String Theory and Non-Riemannian Geometry

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String theory, Gravity and Cosmology

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Prologue

- Ever since the birth of General Relativity, Riemannian geometry has been the mathematical paradigm for modern physics. The metric, $g_{\mu\nu}$, is privileged to be the fundamental variable that provides a concrete tool to address the notion of 'spacetime'.

- However, string theory suggests to put a two-form gauge potential, $B_{\mu\nu}$, and a scalar dilaton, ϕ, on an equal footing along with the metric: Forming the closed string massless sector, they are ubiquitous in all string theories, and are transformed to one another under T-duality.

- By now, Double Field Theory has evolved to achieve its own autonomy statute, perhaps as an alternative gravitational theory to GR. Postulating the $\mathbf{O}(D, D)$ symmetry as the fundamental principle, GR and the Einstein Field Equations are unambiguosly augmented.

- Further, it turns out that DFT encompasses not only the Riemannian geometry but also non-Riemannian ones where the notion of Riemannian metric ceases to exist.

- In this talk, after reviewing these aspects of DFT, I will introduce my latest work with Shigeki Sugimoto (arXiv:2008.03084, PRL), where we examined some quantum consistency of the non-Riemannian geometries as novel backgrounds of string theory.

$\mathbf{O}(D,D)$ Symmetry Principle

- Working hypothesis is to view an $\mathbf{O}(D,D)$ invariant metric, $\mathcal{J}_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and an $\mathbf{O}(D,D)$ covariant generalized metric, $\mathcal{H}_{MN}$, as fundamental entities.

- The generalized metric should satisfy defining properties:

$$\mathcal{H}_{MN} = \mathcal{H}_{NM}\,, \qquad\qquad \mathcal{H}_M{}^K \mathcal{H}_N{}^L \mathcal{J}_{KL} = \mathcal{J}_{MN}\,.$$

- Combing the two, we have a pair of projectors (orthogonal and complete),

$$P_{MN} = \tfrac{1}{2}(\mathcal{J}_{MN} + \mathcal{H}_{MN})\,, \qquad\qquad \bar{P}_{MN} = \tfrac{1}{2}(\mathcal{J}_{MN} - \mathcal{H}_{MN})\,,$$

- Further, taking the 'square root' of each projector,

$$P_{MN} = V_M{}^p V_N{}^q \eta_{pq}\,, \qquad\qquad \bar{P}_{MN} = \bar{V}_M{}^{\bar{p}} \bar{V}_N{}^{\bar{q}} \bar{\eta}_{\bar{p}\bar{q}}\,,$$

  we obtain a pair of DFT-vielbeins which meet their own defining properties,

$$V_{Mp} V^M{}_q = \eta_{pq}\,, \qquad \bar{V}_{M\bar{p}} \bar{V}^M{}_{\bar{q}} = \bar{\eta}_{\bar{p}\bar{q}}\,, \qquad V_{Mp} \bar{V}^M{}_{\bar{q}} = 0\,.$$

- Besides, there is an $\mathbf{O}(D,D)$ singlet dilaton, d, which can be gauge fixed by diffeomorphisms.

Semi-covariant formalism

w/ Imtak Jeon and Kanghoon Lee 2010, 2011

- In GR, the Christoffel symbol is the unique metric-compatible connection, $\nabla_{\lambda} g_{\mu\nu} = 0$, which satisfies either a torsionless condition, or an alternative condition that the metric is the only ingredient to form the connection.

- Similarly, the connection in DFT can be uniquely fixed

$$\Gamma_{LMN}=2\big(P\partial_L P\bar{P}\big)_{[MN]}+2\Big(\bar{P}_{[M}{}^{J}\bar{P}_{N]}{}^{K}-P_{[M}{}^{J}P_{N]}{}^{K}\Big)\partial_J P_{KL}-\tfrac{4}{D-1}\Big(\bar{P}_{L[M}\bar{P}_{N]}{}^{K}+P_{L[M}P_{N]}{}^{K}\Big)\Big(\partial_K d+(P\partial^J P\bar{P})_{[JK]}\Big)$$

while the compatibility holds,

$$\nabla_L \mathcal{J}_{MN} = 0\,, \qquad \nabla_L \mathcal{H}_{MN} = 0\,, \qquad \nabla_L d = -\tfrac{1}{2} e^{2d} \nabla_L \left(e^{-2d}\right) = 0\,.$$

- Further, spin connections for twofold local Lorentz symmetries can be determined,

$$\Phi_{Mpq} = V^{N}{}_{p} \nabla_M V_{Nq}\,, \qquad \bar{\Phi}_{M\bar{p}\bar{q}} = \bar{V}^{N}{}_{\bar{p}} \nabla_M \bar{V}_{N\bar{q}}$$

by requiring that Master derivative,

$$\mathcal{D}_M = \partial_M + \Gamma_M + \Phi_M + \bar{\Phi}_M = \nabla_M + \Phi_M + \bar{\Phi}_M$$

should be compatible with the vielbeins,

$$\mathcal{D}_M V_{Np} = \nabla_M V_{Np} + \Phi_{Mp}{}^{q} V_{Nq} = 0\,, \qquad \mathcal{D}_M \bar{V}_{N\bar{p}} = \nabla_M \bar{V}_{N\bar{p}} + \bar{\Phi}_{M\bar{p}}{}^{\bar{q}} \bar{V}_{N\bar{q}} = 0\,.$$

These spin connections are essentially the ‘generalized fluxes’ à la Aldazabala, Marques, Nunez, and Grana.

