4-point function from conformally coupled scalar in AdS_6

Jae-Hyuk Oh

Hanyang University

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Overview

1. Conformal correlators in momentum space
2. Holographic scalar n-point functions
3. Holographic correlation functions and CWI

Motivations(Momentum space conformal correlators) I

Motivations 1

It is very clear how the momentum flows in the interaction vertices. One may construct or say analyze the correlation functions by a Feynman-like diagrammatic language[Skenderis et. al.].

Motivations 2

One may also develop a boothtrap or conformal block techniques in momentum space[Marc Gilloz].

Motivations 3

Holographic conformal correlation functions.

Methods to get correlation in momentum space I

Method 1: Dirrect Fourier tranform

Translation and rotation symmetries

Translation symmetry is trivially satisfied in momentum space. For $SO(d)$ rotation symmetry, spatial indices are all contracted in the correlation function.

Method 2: Conformal Ward Identities Dilatation Ward identity is $\mathcal{D}\left(p_j, \frac{\partial}{\partial p_j}\right)\Phi(p_1, ..., p_n) = 0$, where Φ is the n-point correlation function in momentum space, the momenta are constrained to accept momentum conservation $\sum_{j=1}^{n} \vec{p}_j = 0$. The differential operator $\mathcal{D}$ is given by

$$\mathcal{D} = \sum_{j=1}^{n-1} p_j^a \frac{\partial}{\partial p_j^a} + \Delta', \tag{1}$$

where the $\Delta' = -\sum_{i=1}^{n} \Delta_i + (n-1)d$.

Methods to get correlation in momentum space II

Special conformal symmetry

Special conformal Wand identity is $\mathcal{K}^k\left(p_j, \frac{\partial}{\partial p_j}\right)\Phi(p_1, ..., p_n) = 0$, where Φ is the n-point correlation function in momentum space, the momenta are constrained to accept momentum conservation $\sum_{j=1}^{n}\vec{p}_j = 0$. In fact, the operator $\mathcal{K}^k$ is an differential operator with respect to $n-1$ independent momenta. More preceisely, $\mathcal{K}^k \equiv \sum_{j=1}^{n-1}\mathcal{K}_j^k$ and

$$\mathcal{K}_j^k = 2(\Delta_j - d)\frac{\partial}{\partial p_j^k} + p_j^k\frac{\partial^2}{\partial p_j^a \partial p_j^a} - 2p_j^a\frac{\partial}{\partial p_j^k \partial p_j^a}, \tag{2}$$

where the Δ_j is the conformal dimension of the j-th operator.

Momentum space conformal correations functions I

2-point function

$$\langle O(p_1)O(p_2)\rangle = (2\pi)^d \delta^{(d)}(p_1+p_2)\langle\langle O(p_1)O(p_2)\rangle\rangle, \tag{3}$$

where

$$\langle\langle O(p_1)O(p_2)\rangle\rangle = \frac{C_2 \pi^{d/2} 2^{d-2\Delta}\Gamma\left(\frac{d-2\Delta}{2}\right)}{\Gamma(\Delta)}|p|^{2\Delta-d}, \tag{4}$$

and $|p| = |p_1| = |p_2|$. Some special cases are there if $2\Delta - d = \pm 1$ as

$$\sim |p| \text{ for } 2\Delta - \mathrm{d} = +1 \quad \text{or} \quad \sim |p|^{-1} \text{for } 2\Delta - \mathrm{d} = -1. \tag{5}$$

Momentum space conformal correations functions II

3-point function

The form of the 3-point function is an integral form being given by

$$\begin{aligned}\langle\langle O(p_1)O(p_2)O(p_3)\rangle\rangle &= C_3|p_1|^{\Delta_1-\frac{d}{2}}|p_2|^{\Delta_2-\frac{d}{2}}|p_3|^{\Delta_3-\frac{d}{2}}\int_0^\infty dx\ x^{\frac{d}{2}-1} \\ &\times \left\{\prod_{j=1}^{3} K_{\Delta_j-\frac{d}{2}}(|p_j|x)\right\}, \end{aligned} \tag{6}$$

where the K represents modified Bessel function and C_3 is a constant.

