TTbar deformed 1d Bose gas

Yunfeng Jiang 江云峰

CERN

@APCPT, Pohang, Korea
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Based on the works

Y.Jiang, arXiv: **2011.00637**

B.Pozsgay, Y. Jiang, G. Takacs, arXiv: **1911.11118**

TTbar deformation

An irrelevant def

From IR to UV

Novel UV behavior

TTbar deformation

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From IR to UV

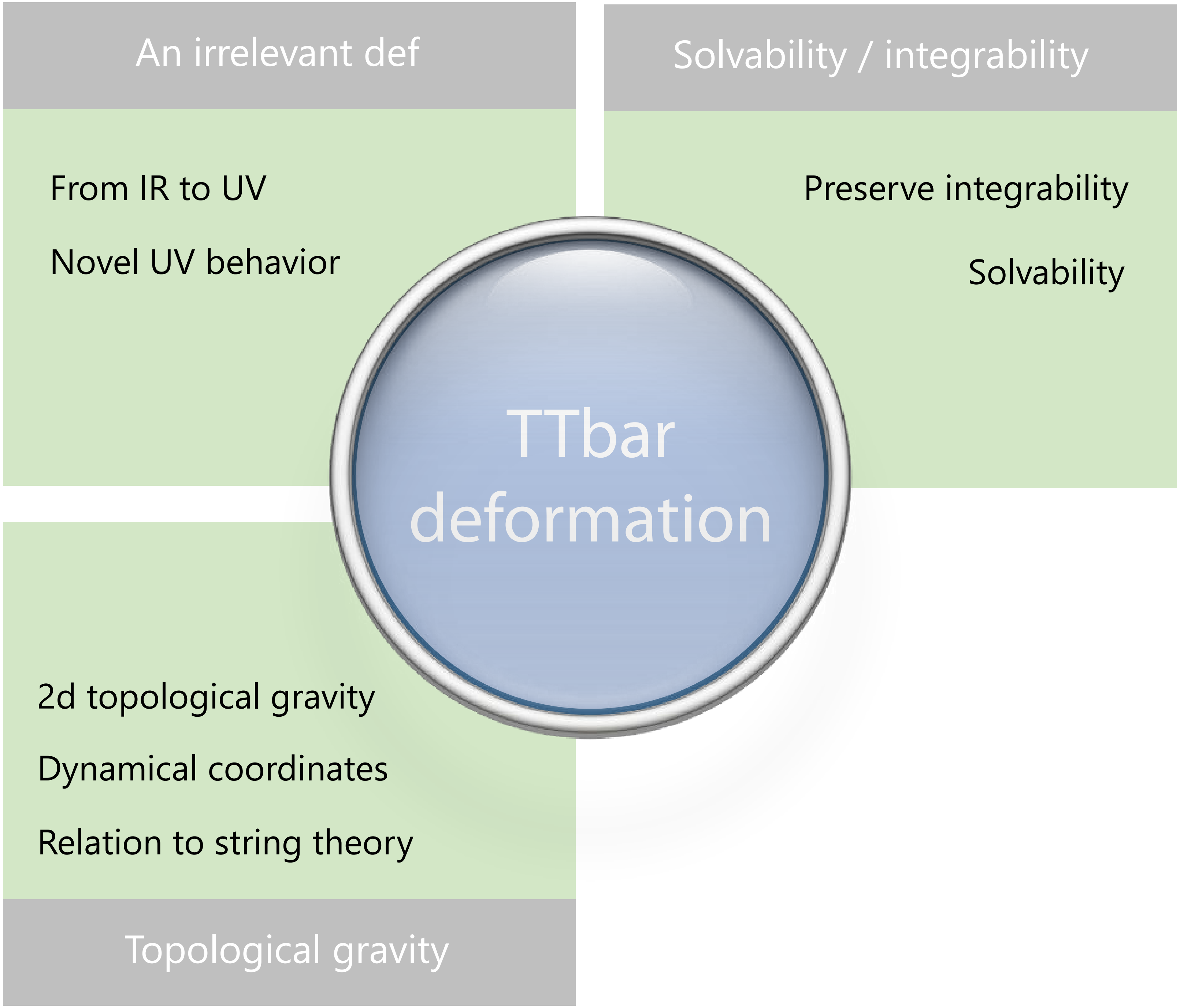