Semi-covariant formalism

w/ Imtak Jeon and Kanghoon Lee 2010, 2011

- Semi-covariant Riemann curvature :

$$S_{KLMN} = S_{[KL][MN]} = S_{MNKL} := \tfrac{1}{2}\left(R_{KLMN} + R_{MNKL} - \Gamma^{J}{}_{KL}\Gamma_{JMN}\right) , \qquad S_{[KLM]N} = 0 ,$$

where R_{ABCD} denotes the ordinary "field strength", $R_{CDAB}=\partial_A\Gamma_{BCD}-\partial_B\Gamma_{ACD}+\Gamma_{AC}{}^{E}\Gamma_{BED}-\Gamma_{BC}{}^{E}\Gamma_{AED}$.

By construction, it varies as $\delta S_{ABCD}=\nabla_{[A}\delta\Gamma_{B]CD}+\nabla_{[C}\delta\Gamma_{D]AB}$, hence good for variational principle.

- Semi-covariance means, with $\mathcal{P}_{LMN}{}^{EFG}=P_L{}^E P_{[M}{}^{[F}P_{N]}{}^{G]}+\frac{2}{P_K{}^K-1}P_{L[M}P_{N]}{}^{[F}P^{G]E}$,

$$\delta_\xi\left(\nabla_L T_{M_1\cdots M_n}\right) = \hat{\mathcal{L}}_\xi\left(\nabla_L T_{M_1\cdots M_n}\right)+\sum_{i=1}^{n} 2(\mathcal{P}+\bar{\mathcal{P}})_{LM_i}{}^{NEFG}\partial_E\partial_F\xi_G\, T_{M_1\cdots M_{i-1}NM_{i+1}\cdots M_n}$$

$$\delta_\xi S_{KLMN} = \hat{\mathcal{L}}_\xi S_{KLMN} + 2\nabla_{[K}\left[(\mathcal{P}+\bar{\mathcal{P}})_{L][MN]}{}^{EFG}\partial_E\partial_F\xi_G\right] + 2\nabla_{[M}\left[(\mathcal{P}+\bar{\mathcal{P}})_{N][KL]}{}^{EFG}\partial_E\partial_F\xi_G\right]$$

$$\delta_\xi\Gamma_{CAB} = \hat{\mathcal{L}}_\xi\Gamma_{CAB} + 2\left[(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE} - \delta_C^F\delta_A^D\delta_B^E\right]\partial_F\partial_{[D}\xi_{E]}$$

- The red-colored anomalies can be easily projected out to give fully covariant objects, *e.g.*

$$\mathcal{D}_p T_{\bar{q}} = \nabla_L T_M V^L{}_p \bar{V}^M{}_{\bar{q}} , \qquad S_{p\bar{q}} = S_{MN} V^M{}_p \bar{V}^N{}_{\bar{q}} \quad (\text{Ricci}) , \qquad S_{(0)} = S_{pq}{}^{pq} - S_{\bar{p}\bar{q}}{}^{\bar{p}\bar{q}} \quad (\text{scalar})$$

$$\gamma^p\mathcal{D}_p\rho , \quad \mathcal{D}_{\bar{p}}\rho \quad (\text{Dirac}) , \qquad \mathcal{F}_{p\bar{q}} = \left(\nabla_M W_N - \nabla_N W_M - i\left[W_M, W_N\right]\right) V^M{}_p \bar{V}^N{}_{\bar{q}} \quad (\text{Yang–Mills})$$

$$\mathcal{D}_\pm\mathcal{C} = \gamma^p\mathcal{D}_p\mathcal{C} \pm \gamma^{(D+1)}\mathcal{D}_{\bar{p}}\mathcal{C}\bar{\gamma}^{\bar{p}} , \quad (\mathcal{D}_\pm)^2 = 0 \quad \Longrightarrow \quad \mathcal{F} = \mathcal{D}_+\mathcal{C} \quad (\text{RR flux}) .$$

Einstein Equations from General Covariance

w/ S. Angus and K. Cho 1804.00964

- Let us consider a DFT action coupled to generic matter, Υ_a (*e.g.* RR, fermions, or tachyon),

$$\text{Action} = \int_\Sigma e^{-2d}\left[\tfrac{1}{16\pi G}S_{(0)} + L_{\rm matter}\left(\Upsilon_a, \mathcal{D}_A\Upsilon_b\right)\right]$$

Deriving the variation of the action induced by all the fields, d, V_{Ap}, $\bar{V}_{Ap}$, Υ_a ,

$$\delta\text{Action} = \int_\Sigma e^{-2d}\left[\tfrac{1}{4\pi G}\bar{V}^{M\bar{q}}\delta V_M{}^p(S_{p\bar{q}} - 8\pi G K_{p\bar{q}}) - \tfrac{1}{8\pi G}\delta d(S_{(0)} - 8\pi G T_{(0)}) + \delta\Upsilon_a\frac{\delta L_{\rm matter}}{\delta\Upsilon_a}\right]$$

we naturally define

$$K_{p\bar{q}} := \tfrac{1}{2}\left(V_{Mp}\frac{\delta L_{\rm matter}}{\delta\bar{V}_M{}^{\bar{q}}} - \bar{V}_{M\bar{q}}\frac{\delta L_{\rm matter}}{\delta V_M{}^p}\right) = -2V_{Mp}\bar{V}_{N\bar{q}}\frac{\delta L_{\rm matter}}{\delta\mathcal{H}_{MN}}\,, \qquad T_{(0)} := e^{2d}\times\frac{\delta\left(e^{-2d}L_{\rm matter}\right)}{\delta d}$$