Momentum space conformal correations functions III

Some simplest cases of the 3-point functions 1

When $d = 2\Delta_j \pm 1$, for all the $_j$s, then the Bessel becomes simple as

$$K_{\Delta_j - \frac{d}{2}}(|p_j|x) = \sqrt{\frac{\pi}{2|p_j|x}}e^{-|p_j|x}. \tag{7}$$

For $\Delta_+ = \frac{d+1}{2}$ operator,

$$\langle\langle O_{\Delta_+}(p_1)O_{\Delta_+}(p_2)O_{\Delta_+}(p_3)\rangle\rangle = \int_0^\infty dx x^{\frac{d-5}{2}} e^{-(|p_1|+|p_2|+|p_3|)x} \tag{8}$$

For $d = 5$ and $\Delta_+ = 3$, the 3-point function is

$$= \frac{C_3^{123}}{|p_1| + |p_2| + |p_3|} \tag{9}$$

For $d = 3$ and $\Delta_+ = 2$, it is $\tilde{C}_3^{123}$, a const.

Momentum space conformal correations functions IV

Some simplest cases of the 3-point functions 2

For $\Delta_- = \frac{d-1}{2}$ operator,

$$\langle\langle O_{\Delta_-}(p_1) O_{\Delta_-}(p_2) O_{\Delta_-}(p_3)\rangle\rangle = (|p_1||p_2||p_3|)^{-1} \int_0^\infty dx x^{\frac{d-5}{2}} e^{-(|p_1|+|p_2|+|p_3|)x} \tag{10}$$

For $d = 5$ and $\Delta_- = 2$, the 3-point function is

$$= \frac{C_3^{123}}{|p_1||p_2||p_3|(|p_1| + |p_2| + |p_3|)} \tag{11}$$

For $d = 3$ and $\Delta_- = 1$, it is

$$= \frac{\tilde{C}_3^{123}}{|p_1||p_2||p_3|}. \tag{12}$$

Another observation I

4-point function

$$\langle\langle O_{\Delta_1}(p_1)O_{\Delta_2}(p_2)O_{\Delta_3}(p_3)O_{\Delta_4}(p_4)\rangle\rangle \tag{13}$$
$$= \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_3}{(2\pi)^d}\frac{f(u,v)}{Den_3(q_j,p_k)},$$

where

$$Den_3 = |q_3|^{2\delta_{12}+d}|q_2|^{2\delta_{13}+d}|q_1|^{2\delta_{23}+d}|p_1+q_2-q_3|^{2\delta_{14}+d} \tag{14}$$
$$\times|p_2+q_3-q_1|^{2\delta_{24}+d}|p_3+q_1-q_2|^{2\delta_{34}+d},$$

$$u = \frac{|q_1||p_1+q_2-q_3|}{|q_2||p_2+q_3-q_1|}, \quad \text{and} \quad u = \frac{|q_2||p_2+q_3-q_1|}{|q_3||p_3+q_1-q_2|}, \tag{15}$$

kinds of cross ratios, $\delta_{ij} = \frac{\sum_{k=1}^4 \Delta_k}{3} - \Delta_i - \Delta_j$[Skenderis et. al.].

Another observation II

4-K integral form

A certain 4-point function is a form of 4-K(Bessel) integral form[Skederis et al., Claudio Corianò et al.] This 4-K integral form is also obtained from a holographic model with scalar theory with quartic interaction in AdS spacetime.

n-point functions

$$\Phi(p_1, p_2, ..., p_n) = \frac{C_n^{12...n}}{\left(\sum_{i=1}^{n-1} |p_i| + \left|\sum_{j=1}^{n-1} p_j\right|\right)^{(n-1)d-\frac{n}{2}(d+1)}}, \tag{16}$$

satisfies conformal Ward identities and the $\Phi(p_1, p_2, ..., p_n) \equiv \langle O_{\Delta_+}(p_1) O_{\Delta_+}(p_2) ... O_{\Delta_+}(p_n) \rangle$, a kind of conformal correlation functions and again $\Delta_+ = \frac{d+1}{2}$[J.OH]. Did not address the cross ratio issue yet.

Holography? I

Holography do what for this issue?

Holographic model: We start with the conformally scalar field theory defined in (Euclidean)AdS spacetime, given by

$$S = \int_{r>\epsilon} drd^dx\sqrt{g}\mathcal{L}(\phi, \partial\phi) + S_B, \tag{17}$$

where the spacetime is descrived by $d+1$-dimensional Euclidean AdS metric as

$$ds^2 = g_{MN}dx^M dx^N = \frac{1}{r^2}\left(dr^2 + \sum_{i=1}^{d} dx^i dx^i\right) \tag{18}$$

The S_B is a collection of boundary terms which is designed for a well defined variational problem of the theory The conformally coupled scalar field Lagrangian denstiy is given by

$$\mathcal{L}(\phi, \partial\phi) = \frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^{\frac{2(d+1)}{d-1}}, \tag{19}$$

where the Latin(capital) indices M, N... run from 1 to $d+1$, where the coordinate x^{d+1} denotes the AdS radial variable r. The mass of the conformally coupled scalar is not arbitrary, which should be

$$m^2 = -\frac{d^2-1}{4}, \tag{20}$$

where the mass term is originated from the background curvature scalar of the AdS space.

