ depends on local operator sources

Matter correlators — cont'd

Applying random geometry calculus to correlators

$$\left\langle \prod_{a=1}^{n} \mathcal{O}_{\Delta_{a}}(x_{a}) \right\rangle_{\delta\mu} = \mathcal{N}^{-1} \int d\alpha \exp\left[-\frac{1}{4\delta\mu} \int d^{2}x \alpha_{i} \Box \alpha_{i} + \ln\left\langle \prod_{a=1}^{n} \mathcal{O}_{\Delta_{a}}(x_{a} + \alpha_{a}) \right\rangle_{\text{CFT}} + \sum_{a=1}^{n} \frac{\Delta_{a}}{2} \partial_{k} \alpha_{k}(x_{a}) \right]_{\text{HS Gaussian}} + \frac{\ln\left\langle \prod_{a=1}^{n} \mathcal{O}_{\Delta_{a}}(x_{a} + \alpha_{a}) \right\rangle_{\text{CFT}}}{0 \text{th order correlator}} + \sum_{a=1}^{n} \frac{\Delta_{a}}{2} \partial_{k} \alpha_{k}(x_{a}) \right]_{\text{Jacobian}}$$

where ϕ can be neglected for leading log corrections

+ Saddle point value of α_i (cf. an example of operator-dependent evolving geometry)

$$\alpha_i(x) = \frac{\delta\mu}{\pi} \sum_{a=1}^n \ln \frac{|x - x_a|}{\varepsilon} \partial_{x_a^i} \ln \left\langle \prod_{b=1}^n \mathcal{O}(x_b) \right\rangle - \frac{\delta\mu}{2\pi} \sum_{a=1}^n \Delta_a \frac{(x - x_a)_i}{|x - x_a|^2}$$

Saddle point action (focusing on leading logarithmic corrections)

$$e^{-S_{\text{saddle}}} = \left\langle \prod_{a=1}^{n} \mathscr{O}_{\Delta_{a}}(x_{a}) \right\rangle_{\text{CFT}} e^{\frac{\delta\mu}{2\pi}\sum_{A\neq B}\ln\frac{|x_{A}-x_{B}|}{\varepsilon}\frac{\partial}{\partial x_{A}^{i}}\ln\left\langle \prod_{a=1}^{n} \mathscr{O}_{\Delta_{a}}(x_{a})\right\rangle_{\text{CFT}}\frac{\partial}{\partial x_{B}^{i}}\ln\left\langle \prod_{a=1}^{n} \mathscr{O}_{\Delta_{a}}(x_{a})\right\rangle_{\text{CFT}}}$$

Matter correlators — cont'd

2pt functions to the 1st order

$$\left\langle \mathcal{O}_{\Delta}(x_a) \mathcal{O}_{\Delta}(x_b) \right\rangle_{\delta\mu} = \frac{1}{|x_a - x_b|^{2\Delta}} \left| 1 - \frac{4\delta\mu\Delta^2}{\pi} \frac{\ln(|x_a - x_b|/\varepsilon)}{|x_a - x_b|^2} \right| + \mathcal{O}(\delta\mu^2)$$

Kraus-Liu-Marolf, Cardy

for 2nd order, see Song He's talk

- Logarithmic corrections characteristic of $T\bar{T}$ deformation
- ◆ Neither cutoff AdS nor mixed b.c. proposals able to reproduce log corrections → Belief: proposals valid for pure gravity w/o matter?
 McGough-Mezei-Verlinde, Guica-Monten
- Log absent in stress tensor 2pt (to the 2nd order) and 3pt (to the 1st order) functions
- ✦ A stress tensor 4pt function receives logarithmic corrections (see below)

Matter correlators — cont'd

Higher order corrections by PDE

$$\frac{\partial}{\partial \mu} \ln \left\langle \prod_{a=1}^{n} \mathscr{O}_{\Delta_{a}}(x_{a}) \right\rangle_{\mu} = \frac{1}{2\pi} \sum_{A \neq B} \ln \frac{|x_{A} - x_{B}|}{\varepsilon} \frac{\partial}{\partial x_{A}^{i}} \ln \left\langle \prod_{a=1}^{n} \mathscr{O}_{\Delta_{a}}(x_{a}) \right\rangle_{\mu} \frac{\partial}{\partial x_{B}^{i}} \ln \left\langle \prod_{a=1}^{n} \mathscr{O}_{\Delta_{a}}(x_{a}) \right\rangle_{\mu} + \frac{1}{2\pi} \sum_{A \neq B} \ln \frac{|x_{A} - x_{B}|}{\varepsilon} \frac{\partial^{2}}{\partial x_{A}^{i} \partial x_{B}^{i}} \ln \left\langle \prod_{a=1}^{n} \mathscr{O}_{\Delta_{a}}(x_{a}) \right\rangle_{\mu}$$
Contribution from Gaussian fluctuations about the saddle point