Novel UV behavior

Solvability / integrability

Preserve integrability

Solvability

TTbar deformation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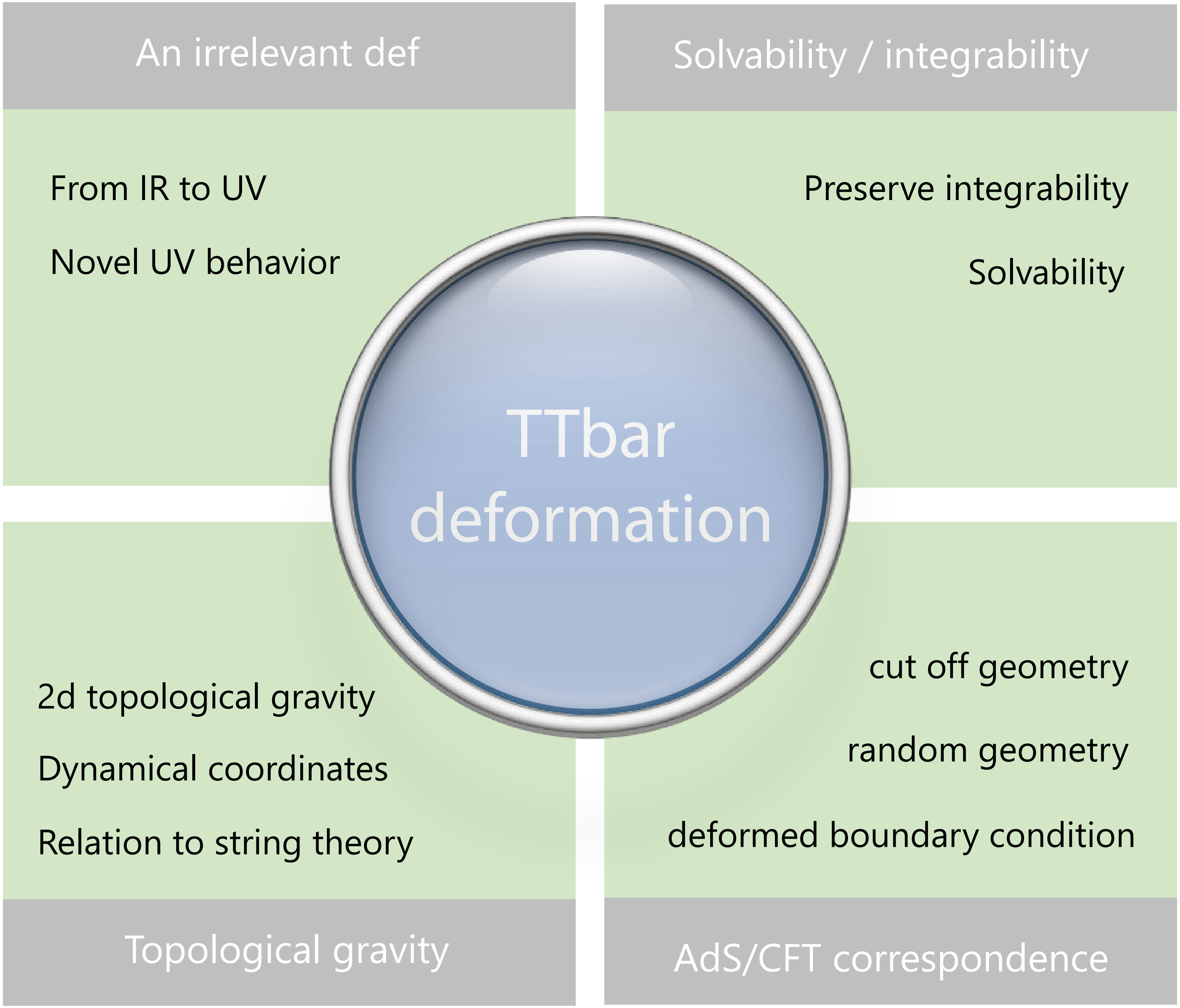
TTbar deformation
2d topological gravity
Dynamical coordinates
Relation to string theory
Topological gravity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cut off geometry
random geometry
deformed boundary condition
AdS/CFT correspondence