- The diffeomorphic invariance of the action,

$$0 = \int_\Sigma e^{-2d}\left[\tfrac{1}{8\pi G}\xi^N\mathcal{D}^M\left[4V_{[M}{}^p\bar{V}_{N]}{}^{\bar{q}}(S_{p\bar{q}} - 8\pi G K_{p\bar{q}}) - \tfrac{1}{2}\mathcal{J}_{MN}(S_{(0)} - 8\pi G T_{(0)})\right] + \delta_\xi\Upsilon_a\frac{\delta L_{\rm matter}}{\delta\Upsilon_a}\right]$$

further guides us to identify the Einstein curvature, w/ S. Rey, W. Rim, and Y. Sakatani 2015

$$G_{MN} := 4V_{[M}{}^p\bar{V}_{N]}{}^{\bar{q}}S_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{MN}S_{(0)}\,, \qquad \nabla_M G^{MN} = 0 \qquad \text{(off-shell)}\,,$$

and the Energy-Momentum tensor,

$$T_{MN} := 4V_{[M}{}^p\bar{V}_{N]}{}^{\bar{q}}K_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{MN}T_{(0)}\,, \qquad \mathcal{D}_M T^{MN} = 0 \qquad \text{(on-shell)}\,.$$

- **Equating them, we obtain the Einstein Double Field Equations:** $G_{MN} = 8\pi G T_{MN}$

Question: Is DFT a mere reformulation of SUGRA in an $\mathbf{O}(D,D)$ manifest fashion?

The answer would be (and had been) yes, if we employ the well-known parametrization,

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix} \qquad e^{-2d} = \sqrt{|g|}e^{-2\phi}$$

Giveon, Rabinovici, Veneziano '89, Duff '90

Upon this parametrization, EDFEs, $G_{MN} = 8\pi G T_{MN}$, unify

$$\begin{aligned} R_{\mu\nu} + 2\bigtriangledown_{\mu}(\partial_{\nu}\phi) - \tfrac{1}{4}H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma} &= 8\pi G K_{(\mu\nu)} \\ e^{2\phi}\bigtriangledown^{\rho}\left(e^{-2\phi}H_{\rho\mu\nu}\right) &= 16\pi G K_{[\mu\nu]} \\ R + 4\Box\phi - 4\partial_{\mu}\phi\partial^{\mu}\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} &= 8\pi G T_{(0)} \end{aligned}$$

which implies Stringy Newton Gravity in a non-relativistic limit ($D = 4$),

$$\nabla^2\Phi_{\mathrm{Newton}} = 4\pi G\rho + \mathbf{H}\cdot\mathbf{H}\,, \quad \nabla\cdot\mathbf{H} = \mathbf{0}\,, \quad \nabla\times\mathbf{H} = \mathbf{4\pi G\,K}$$

$\Rightarrow$ ***H*-flux as dark matter,** w/ Kevin Morand and Kyungho Cho 2019

- The truth is that, DFT works perfectly fine with any generalized metric that satisfies the defining properties: $\mathcal{H}_{MN} = \mathcal{H}_{NM}$, $\mathcal{H}_M{}^K\mathcal{H}_N{}^L\mathcal{J}_{KL} = \mathcal{J}_{MN}$.

  And the above famous parametrization is not the most general solution to them.

  Hence the answer to the question can be **negative.**

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Classification

w/ Kevin Morand 1707.03713

The most general parametrizations of the generalized metric, $\mathcal{H}_{MN}=\mathcal{H}_{NM}$, $\mathcal{H}_M{}^K\mathcal{H}_N{}^L\mathcal{J}_{KL}=\mathcal{J}_{MN}$, can be classified by two non-negative integers, $(n,\bar{n})$, $0\leq n+\bar{n}\leq D$:

$$\mathcal{H}_{MN}=\left(\begin{array}{cc} H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda}+Y_i^\mu X_\lambda^i-\bar{Y}_{\bar{\imath}}^\mu\bar{X}_\lambda^{\bar{\imath}} \\ B_{\kappa\rho}H^{\rho\nu}+X_\kappa^iY_i^\nu-\bar{X}_\kappa^{\bar{\imath}}\bar{Y}_{\bar{\imath}}^\nu & K_{\kappa\lambda}-B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda}+2X^i_{(\kappa}B_{\lambda)\rho}Y_i^\rho-2\bar{X}^{\bar{\imath}}_{(\kappa}B_{\lambda)\rho}\bar{Y}_{\bar{\imath}}^\rho \end{array}\right)$$

$$=\left(\begin{array}{cc} 1 & 0 \\ B & 1 \end{array}\right)\left(\begin{array}{cc} H & Y_i(X^i)^T-\bar{Y}_{\bar{\imath}}(\bar{X}^{\bar{\imath}})^T \\ X^i(Y_i)^T-\bar{X}^{\bar{\imath}}(\bar{Y}_{\bar{\imath}})^T & K \end{array}\right)\left(\begin{array}{cc} 1 & -B \\ 0 & 1 \end{array}\right)$$

i) Symmetric and skew-symmetric fields : $\quad H^{\mu\nu}=H^{\nu\mu},\quad K_{\mu\nu}=K_{\nu\mu},\quad B_{\mu\nu}=-B_{\nu\mu}\,;$

ii) Two kinds of zero eigenvectors: with $i,j=1,2,\cdots,n$ & $\bar{\imath},\bar{\jmath}=1,2,\cdots,\bar{n}$,

$$H^{\mu\nu}X_\nu^i=0\,,\qquad H^{\mu\nu}\bar{X}_\nu^{\bar{\imath}}=0\,,\qquad K_{\mu\nu}Y_j^\nu=0\,,\qquad K_{\mu\nu}\bar{Y}_{\bar{\jmath}}^\nu=0\,;$$

iii) Completeness relation: $\qquad H^{\mu\rho}K_{\rho\nu}+Y_i^\mu X_\nu^i+\bar{Y}_{\bar{\imath}}^\mu\bar{X}_\nu^{\bar{\imath}}=\delta^\mu{}_\nu.$

– Both $H^{\mu\nu}$ and $K_{\mu\nu}$ have the signature, $(t,s,n+\bar{n})$ for temporal, spatial, and non-Riemannian dimensions.