Properties of the theory I

- In mass range of

$$-\frac{d^2}{4} \leq m^2 \leq -\frac{d^2}{4} + 1, \tag{21}$$

  (1)standard and (2)alternative quantization schemes are possible. Due to the special value of the mass term, we only have two possible dual operators: $O_{\Delta_+}(x)$ and $O_{\Delta_-}(x)$[The first motivation].
- Standard quantization: Dual field theory operator $\to O_{\Delta_+}(x)$, where $\Delta_+ = \frac{d+1}{2}$.
- Alternative quantization: Dual field theory operator $\to O_{\Delta_-}(x)$, where $\Delta_- = \frac{d-1}{2}$.

- By a field redefinition, $\phi(x) = r^{\frac{d-1}{2}f(x)}$,

$$S = \int_{r>\epsilon} dr d^d x \left(\frac{1}{2}\delta^{MN}\partial_M f(x)\partial_N f(x) + \frac{\lambda}{4} f^{\frac{2(d+1)}{d-1}}(x)\right) + \frac{d-1}{2}\int d^d x \, \frac{f^2(x)}{2r}\Big|_{\epsilon}^{\infty} + S_B, \quad (22)$$

  which is massless scalar theory in flat $(d+1)$-dimensional Euclidean space, $\mathbb{R}^d \times R_+$

- The divergent piece in equation(22) is eliminated by a counter term action:

$$S_{ct} = \frac{d-1}{2}\int d^d x \sqrt{\gamma}\phi^2(x), \quad (23)$$

  where γ is determinant of the $\gamma_{\mu\nu}$, which is defined as $\gamma_{\mu\nu} = \frac{\partial x^M}{\partial x^\mu}\frac{\partial x^N}{\partial x^\nu} g_{MN}$ at $r = \epsilon$.

- The solutions of the equation of motion of the theory are rather simple and so relatively easy to study. At the zeroth order in λ, we have the two independent solutions as

$$f_1(p_i, r) = F_p e^{-|p|r} \quad \text{or} \quad f_2(p_i, r) = g_p e^{|p|r}, \tag{24}$$

  where the coefficients F_p and g_p are arbitrary (boundary directional) momentum "p_i" dependent functions. We solve the system with power expansion order by order in λ[The second motivation].
- *Regularity Conditions* The ϕ_1 is regular everywhere but the ϕ_2 does not. The stress energy tensor shows divergence at $r = \infty$, the Poincare horizon unless the absolute value of the momentum $|p|$ vanishes. Therefore, $g_p = 0$.
- *Boundary Conditions* Consider a case of Dirichlet B.C. i.e. $\delta\phi = 0$.

Properties of the theory IV

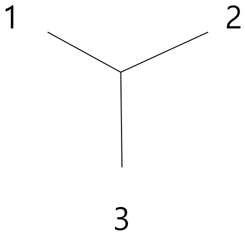
- *The self − interacton vertax* The action with a new field ϕ is a massless scalar field theory with a peculiar self interaction $\sim \phi^{\frac{2(d+1)}{d-1}}$. For a model being a possilbe quantum theory, the power of the self interaction becomes an integral number.
- when $d = 3$ corresponding to ϕ^4 self interaction.
- when $d = 5$ corresponding to ϕ^3 self interaction.

Holographic correlation functions I

2- and 3- point functions

- $\langle O_\Delta(k_1) O_\Delta(k_2)\rangle = \frac{|k_1|}{2}\delta^{(5)}(k_1+k_2)$.
- $\langle O_\Delta(k_1) O_\Delta(k_2) O_\Delta(k_3)\rangle = \frac{\lambda_5}{3\cdot(\sum_{i=1}^{3}|k_i|)}\delta^{(5)}(k_1+k_2+k_3)$, where $\lambda_5 = \frac{3\lambda}{4(2\pi)^{5/2}}$