The question

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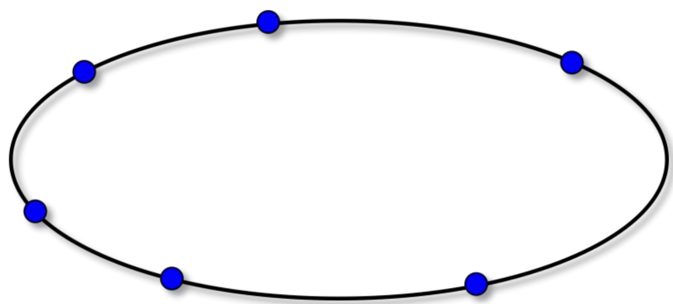
Particles moving in 1d



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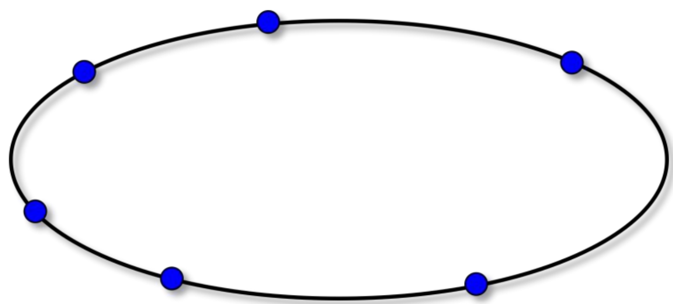

Hamiltonian

$$H = -\frac{\hbar^2}{2m}\sum_{k=1}^{N}\frac{\partial^2}{\partial x_i^2} + V(x_1, \ldots, x_N)$$

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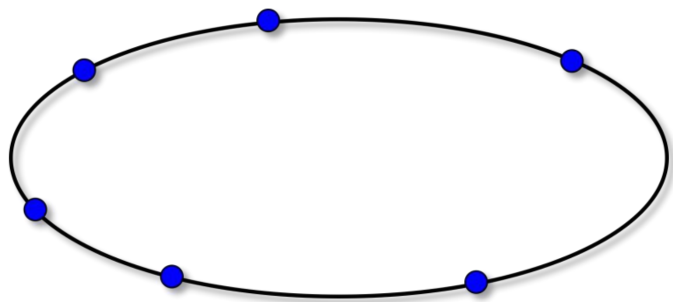

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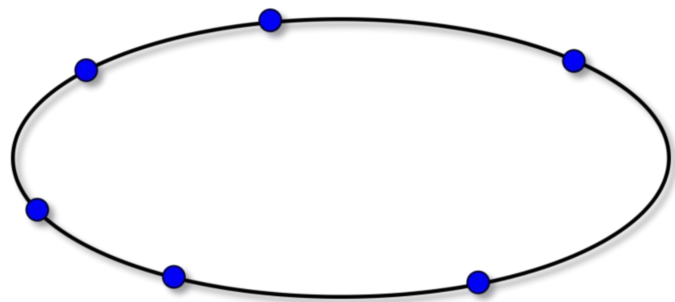
Question Can we TTbar deform it ? How ?

Why do we want to do that ?



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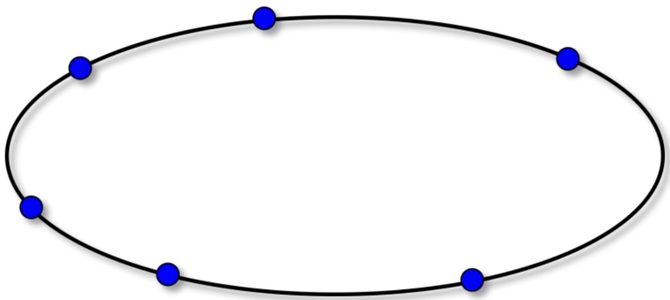

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Can define such deformations for such kind of model ?

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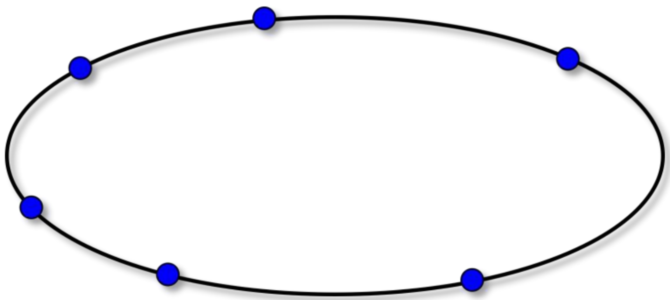

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Share same features TTbar for relativistic QFT, but in a simpler set-up.