– **The underlying coset is** $\frac{\mathbf{O}(D,D)}{\mathbf{O}(t+n,s+n)\times\mathbf{O}(s+\bar{n},t+\bar{n})}$ **with dimensions** $D^2-(n-\bar{n})^2$**, while** $\mathcal{H}_M{}^M=2(n-\bar{n})$.

Classification

w/ Kevin Morand 1707.03713

I. $(n,\bar{n})=(0,0)$ corresponds to the Riemannian case or Generalized Geometry à la Hitchin.

II. $(n,\bar{n})\neq(0,0)$: Non-Riemannian. Examples include

- $(1,0)$ Newton–Cartan gravity, $\mathrm{d}s^2=-c^2\mathrm{d}t^2+\mathrm{d}\mathbf{x}^2$, $\lim_{c\to\infty} g^{-1}$ is finite & degenerate
- $(1,1)$ Stringy/torsional Newton–Cartan (curved), Gomis–Ooguri non-relativistic string (flat)
  Andringa, Bergshoeff, Gomis, de Roo 2012; Harmark, Hartong, Obers 2017; w/ Melby-Thompson, Meyer, Ko 2015; Blair 2019
- $(D{-}1,0)$ ultra-relativistic Carroll gravity, $\mathrm{d}\tau^2=-\mathrm{d}t^2+c^{-2}\mathrm{d}\mathbf{x}^2$, $\lim_{c\to 0} g^{-1}$ is finite & degenerate
- $(n,\bar{n})$ with $n+\bar{n}=D$: maximally non-Riemannian with no time and no space.

  In particular, $(D,0)$ or $(0,D)$ is uniquely given as $\mathcal{H}=\pm\mathcal{J}$ with trivial coset, $\frac{\mathbf{O}(D,D)}{\mathbf{O}(D,D)}$.
  These two are the perfectly $\mathbf{O}(D,D)$-symmetric vacua of DFT with no moduli.

  "*Spacetime emerges after SSB of $\mathbf{O}(D,D)$, identifying $\{g,B\}$ as Nambu–Goldstone boson moduli.*"
  Berman, Blair, and Otsuki 2019

– Generically, on worldsheet, string becomes chiral and anti-chiral over the n and $\bar{n}$ dimensions:

$$X^i_\mu\,\partial_+x^\mu(\tau,\sigma)=0\,,\qquad\qquad \bar{X}^{\bar{\imath}}_\mu\,\partial_-x^\mu(\tau,\sigma)=0\,.$$

– Further, analysis on Killing equations, $\hat{\mathcal{L}}_\xi\mathcal{H}_{MN}=8\bar{P}_{(M}{}^{[K}P_{N)}{}^{L]}\nabla_K\xi_L=0$, reveals that non-Riemannian isometries are supertranslational. w/ Chris Blair and Gerben Oling, in preparation.

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Fluctuation Analysis

w/ Melby-Thompson, Meyer, Ko 2015, and w/ K. Cho 2019

- Linearized Einstein Double Field Equations for vacuum, $G_{MN} = 0$, are

$$(P^{MN} - \bar{P}^{MN})\nabla_M \partial_N \delta d - \tfrac{1}{4}\nabla_M \nabla_N \delta\mathcal{H}^{MN} = 0 \qquad (\text{scalar})$$

$$P_M{}^K \bar{P}_N{}^L \nabla_K \partial_L \delta d + \tfrac{1}{8}(P_M{}^K \bar{\Delta}_N{}^L - \Delta_M{}^K \bar{P}_N{}^L)\delta\mathcal{H}_{KL} = 0 \qquad (\text{Ricci})$$

which are invariant under the diffeomorphically inherited transformations of the fluctuations,

$$\delta_\xi(\delta d) = \hat{\mathcal{L}}_\xi d\,, \qquad \delta_\xi(\delta\mathcal{H}_{MN}) = \hat{\mathcal{L}}_\xi \mathcal{H}_{MN}\,.$$

Note $\Delta_K{}^L = P_K{}^L P^{MN}\nabla_M\nabla_N - 2P_K{}^N P^{LM}(\nabla_M\nabla_N - S_{MN})$ and similarly for $\bar{\Delta}_K{}^L$ with $P \leftrightarrow \bar{P}$.

- Analysis of $\delta\mathcal{H}_{MN}$ around a generic $(n, \bar{n})$ non-Riemannian background shows
  - $\delta\mathcal{H}_{MN}$'s form the coset $\frac{\mathbf{O}(D,D)}{\mathbf{O}(t+n,s+n)\times\mathbf{O}(s+\bar{n},t+\bar{n})}$ with dimensions, $D^2 - (n-\bar{n})^2$.
  - $\delta\mathcal{H}_{MN}$'s include those which decrease the 'non-Riemanniânity', *e.g.* $(n,\bar{n}) \to (n-1, \bar{n}-1)$.
    Riemannian spacetime may then emerge out of a maximally non-Riemannian background.
    It also suggests that the various non-Riemannian gravities, such as Newton–Cartan, should better be identified as different solution sectors of DFT rather than viewed as independent theories.

Section condition = Doubled-yet-Gauged

JHP 1304.5946

- DFT necessarily imposes the 'section condition' for $x^M = (\tilde{x}_\mu, x^\nu)$,

$$\partial_M \partial^M = \partial_\mu \tilde{\partial}^\mu + \tilde{\partial}^\mu \partial_\mu = 0$$

which can be generically solved by letting $\tilde{\partial}^\mu = 0$, up to $\mathbf{O}(D,D)$ rotations.

- The section condition is mathematically equivalent to a certain translational invariance:

$$\Phi_s(x) = \Phi_s(x + \Delta)\,, \qquad \Delta^M = \Phi_t \partial^M \Phi_u\,,$$

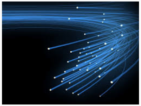
where $\Phi_s, \Phi_t, \Phi_u \in \{\, d, \mathcal{H}_{MN}\,, \xi^M\,, \partial_N d\,, \partial_L \mathcal{H}_{MN}, \cdots \}$, arbitrary functions appearing in DFT, and Δ^M is said to be 'derivative-index-valued'.