3-point function



trace operator.pdf

Figure: 3-point function

4-point function

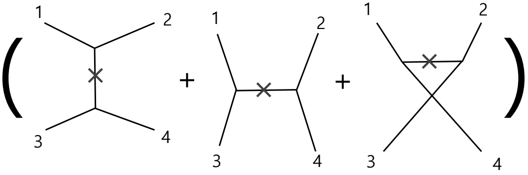
4-point function

$$\langle O_\Delta(k_1)O_\Delta(k_2)O_\Delta(k_3)O_\Delta(k_4)\rangle = -\frac{\lambda_5^2}{6\cdot(\sum_{i=1}^4 |k_i|)}\delta^{(5)}(k_1+k_2+k_3+k_4)$$
$$\times\left(\frac{1}{(|k_1|+|k_2|+|k_1+k_2|)(|k_3|+|k_4|+|k_3+k_4|)} + (k_1\leftrightarrow k_3) + (k_1\leftrightarrow k_4)\right)$$

4-point function

trace operator.pdf

$$\Phi^{(4)}_{(k_1,k_2,k_3,k_4)} = -\frac{3^2}{2(\sum_{i=1}^{4}|k_i|)}\frac{1}{3}$$



$$\equiv -\frac{3}{2}$$

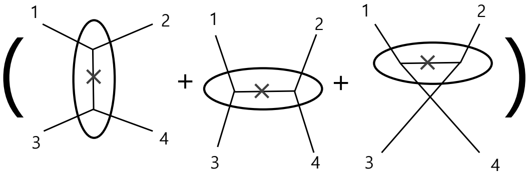


Figure:

Multi-point function

In general the fixed points of the n-trace operators are given by the following condition

$$\Phi^{(n)}_{(k_1,\ldots,k_n)} = -\frac{1}{2(\sum_{i=1}^{n}|k_i|)}\sum_{n'=1}^{n-3}(n'+2)(n-n') \times \mathcal{P}\left\{\Phi^{(n'+2)}_{(k_1,\ldots,k_{n'+1},-\sum_{j=1}^{n'+1}k_j)}\Phi^{(n-n')}_{(k_{n'+2},\ldots,k_{n-1},-\sum_{j=1}^{n-1}k_j,\sum_{j=1}^{n+1}k_j)}\right\} \quad (25)$$

5-point function and more

trace operator.pdf

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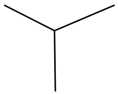

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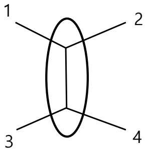

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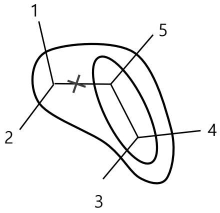

Figure: 5-point function and more

Conformal Ward Identities I

Translation and rotation symmetries

Since the holographic correlation functions are functions of absolute values of a ceratin linear combinations of momenta p_i. This ensures translation and $SO(d)$ rotation symmetries are respected.

Dilatation Ward identity is $\mathcal{D}\left(p_j, \frac{\partial}{\partial p_j}\right)\Phi(p_1, ..., p_n) = 0$, where Φ is the n-point correlation function in momentum space, the momenta are constrained to accept momentum conservation $\sum_{j=1}^{n} \vec{p}_j = 0$. The differential operator $\mathcal{D}$ is given by

$$\mathcal{D} = \sum_{j=1}^{n-1} p_j^a \frac{\partial}{\partial p_j^a} + \Delta', \tag{26}$$

where the $\Delta' = -\sum_{i=1}^{n} \Delta_i + (n-1)d$.

Conformal Ward Identities II

Special conformal symmetry

Special conformal Wand identity is $\mathcal{K}^k\left(p_j, \frac{\partial}{\partial p_j}\right)\Phi(p_1, ..., p_n) = 0$, where Φ is the n-point correlation function in momentum space, the momenta are constrained to accept momentum conservation $\sum_{j=1}^{n}\vec{p}_j = 0$. In fact, the operator $\mathcal{K}^k$ is an differential operator with respect to $n-1$ independent momenta. More preceisely, $\mathcal{K}^k \equiv \sum_{j=1}^{n-1}\mathcal{K}_j^k$ and

$$\mathcal{K}_j^k = 2(\Delta_j - d)\frac{\partial}{\partial p_j^k} + p_j^k\frac{\partial^2}{\partial p_j^a \partial p_j^a} - 2p_j^a\frac{\partial}{\partial p_j^k \partial p_j^a}, \tag{27}$$

where the Δ_j is the conformal dimension of the j-th operator.