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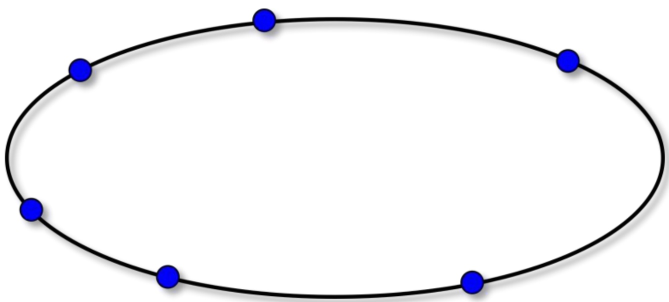

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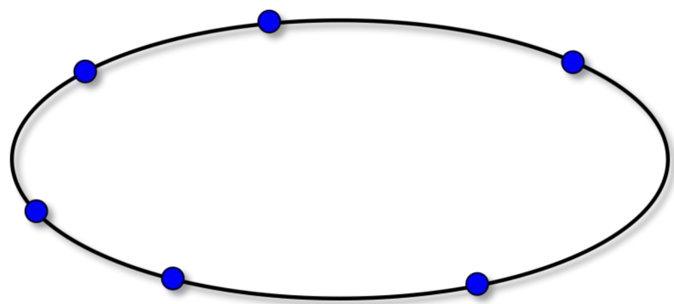
Integrability

A novel type of integrable model that can be interesting

The Lieb-Liniger model



1d Bose gas

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j}^{N} \delta(x_i - x_j)$$

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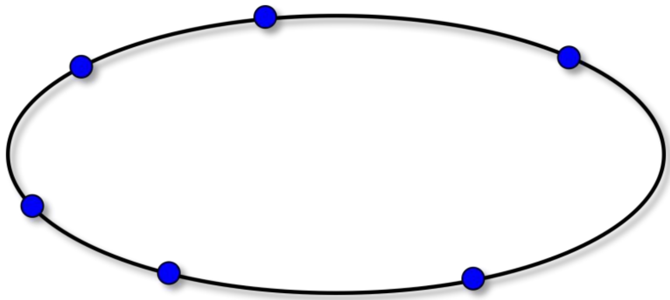

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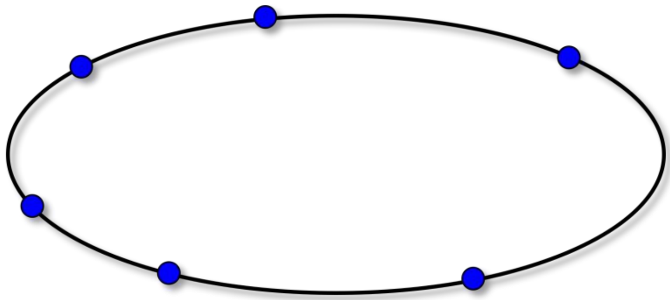

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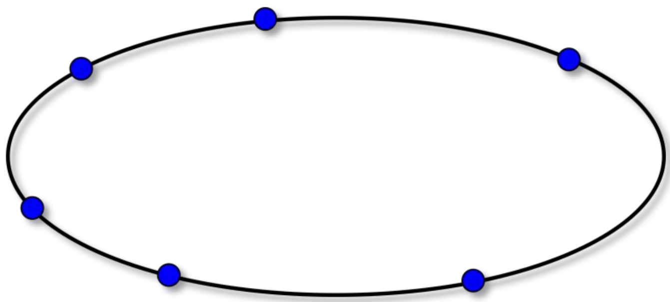

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- Related to other systems (XXZ chain, Sinh-Gordon)
- Realized experimentally by cold atom

II. Bilinear deformations

TTbar deformation

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Definition for QFT

$$\frac{d}{d\lambda} S_\lambda = \int d^2x \, \det\left(T_{\mu\nu}\right)$$

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Actions and path integrals are less common in spin chains and Bose gas

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Two **conserved currents**

$$J^a = (\hat{q}^a, j^a) \qquad\qquad a = 1, 2$$

$$\partial_t \hat{q}^a + \partial_x j^a = 0$$

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Make **anti-symmetric** combination

$$O_{JJ}(x,t) = \epsilon_{ab} J^a J^b(x,t)$$

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3 Specialize to **TTbar deformation**

$$\mathrm{det} T_{ab} = \epsilon_{ab} T^{1a} T^{2b} \quad \Bigg| \quad \begin{array}{ll} T^{1a} = (\hat{h}(x), j_H(x)) & \text{Energy density} \\ T^{2a} = (\hat{p}(x), j_P(x)) & \text{Momentum density} \end{array}$$