► **'Physics' should be invariant under such a shift of the doubled coordinates.**

Doubled coordinates, $x^M = (\tilde{x}_\mu\,, x^\nu)$, are gauged through an equivalence relation,

$$x^M \sim x^M + \Delta^M(x) \quad : \quad \textbf{Coordinate Gauge Symmetry}$$

where Δ^M is derivative-index-valued.



Each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^D$.

- With $\tilde{\partial}^\mu = 0$ and $\Delta^M = c_\mu \partial^M x^\mu$, we note $(\tilde{x}_\mu\,, x^\nu) \sim (\tilde{x}_\mu + c_\mu\,, x^\nu)$.
$\mathbf{O}(D,D)$ then rotates the gauged directions and hence the section.

c.f. Alfonsi 2019, 2020 for formal discussion

- In DFT, the usual coordinate basis of one-forms, $\mathrm{d}x^A$, is not covariant:
  - Neither diffeomorphic covariant,

$$\delta x^M = \xi^M \,, \qquad \delta(\mathrm{d}x^M) = \mathrm{d}x^N \partial_N \xi^M \neq \mathrm{d}x^N(\partial_N \xi^M - \partial^M \xi_N)$$

  - Nor invariant under the coordinate gauge symmetry,

$$\mathrm{d}x^M \quad \longrightarrow \quad \mathrm{d}\big(x^M + \Delta^M\big) \neq \mathrm{d}x^M \,.$$

$\Rightarrow$ The naive contraction, $\mathrm{d}x^M \mathrm{d}x^N \mathcal{H}_{MN}$, is not an invariant scalar nor 'proper length'.

- These problems can be all cured by gauging the one-forms, $\mathrm{d}x^A$, explicitly,

$$Dx^M := \mathrm{d}x^M - \mathcal{A}^M \,, \qquad \mathcal{A}^M \partial_M = 0 \quad \text{(derivative-index-valued)}\,.$$

Dx^M is covariant:

$$\delta x^M = \Delta^M \,, \quad \delta \mathcal{A}^M = \mathrm{d}\Delta^M \qquad \Longrightarrow \qquad \delta(Dx^M) = 0\,;$$

$$\delta x^M = \xi^M \,, \quad \delta \mathcal{A}^M = \partial^M \xi_N (\mathrm{d}x^N - \mathcal{A}^N) \quad \Longrightarrow \quad \delta(Dx^M) = Dx^N(\partial_N \xi^M - \partial^M \xi_N)\,.$$

– Concretely, setting $\tilde{\partial}^\mu = 0$ and $\mathcal{A}^M = A_\lambda \partial^M x^\lambda = (A_\mu, 0)$, we get $Dx^M = (\mathrm{d}\tilde{x}_\mu - A_\mu, \mathrm{d}x^\nu)\,.$

Proper Length & Point Particle

w/ S. Ko, M. Suh 2016 and w/ T. Basile, E. Joung 2019

- With $Dx^M = \mathrm{d}x^M - \mathcal{A}^M$, it is possible to define the 'proper length' through a path integral,

$$\textbf{Proper Length} := -\ln\left[\int \mathcal{D}\mathcal{A}\ \exp\left(-\int\sqrt{Dx^M Dx^N \mathcal{H}_{MN}}\right)\right].$$

– With $\tilde{\partial}^\mu = 0$, $\mathcal{A}^M = (A_\mu, 0)$, and the decomposition, $A_\mu = (KH + X^i Y_i + \bar{X}^{\bar{\imath}}\bar{Y}_{\bar{\imath}})_\mu{}^\nu A_\nu$,

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– Essentially, $(KHA)_\mu$ leads to Gaussian integral, while $(X\cdot YA)_\nu$ and $(\bar{X}\cdot\bar{Y}A)_\mu$ are Lagrange multipliers to <u>freeze</u> the non-Riemannian dimensions: $X^i_\mu \mathrm{d}x^\mu = 0$, $\bar{X}^{\bar{\imath}}_\mu \mathrm{d}x^\mu = 0$

Proper Length then reduces consistently to $\int\sqrt{\mathrm{d}x^\mu \mathrm{d}x^\nu K_{\mu\nu}(x)}$, which is independent of $\tilde{x}_\mu$. Hence, it measures the distance between two gauge orbits, as desired.

- This line of thinking readily leads to a completely covariant particle action (Faddeev–Popov),

$$S_{\mathrm{particle}} = \int \mathrm{d}\tau\ \tfrac{1}{2}e^{-1}D_\tau x^M D_\tau x^N \mathcal{H}_{MN}(x) - \tfrac{1}{2}m^2 e + k_M\mathcal{A}^M + k(e-1) + \tfrac{1}{2}\theta_M\dot{\theta}^M + \sum_{a=1}^{2}\tfrac{1}{2}\vartheta_a\dot{\vartheta}^a$$

where $\theta^M = (C_\mu, B^\nu)$ and $\vartheta^a = (c, b)$. This is a constrained system, and the relevant Dirac bracket coincides with the graded Poisson bracket introduced by Deser and Sämann 2016.

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Doubled-yet-Gauged String

- The formalism extends to string: Hull 2006; w/ Kanghoon Lee 2013

$$S_{\rm string} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \; -\tfrac{1}{2}\sqrt{-h}h^{\alpha\beta}D_\alpha x^M D_\beta x^N \mathcal{H}_{MN}(x) - \epsilon^{\alpha\beta}D_\alpha x^M \mathcal{A}_{\beta M}$$

which is manifestly $\mathbf{O}(D,D)$ symmetric, doubled target spacetime diffeomorphism covariant, the coordinate gauge symmetry invariant, and worldsheet diffeomorphism invariant.