This is more concisely expressed as

$$\frac{\partial}{\partial\mu} \left\langle \prod_{a=1}^{n} \mathcal{O}_{\Delta_{a}}(x_{a}) \right\rangle_{\mu} = \frac{1}{2\pi} \sum_{A \neq B} \ln \frac{|x_{A} - x_{B}|}{\varepsilon} \frac{\partial^{2}}{\partial x_{A}^{i} \partial x_{B}^{i}} \left\langle \prod_{a=1}^{n} \mathcal{O}_{\Delta_{a}}(x_{a}) \right\rangle_{\mu}$$

In momentum space, 2pt functions solved to

$$\left\langle \tilde{\mathcal{O}}_{\Delta}(k)\tilde{\mathcal{O}}_{\Delta}(-k) \right\rangle_{\mu} \propto k^{2(\Delta-1)-\frac{\mu}{4\pi}k^2\ln(k^2\varepsilon^2)}$$

other method by Cardy

Note! All orders in μ

T \overline{T} -deformed Liouville action

2pt, 3pt stress tensor correlators Kraus-Liu-Marolf, Aharony-Vaknin

- In path integral formulation, CFT stress tensor correlators calculated from Liouville (Polyakov) action
- + Idea: HS Gaussian average over Liouville action on curved spaces

$$e^{-S_{\delta\mu}[g]} = \mathcal{N}^{-1} \int [d\alpha] [d\phi] \exp\left[-\frac{1}{8\delta\mu} \int d^2x \sqrt{g} E^{ik} E^{jl} h_{ij} h_{kl} - \underbrace{S_L[g+h]}_{\text{CFT}}\right]$$

where Liouville action and metric fluctuation h_{ii}

$$S_L[g] = \frac{c}{96\pi} \int d^2 x \sqrt{g} R \Box^{-1} R , \qquad h_{ij} = \nabla_i \alpha_j + \nabla_j \alpha_i + g_{ij} \left(2\phi - \nabla_k \alpha^k \right)$$

Deformed Liouville action = saddle point action

$$S_{\delta\mu}[g] = S_L[g] - \frac{\delta\mu c^2}{(48\pi)^2} \int d^2x \sqrt{g} R \left(1 - \nabla^k \frac{1}{\Box_v + R/2} \nabla_k\right) R$$
$$\underbrace{= \delta S_L[g]}$$

 $\delta S_{\text{fluct}}[g] \sim \mathcal{O}(\delta \mu c)$ is divergent & renormalized to zero

Algorithm to compute stress tensor correlators on \mathbb{R}^2

$$-S_{\delta\mu}[g] = -S_{\delta\mu}[\underbrace{\delta}_{\ell} + h] = \frac{1}{2!} \int d^2x \int d^2x' h_{ij}(x) h_{kl}(x') \langle T^{ij}(x)T^{kl}(x') \rangle_{\delta\mu} + \cdots$$

on \mathbb{R}^2

with $h_{ij} = \partial_i \alpha_j + \partial_j \alpha_i + \delta_{ij} \left(2\phi - \partial_k \alpha^k \right)$ (abusing notation) $\int d^2 x \, h_{ij} T^{ij} = -\frac{1}{\pi} \int d^2 z \left(\partial \alpha \overline{T} + \overline{\partial} \overline{\alpha} T + \phi \Theta \right)$

***** Conformal gauge $x \equiv (z, \overline{z}) \mapsto \tilde{x} \equiv (\tilde{z}, \tilde{\overline{z}})$ for practical computations

$$ds^{2} = dz \, d\bar{z} + 2 \left[\partial \alpha(x) \, dz^{2} + \bar{\partial}\bar{\alpha}(x) \, d\bar{z}^{2} + \phi(x) \, dz \, d\bar{z} \right] = e^{2\Psi(\tilde{x})} d\tilde{z} \, d\tilde{\bar{z}}$$

solving it for coordinates $(\tilde{z}, \tilde{\bar{z}})$ and Weyl factor $\Psi(\tilde{x})$

$$\tilde{z} = z + \delta z(\alpha, \bar{\alpha}, \phi) , \quad \tilde{\bar{z}} = \bar{z} + \delta \bar{z}(\alpha, \bar{\alpha}, \phi) , \quad \Psi(\tilde{x}) = \underbrace{\Phi(x)}_{\equiv \phi - \partial \bar{\alpha} - \bar{\partial} \alpha} + \delta \Phi(\alpha, \bar{\alpha}, \phi)$$

Deformed Liouville action (to the 4th order) in conformal gauge

• No 1st order quadratic corrections \longrightarrow 2pt corrections $\sim O(\delta \mu^2)$