Bilinear deformation

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$$\det T_{ab}(x) = \hat{h}(x) j_P(x) - \hat{p}(x) j_H(x)$$

TTbar deformation for **general system**

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Define it for spin chains ?

$$H = \sum_k \hat{h}(k)$$

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$$\boxed{\partial_t \hat{q}(x) = i[H, \hat{q}(x)] = -\partial_x j(x)}$$

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Cannot define TTbar for spin chain, but other bilinear deformations

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Related to the **discrete** nature of spin chains

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We can define **TTbar for 1d Bose gas**

Factorization property

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Flow equation for spectrum

$$\frac{d}{d\lambda}E_n = Q_a\langle n|j_b|n\rangle - Q_b\langle n|j_a|n\rangle$$

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For **integrable models**, flow equation can be written down and solved.

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BBL construction can be generalized to **Bose gas**

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Theorem

$$i[X_{JJ}, H] = \int_0^R O_{JJ} dx - [\hat{Q}_1 j_2(0) - \hat{Q}_2 j_1(0)]$$

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Theorem

The two deformations are **the same** in **infinite volume**.

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In particular, it **preserves integrability**

Bethe ansatz

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Eigenstates

N-particle state constructed by Bethe ansatz

$$|\{u_1, u_2, \ldots, u_N\}\rangle$$

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Dispersion relations

For relativistic QFT

$$e(u) = m\cosh u \qquad p(u) = m\sinh u$$

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For Bose gas

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Dispersion relations

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3

Bethe equations

$$e^{ip(u_j)R} \prod_{k \neq j}^{N} S(u_j, u_k) = 1$$

S-matrix deformation

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Under bilinear deformation

$$S(u,v) \mapsto S(u,v) \times e^{-i\lambda[h_a(u)h_b(v)-h_b(u)h_a(v)]}$$

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CDD-like factors

$h_a(u)$ related to **charges** of a single particle

$$Q_a|\mathbf{u}_N\rangle = \sum_{j=1}^{N} h_a(u_j)|\mathbf{u}_N\rangle$$

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This modifies Bethe equations, takes into account **finite-size effects**.

III. Finite size spectrum

A simpler deformation

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$$\{Q\} = \{Q_0, Q_1, Q_2, \ldots\}$$

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The **first three** are universal

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Infinite bilinear deformations

$$O_{a,b} = [Q_a Q_b]$$

A simpler deformation

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TTbar deformation [Next-to-simplest]

$$T\bar{T} = O_{1,2}$$

A simpler deformation

Integrable models [Infinite conserved charges]

$$\{Q\} = \{Q_0, Q_1, Q_2, \ldots\}$$

The **first three** are universal

$Q_0 = \hat{N}$	particle number
$Q_1 = \hat{P}$	momentum
$Q_2 = \hat{H}$	energy

What about **the simplest** one ? see also [Cardy and Doyon 2020]

$$O_{0,1} = [\hat{N}\hat{P}]$$

S-matrix of LL model

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S-matrix of Lieb-Liniger model

$$S_{\mathrm{LL}}(u,v) = \frac{u - v - ic}{u - v + ic}$$

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Phase shift

$$\theta(u,v) = -i\log S(u,v)$$

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S-matrix of Lieb-Liniger model

$$S_{\mathrm{LL}}(u,v)=\frac{u-v-ic}{u-v+ic}$$

Phase shift

$$\theta(u,v)=-i\log S(u,v)$$

Consider the **free boson limit** $c\to 0$

$$\lim_{c\to 0}\theta(u,v)=-\pi\,\mathrm{sgn}(u-v)$$

Hardcore deformation

Hardcore deformation

Recall

$$S(u,v) \mapsto S(u,v) \times e^{-i\lambda[h_a(u)h_b(v)-h_b(u)h_a(v)]}$$

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$$h_0(u) = 1 \qquad\qquad h_1(u) = u$$

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We find

$$\theta_\lambda(u,v) = -\pi\,\mathrm{sgn}(u-v) + \lambda(u-v)$$

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The hard rod gas

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$$H_{\mathrm{HR}} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{i<j}^{N} v(x_i - x_j) \qquad v(x) = \begin{cases} \infty & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$$