– Classically, upon a generic $(n,\bar{n})$ non-Riemannian backgrounds, after integrating out the auxiliary gauge potential —quadratic in $(KHA)_\mu$ and linear in $(X{\cdot}YA)_\mu, (\bar{X}{\cdot}\bar{Y}A)_\mu$—

$$S_{\rm string} \;\Rightarrow\; \frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma \; -\tfrac{1}{2}\sqrt{-h}h^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu K_{\mu\nu} + \tfrac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu B_{\mu\nu} + \tfrac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha \tilde{x}_\mu \partial_\beta x^\mu$$

and string becomes chiral and anti-chiral over the n and $\bar{n}$ dimensions respectively,

$$X_\mu^i\left(\partial_\alpha x^\mu + \frac{1}{\sqrt{-h}}\epsilon_\alpha{}^\beta\partial_\beta x^\mu\right) = 0\,, \qquad \bar{X}_\mu^{\bar{\imath}}\left(\partial_\alpha x^\mu - \frac{1}{\sqrt{-h}}\epsilon_\alpha{}^\beta\partial_\beta x^\mu\right) = 0\,.$$

– κ-symmetric Green–Schwarz superstring extension unifies IIA & IIB JHP 1609.04265

$$S_{\rm GS} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \; -\tfrac{1}{2}\sqrt{-h}h^{\alpha\beta}\Pi_\alpha^M \Pi_\beta^N \mathcal{H}_{MN} - \epsilon^{\alpha\beta}D_\alpha x^M\left(\mathcal{A}_{\beta M} - i\Sigma_{\beta M}\right)$$

where $\Pi_\alpha^M = D_\alpha x^M - i\Sigma_\alpha^M$, $\Sigma_\alpha^M = \bar{\theta}\gamma^M\partial_\alpha\theta + \bar{\theta}'\bar{\gamma}^M\partial_\alpha\theta'$ (IIA & IIB distinguished by $V_{Ap}, \bar{V}_{B\bar{q}}$).

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BRST quantization 1/4

w/ Shigeki Sugimoto 2008.03084

- Toward the BRST quantization, we fix the background to be constant flat and maximally non-Riemannian: $\frac{\mathbf{O}(D,D)}{\mathbf{O}(n,n)\times\mathbf{O}(\bar{n},\bar{n})}$, $n+\bar{n}=D$. With the decomposition of $\mu=(i,\bar{\imath})$, performing a field redefinition of the potential, $A_{\alpha\mu}$, to a coordinate gauge symmetry invariant quantity,

$$p_{\alpha i}:=\partial_\alpha\tilde{x}_i-A_{\alpha i}\,,\qquad\qquad p_{\alpha\bar{\imath}}:=A_{\alpha\bar{\imath}}-\partial_\alpha\tilde{x}_{\bar{\imath}}\,,$$

the doubled-yet-gauged string Lagrangian takes the form,

$$\mathcal{L}_0=-\sqrt{-h}\Big(p_{\alpha i}h_+^{\alpha\beta}\partial_\beta x^i+p_{\alpha\bar{\imath}}h_-^{\alpha\beta}\partial_\beta x^{\bar{\imath}}\Big)+\epsilon^{\alpha\beta}\partial_\alpha\tilde{x}_\mu\partial_\beta x^\mu$$

where $h_\pm^{\alpha\beta}=\frac{1}{2}\big(h^{\alpha\beta}\pm\frac{1}{\sqrt{-h}}\epsilon^{\alpha\beta}\big)$ are 2×2 chirality projectors on worldsheet, and $p_{\alpha\mu}$'s are evidently the Lagrange multipliers, imposing $h_+^{\alpha\beta}\partial_\beta x^i=0$ and $h_-^{\alpha\beta}\partial_\beta x^{\bar{\imath}}=0$.

– It is convenient to parametrize $\sqrt{-h}h^{\alpha\beta}$, which has unit determinant, by two variables,

$$\sqrt{-h}h^{\tau\tau}=-\frac{1}{e}\,,\qquad\sqrt{-h}h^{\tau\sigma}=\frac{\omega}{e}\,,\qquad\sqrt{-h}h^{\sigma\sigma}=e-\frac{\omega^2}{e}\,.$$

We may read off how these transform under $\delta\sigma^\alpha=c^\alpha$, from the standard rule for $\sqrt{-h}h^{\alpha\beta}$:

$$\delta e=c^\alpha\partial_\alpha e+(\partial_\tau c^\tau-\partial_\sigma c^\sigma)e-2\partial_\sigma c^\tau\omega e\,,\quad\delta\omega=c^\alpha\partial_\alpha\omega+(\partial_\tau c^\tau-\partial_\sigma c^\sigma)\omega+\partial_\tau c^\sigma-\partial_\sigma c^\tau(\omega^2+e^2)\,.$$

– From the orthogonality of $h_+^{\alpha\beta}$ and $h_-^{\alpha\beta}$, the Lagrangian possesses an extra gauge symmetry,

$$\left\{\begin{array}{l}\delta p_{\alpha i}=\hat{C}_{\beta i}h_-^{\beta}{}_\alpha\\[6pt]\delta p_{\alpha\bar{\imath}}=\hat{C}_{\beta\bar{\imath}}h_+^{\beta}{}_\alpha\end{array}\right.\quad\Longleftrightarrow\quad\left\{\begin{array}{l}\delta p_{\pm i}=(\omega-e\pm1)C_i\\[6pt]\delta p_{\pm\bar{\imath}}=(\omega+e\pm1)C_{\bar{\imath}}\end{array}\right.$$

BRST quantization 2/4

- Fixing all the gauges,

$$e \equiv 1\,, \qquad \omega \equiv 0\,, \qquad \tilde{x}_\mu \equiv 0\,, \qquad p_{-i} \equiv 0\,, \qquad p_{+\bar{\imath}} \equiv 0$$

  the full Lagrangian with Faddeev–Popov ghosts is

$$\mathcal{L}_{\textbf{full}} = \mathcal{L}_0 - i\delta_{\mathbf{B}}\Big(\ln e\, b_e + \omega b_\omega + \tilde{x}_\mu \tilde{B}^\mu + p_{-i}B^i + p_{+\bar{\imath}}B^{\bar{\imath}}\Big)$$

  where $\{b_e, b_\omega, \tilde{B}^\mu, B^i, B^{\bar{\imath}}\}$ are anti-ghosts, and $\delta_{\mathbf{B}}$ denotes the nilpotent BRST transformation.