Kraus-Liu-Marolf

◆ 2d Green's function $\frac{1}{\partial \bar{\partial}} f(z) = \frac{1}{2\pi} \int d^2 z' \ln |z - z'|^2 f(z')$ potential source of logarithmic corrections

Stress tensor correlators from deformed Liouville action (up to contact terms)

- + 2pt functions ~ $\mathcal{O}(\delta\mu^2)$
- ✦ 3pt functions

Kraus-Liu-Marolf Aharony-Vaknin

$$\left\langle \Theta(z_1) T(z_2) \bar{T}(z_3) \right\rangle = \frac{\pi^3 \delta^3(\delta S_L)}{\delta \phi(z_1) \delta \bar{\partial} \bar{\alpha}(z_2) \delta \partial \alpha(z_3)} = -\frac{\delta \mu c^2}{4\pi} \frac{1}{z_{12}^4 \bar{z}_{13}^4} \left\langle T(z_1) \bar{T}(z_2) \bar{T}(z_3) \right\rangle = \frac{\pi^3 \delta^3(\delta S_L)}{\delta \bar{\partial} \bar{\alpha}(z_1) \delta \partial \alpha(z_2) \delta \partial \alpha(z_3)} = -\frac{\delta \mu c^2}{3\pi} \frac{1}{z_{12}^3 \bar{z}_{23}^5} + (z_2 \leftrightarrow z_3)$$

4pt functions (new to our knowledge)

$$\langle T(z_1)T(z_2)\bar{T}(z_3)\Theta(z_4)\rangle = -\frac{\pi^4 \delta^4(\delta S_L)}{\delta\bar{\partial}\bar{\alpha}(z_1)\delta\bar{\partial}\bar{\alpha}(z_2)\delta\partial\alpha(z_3)\delta\phi(z_4)} = -\frac{\delta\mu c^2}{2\pi} \frac{1}{z_{41}^2 z_{42}^2 z_{12}^2 \bar{z}_{34}^4} \langle T(z_1)T(z_2)T(z_3)\bar{T}(z_4)\rangle = \frac{\delta\mu c^2}{\underline{12\pi}} \left[\frac{1}{z_{12}^2 z_{13}^3 z_{23}^2} + \frac{1}{z_{13}^3 z_{13}^2 z_{23}^2} \right] \frac{1}{\bar{z}_{14}^3} + \mathbf{perm}(z_1, z_2, z_3) * Expected to be 6\pi from conformal perturbation theory and deformed OPE \langle T(z_1)T(z_2)\bar{T}(z_3)\bar{T}(z_4)\rangle = \frac{2\delta\mu c^2}{\pi z_{12}^5 \bar{z}_{34}^5} \left[\frac{z_{12}z_{43}}{z_{31}z_{41}} + \frac{z_{12}z_{43}}{z_{32}z_{42}} + \frac{\bar{z}_{34}\bar{z}_{21}}{\bar{z}_{13}\bar{z}_{23}} + \frac{\bar{z}_{34}\bar{z}_{21}}{\bar{z}_{14}\bar{z}_{24}} + 2\ln \left| \frac{z_{13}z_{24}}{z_{14}z_{23}} \right|^2 \right] cross ratio$$

deformation to OPE (non-local)

To-do list

- Higher-order (in $\delta\mu$) deformed Liouville action
- Non-perturbative trace Θ 2pt function (non-unitarity): Haruna-Ishii-Kawai-Sakai-Yoshida

$$\langle \Theta(x)\Theta(0)\rangle \sim + \operatorname{ve}\left(x \gg \sqrt{|\mu|}\right), \quad -\operatorname{ve}\left(x \ll \sqrt{|\mu|}\right)$$

EE (short-distance beyond perturbation in μ), chaos/OTOC, random geometry formulation for higher dimensional *TT* (Taylor, Hartman et. al.) etc.
 for EE, see Hashimoto and Nishida's talks see Tateo's talk for 4D generalization

"Conjecture" on Lyapunov exp $\lambda_L(\mu) = \frac{2\pi}{\beta(\mu)} = \frac{4G\beta}{\mu} \left(\sqrt{1 + \frac{\pi\mu}{G\beta^2}} - 1 \right)$ less chaotic for $\mu > 0$ more chaotic for $\mu < 0$ w/ $\lambda_{\text{max}} = 4\pi/\beta_{\text{Hag}}$

- Application to Liouville & Toda theories: Via AGT what is the $\mathcal{N} = 2$ SQCD counterpart of the $T\bar{T}$ deformation?
- Any insights into complex energy for $\mu > 0$ and Hagedorn for $\mu < 0$ with the help of gravity dual? for a new "rod" picture of Bose gas, see Jiang & Doyon's talks

Thank you!