Describes a gas of hard rods with length $a > 0$

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A known **integrable model**, with phase shift [Sutherland 1971]

$$\theta_{\mathrm{HR}}(u, v) = -\pi \,\mathrm{sgn}(u - v) - a(u - v)$$

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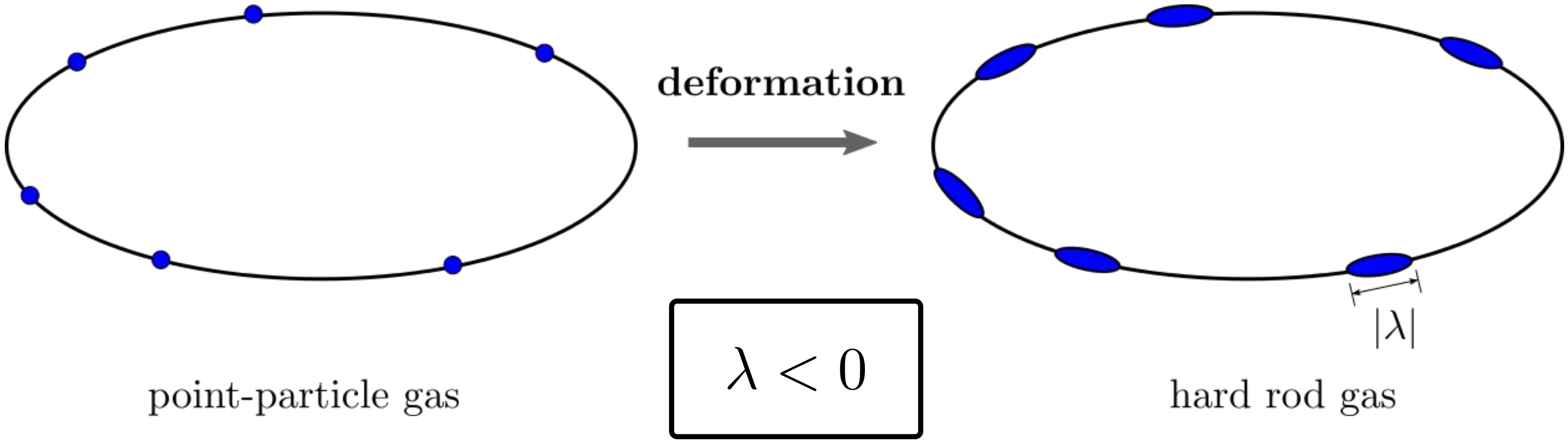
Compare to $O_{0,1}$ deformation

$$\theta_{\lambda}(u,v) = -\pi\,\mathrm{sgn}(u-v) + \lambda(u-v)$$

see also [Cardy and Doyon 2020]



The deformation changes length of the ring by $|\lambda|N$

$\lambda > 0$	Length is increased
$\lambda < 0$	Length is decreased

Spectral flow equation

Spectral flow equation

$$\partial_\lambda E_N = N\partial_R E_N - P_N \langle \mathbf{u}_N | j_{\hat{N}} | \mathbf{u}_N \rangle$$

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Zero-momentum sector

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General solution

$$E_N(R, \lambda) = E_N(R + \lambda N, 0)$$

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General solution

$$E_N(R,\lambda) = E_N(R+\lambda N, 0)$$

- For $\lambda > 0$, the spectrum is well-defined.
- For $\lambda < 0$, there is a critical value $\lambda_c = -R/N$

TTbar deformation

Flow equation

$$\partial_\lambda E_N = E_N \partial_R E_N - P_N \langle \mathbf{u}_N | j_H | \mathbf{u}_N \rangle$$

TTbar deformation

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Mean value of currents

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Conserved charge	Current density

TTbar deformation

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Mean value of currents

Conserved charge	Current density
$Q_a \vert \mathbf{u}_N \rangle = \sum_{j=1}^{N} h_a(u_j) \vert \mathbf{u}_N \rangle$	$\langle \mathbf{u}_N \vert j_a \vert \mathbf{u}_N \rangle = \mathbf{e}' \cdot G^{-1} \cdot \mathbf{h}_a$

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where $e'(u) = de(u)/du$ and G_{jk} is the Gaudin matrix