- After integrating out all the trivially-decoupled component fields, the full Lagrangian reduces to

$$\mathcal{L}_{\textbf{red.}} = 2\big(p_{+i}\partial_- x^i + p_{-\bar{\imath}}\partial_+ x^{\bar{\imath}} + ib_{++}\partial_- c^+ + ib_{--}\partial_+ c^-\big)$$

  with left-moving $\{x^i, p_{+i}, c^+, b_{++}\}$ and right-moving $\{x^{\bar{\imath}}, p_{-\bar{\imath}}, c^-, b_{--}\}$.

  Naturally, the bosonic component fields form $D = n + \bar{n}$ pairs of $\beta\gamma$-system:
  $\beta_i \equiv p_{+i}$, $\bar{\beta}_{\bar{\imath}} \equiv p_{-\bar{\imath}}$ (conformal weight 1) and $\gamma^j \equiv x^j$, $\bar{\gamma}^{\bar{\jmath}} \equiv x^{\bar{\jmath}}$ (conformal weight 0).
  Each pair contributes to a central charge by two.

BRST quantization 3/4

w/ Shigeki Sugimoto 2008.03084

- The BRST charge decomposes, $Q_{\mathbf{B}} = Q_{\mathbf{L}} + Q_{\mathbf{R}}$, with

$$Q_{\mathbf{L}} = \oint \mathrm{d}\sigma \; \beta_i \partial_+ \gamma^i c^+ + i(b_{++}\partial_+ c^+)c^+ = : \sum_{m,n=-\infty}^{\infty} n\left(-i\beta_{mi}\gamma^i_n + b_m c_n\right) c_{-m-n} : \; - a\, c_0$$

  and mirroring expression for $Q_{\mathbf{R}}$.

  The normal ordering constant arises upon quantization: $[\gamma^i_m, \beta_{nj}] = i\delta^i{}_j \delta_{m+n}\,,\ \{b_m, c_n\} = \delta_{m+n}\,.$

- The BRST charges, $Q_{\mathbf{L}}, Q_{\mathbf{R}}$, are nilpotent, if and only if $n = \bar{n} = 13$, implying the usual critical dimension, $D = 26$, since the central charges are $\mathbf{c}_{\mathbf{L}} = 2n{-}26$ and $\mathbf{c}_{\mathbf{R}} = 2\bar{n}{-}26$, both of which should vanish.

- Physical states are annihilated by $Q_{\mathbf{L}}$ and the anti-ghost zero mode b_0 (mirrored by the right-moving sector). Their anti-commutator is

$$L_0 = \{b_0, Q_{\mathbf{B}}\} = N_\beta + N_\gamma + N_b + N_c - a$$

  where the level-counting operators are

$$N_\beta = \textstyle\sum_{p=1}^{\infty} - ip\beta_{-pi}\gamma^i_p\,, \quad N_\gamma = \sum_{p=1}^{\infty} ip\gamma^i_{-p}\beta_{pi}\,, \quad N_b = \sum_{p=1}^{\infty} pb_{-p}c_p\,, \quad N_c = \sum_{p=1}^{\infty} pc_{-p}b_p\,.$$

  These are all positive semi-definite. Hence, the vanishing of L_0 on physical states means a drastic truncation of the entire string spectrum to just one level.

  From $\langle 0|[L_1, L_{-1}]|0\rangle = -2$ with $L_n = \{Q_{\mathbf{L}}, b_n\}$, we identify the level to be unity, $a = 1$.

BRST quantization 4/4

w/ Shigeki Sugimoto 2008.03084

- In the end, the physical states consist of four sectors, with $|k\downarrow\rangle$ satisfying $b_0|k\downarrow\rangle = 0$,

$$\delta\mathcal{H}_{i\bar{\imath}}\,\gamma^{i}_{-1}|k_j\downarrow\rangle\otimes\bar{\gamma}^{\bar{\imath}}_{-1}|k_{\bar{\jmath}}\downarrow\rangle\,,\qquad \delta\mathcal{H}_{i}{}^{\bar{\imath}}\,\gamma^{i}_{-1}|k_j\downarrow\rangle\otimes\bar{\beta}_{-1\bar{\imath}}|k_{\bar{\jmath}}\downarrow\rangle$$

$$\delta\mathcal{H}^{i}{}_{\bar{\imath}}\,\beta_{-1i}|k_j\downarrow\rangle\otimes\bar{\gamma}^{\bar{\imath}}_{-1}|k_{\bar{\jmath}}\downarrow\rangle\,,\qquad \delta\mathcal{H}^{i\bar{\imath}}\,\beta_{-1i}|k_j\downarrow\rangle\otimes\bar{\beta}_{-1\bar{\imath}}|k_{\bar{\jmath}}\downarrow\rangle$$

  which should satisfy on-shell relations for $Q_{\mathbf{B}}$-closedness,

$$k_i\delta\mathcal{H}^{i}{}_{\bar{\imath}} = 0\,,\qquad k_{\bar{\imath}}\delta\mathcal{H}_{i}{}^{\bar{\imath}} = 0\,,\qquad k_i\delta\mathcal{H}^{i\bar{\imath}} = 0\,,\qquad k_{\bar{\imath}}\delta\mathcal{H}^{i\bar{\imath}} = 0$$