When one applies the special conformal Ward identity on the first term of the holographic 4-point function, which gives

$$\mathcal{K}^k \Phi^{(4)}_{k_1,k_2,k_3,k_4} = 2\frac{\{u(3) - u(4)\}\{u(4) + v(4,1)\}}{u^2(3)u^2(4)v^2(4,1)} \times \left\{ \frac{(p_1 + p_2)^k}{|p_1 + p_2|} + \frac{(p_1 + p_2 + p_3)^k}{|p_1 + p_2 + p_3|} \right\}, \tag{28}$$

where

$$u(n) = \sum_{j=1}^{n-1} |p_j| + \left| \sum_{j=1}^{n-1} p_j \right|. \tag{29}$$

Namely, the $u(4) = |p_1| + |p_2| + |p_3| + |p_1 + p_2 + p_3|$.

$$v(E,n) = \left| \sum_{i=1}^{E-1-n} P_i \right| + \sum_{j=E-n}^{E-1} |p_j| + \left| \sum_{k=1}^{E-1} p_k \right|. \tag{30}$$

4-point function $d = 5$, s-channel II

Colinear condition 1

The result above that the (s-channel)4-point function is not a conformal correlation function since it does not satisfy the special confromal Ward identity in general. However, in a certain limit, it does. The right hand side of (28) is proportional to the factor,

$$u(3) - u(4) = |p_1 + p_2| - |p_3| - |p_1 + p_2 + p_3|. \tag{31}$$

Considering momentum conservation, this becomes $|p_3 + p_4| - |p_3| - |p_4|$, which vanishes when $\vec{p}_3$ and $\vec{p}_4$ are colinear.

4-point function $d = 5$, s-channel III

Colinear condition 2

There is another factor

$$\left\{ \frac{(p_1+p_2)^k}{|p_1+p_2|} + \frac{(p_1+p_2+p_3)^k}{|p_1+p_2+p_3|} \right\} = \hat{n}_{12} - \hat{n}_4, \tag{32}$$

where $\hat{n}_{12}$ is a unit vector alnog $\vec{p}_1 + \vec{p}_2$ and $\hat{n}_4$ is a nunit vector along $\vec{p}_4$. If a condition $\hat{n}_{12} = \hat{n}_4$, then the right hand side of (28) vanishes. This means that $\vec{p}_1 + \vec{p}_2$ and $\vec{p}_4$ are colinear.

If one sum up all posible channels(s, t and u), then the above argument is not hold. The only possible limit which makes the 4-point function conformal, is that $\vec{p}_i$ for $i = 1, 2, 3$ and 4 are colinear. In this limit, the 4-point function effectively becomes

$$\to \frac{C_3^{123}}{u^3(4)} = \frac{C_3^{123}}{(|p_1| + |p_2| + |p_3| + |p_1+p_2+p_3|)^3}, \tag{33}$$

4-point function $d = 5$, s-channel IV

which is expected in arXiv:2001.05379. In arXiv:2001.05379, it is proved that n-point correlation function among the scalar operators O_Δ, where $\Delta = \frac{d+1}{2}$ is given by

$$\Phi(p_1, p_2, \ldots, p_n) = \frac{C_3^{123}}{\left(\sum_{i=1}^{n-1} |p_i| + \left|\sum_{j=1}^{n-1} p_j\right|\right)^{(n-1)d - \frac{n}{2}(d+1)}}, \tag{34}$$

and the (33) is recovered when $n = 4$ and $d = 5$. Again, d is spatial dimensionality that the theory defined on and the space is Euclean.
We also point out that one may recognize that in the colinear limit, the n-point holographic correlation functions that we get effectively become

the conformal correlation functions given in arXiv:2001.05379. More precisely,

$$\langle O(p_1)O(p_2)..O(p_n)\rangle = \Phi^{(n)}_{p_1,p_2,\ldots,p_n} \sim \frac{C_n}{\left(\sum_{i=1}^{n-1}|p_i| + |\sum_{i=1}^{n-1} p_i|\right)^{2n-5}}, \tag{35}$$

where the colinear limit denotes that all the external momenta p_is for $i = 1, \ldots, n$ are alined in the same direction and so it is satisfied that $\sum_{i=1}^{n}|p_i| = |\sum_{i=1}^{n} p_i|$. Therefore, in some sense, one may say that the conformally coupled scalar theory in AdS_6 produces conformal correlation functions of a scalar operator O with $\Delta = 3$ in 5-dimensinal Euclidean space as its dual conformal field theory.