  and equivalence relations ($Q_{\mathbf{B}}$-exactness): with divergenceless parameters, $k_i\xi^i = k_{\bar{\imath}}\xi^{\bar{\imath}} = 0$,

$$\delta\mathcal{H}^{i}{}_{\bar{\imath}} \sim \delta\mathcal{H}^{i}{}_{\bar{\imath}} - k_{\bar{\imath}}\xi^{i}\,,\qquad \delta\mathcal{H}_{i}{}^{\bar{\imath}} \sim \delta\mathcal{H}_{i}{}^{\bar{\imath}} + k_i\xi^{\bar{\imath}}\,,\qquad \delta\mathcal{H}_{i\bar{\imath}} \sim \delta\mathcal{H}_{i\bar{\imath}} + k_i\lambda_{\bar{\imath}} - k_{\bar{\imath}}\lambda_i$$

- We have a good reason to denote the physical states by the same symbol as the generalized metric: the $4n\bar{n}$ of $\{\delta\mathcal{H}_{i\bar{\imath}}, \delta\mathcal{H}_{i}{}^{\bar{\imath}}, \delta\mathcal{H}^{i}{}_{\bar{\imath}}, \delta\mathcal{H}^{i\bar{\imath}}\}$ are literally the moduli of the maximally non-Riemannian generalized metric, $\frac{\mathbf{O}(D,D)}{\mathbf{O}(n,n)\times\mathbf{O}(\bar{n},\bar{n})}$, that we have been dealing with.
  On-shell, the fluctuations meet the linearized Einstein Double Field Equations,

$$\partial_i\partial_j\delta\mathcal{H}^{j}{}_{\bar{\imath}} - \partial_{\bar{\imath}}\partial_{\bar{\jmath}}\delta\mathcal{H}_{i}{}^{\bar{\jmath}} + 4\partial_i\partial_{\bar{\imath}}\delta d = 0\,,\quad \partial_i\partial_j\delta\mathcal{H}^{j\bar{\imath}} = 0\,,\quad \partial_{\bar{\imath}}\partial_{\bar{\jmath}}\delta\mathcal{H}^{i\bar{\jmath}} = 0\,,\quad \partial_i\partial_{\bar{\imath}}\delta\mathcal{H}^{i\bar{\imath}} = 0$$

  which enjoy local symmetries inherited from $\hat{\mathcal{L}}_{\xi}\mathcal{H}_{MN}$,

$$\delta(\delta\mathcal{H}^{i}{}_{\bar{\imath}}) = \partial_{\bar{\imath}}\xi^{i}\,,\quad \delta(\delta\mathcal{H}_{i}{}^{\bar{\imath}}) = -\partial_i\xi^{\bar{\imath}}\,,\quad \delta(\delta\mathcal{H}_{i\bar{\imath}}) = \partial_{\bar{\imath}}\lambda_i - \partial_i\lambda_{\bar{\imath}}\,,\quad \delta(\delta d) = -\tfrac{1}{4}(\partial_i\xi^{i} + \partial_{\bar{\imath}}\xi^{\bar{\imath}})$$

– Remarkably, after choosing a gauge, $\delta d = 0$, and restricted to normalizable solutions, the BRST string spectrum agrees with the fluctuation analysis of DFT.

– Comments: *i)* $\delta\mathcal{H}^{i\bar{\imath}}$ may condensate and create Riemannian spacetime. *ii)* DFT = SFT.

Epilogue

– The conventional (or Riemannian) closed string effective action containing a tachyon,

$$\int \mathrm{d}^D x\,\sqrt{-g}e^{-2\phi}\Big[R+4\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}-\tfrac{2(D-26)}{3\alpha'}-\partial_\mu T\partial^\mu T+\tfrac{4}{\alpha'}T^2+\mathcal{O}(T^3)\Big]$$

can be formulated as a DFT coupled to the tachyon,

$$\int e^{-2d}\Big[S_{(0)}-\tfrac{2(D-26)}{3\alpha'}-\mathcal{H}^{MN}\partial_M T\partial_N T+\tfrac{4}{\alpha'}T^2+\mathcal{O}(T^3)\Big]$$

With the choice of the section, $\tilde{\partial}^\mu=0$, the tachyon kinetic term is $\mathcal{H}^{\mu\nu}\partial_\mu T\partial_\nu T$ which obviously vanishes upon the maximally non-Riemannian backgrounds, as $\mathcal{H}^{\mu\nu}=H^{\mu\nu}=0$.

The vanishing kinetic term then may eliminate the tachyonic instability: there is no dynamics for the tachyon to roll down. This agrees with our BRST spectrum analysis and also with a classical intuition for chiral string,

$$x^i(\tau,\sigma)=x^i(0,\tau+\sigma)$$

Namely, it is fixed in space and thus hardly interacts with one another.

– Our BRST charge formula can be extended to a generic $(n,\bar{n})$ non-Riemannian background, to include n pairs of chiral $\beta\gamma$, $\bar{n}$ pairs of anti-chiral $\bar{\beta}\bar{\gamma}$, and ordinary (left-right combined) $D-n-\bar{n}$ number of x^μ. The central charges should be

$$\mathbf{c}_{\mathbf{L}/\mathbf{R}}=D\pm(n-\bar{n})-26\quad\text{(bosonic string)}\,,\qquad\qquad \mathbf{c}_{\mathbf{L}/\mathbf{R}}=D\pm(n-\bar{n})-10\quad\text{(superstring)}$$

Necessarily we require $n=\bar{n}$ and $D=26$ or $D=10$, which might enlarge the string theory landscape far beyond the Riemannian paradigm.

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Thank you