Deformed spectrum

Zero momentum sector

Deformed spectrum

Zero momentum sector

Flow equation

$$\partial_\lambda E_N = E_N \partial_R E_N$$

Deformed spectrum

Zero momentum sector

Compare to

$$\partial_\lambda E_N = N \partial_R E_N$$

Flow equation

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Deformed spectrum

Zero momentum sector

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The inviscid **Burgers' equation**, solved by

$$E_N(R, \lambda) = E_N(R + \lambda E_N, 0)$$

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Zero momentum sector

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The inviscid **Burgers' equation**, solved by

$$E_N(R, \lambda) = E_N(R + \lambda E_N, 0)$$

If we know $E_N(R, 0)$, this gives an **algebraic equation**

Free fermion limit $c \to \infty$ $\theta(u, v) = 0$

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Bethe equations

$$\boxed{u_j R = 2\pi I_j, \qquad j = 1, \cdots, N}$$

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$$E_N(R,0) = \frac{\alpha_N}{R^2}$$

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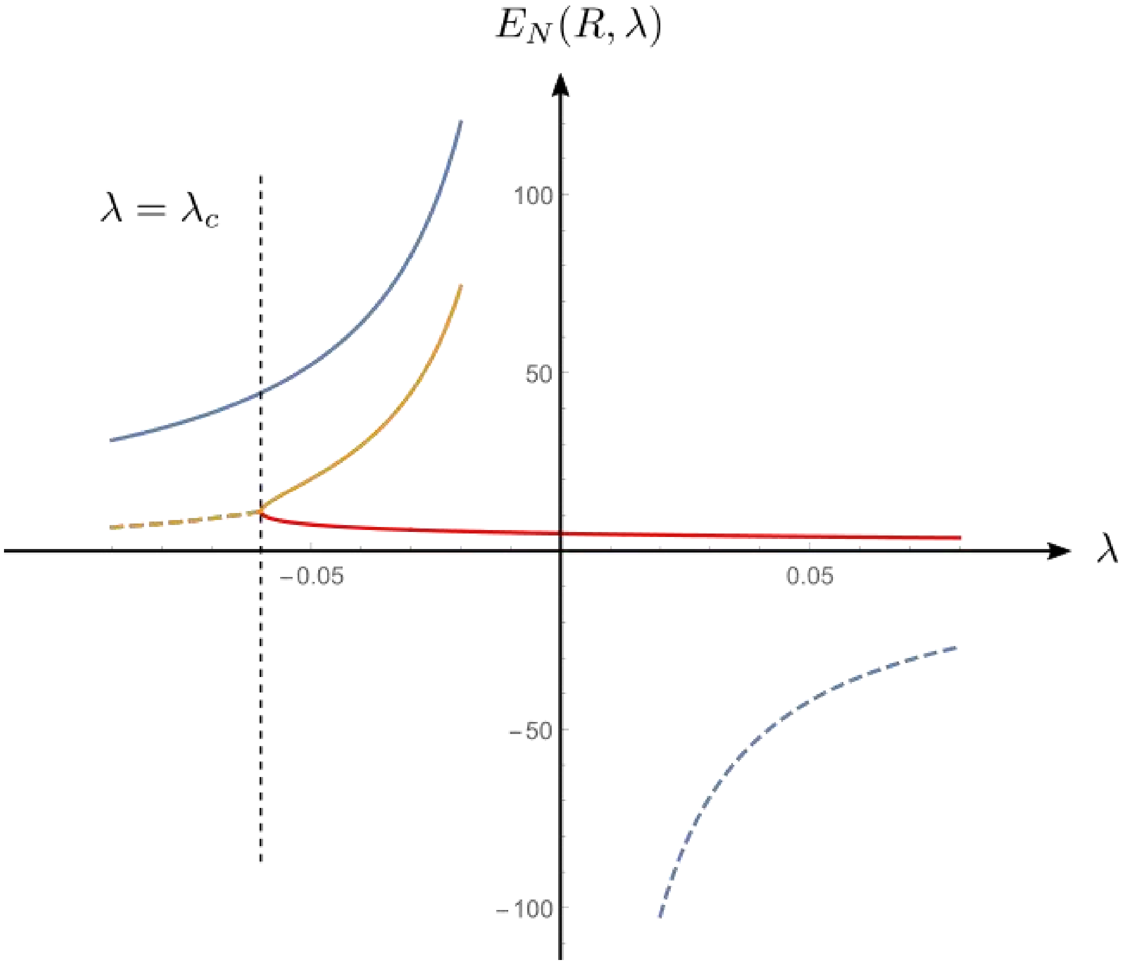
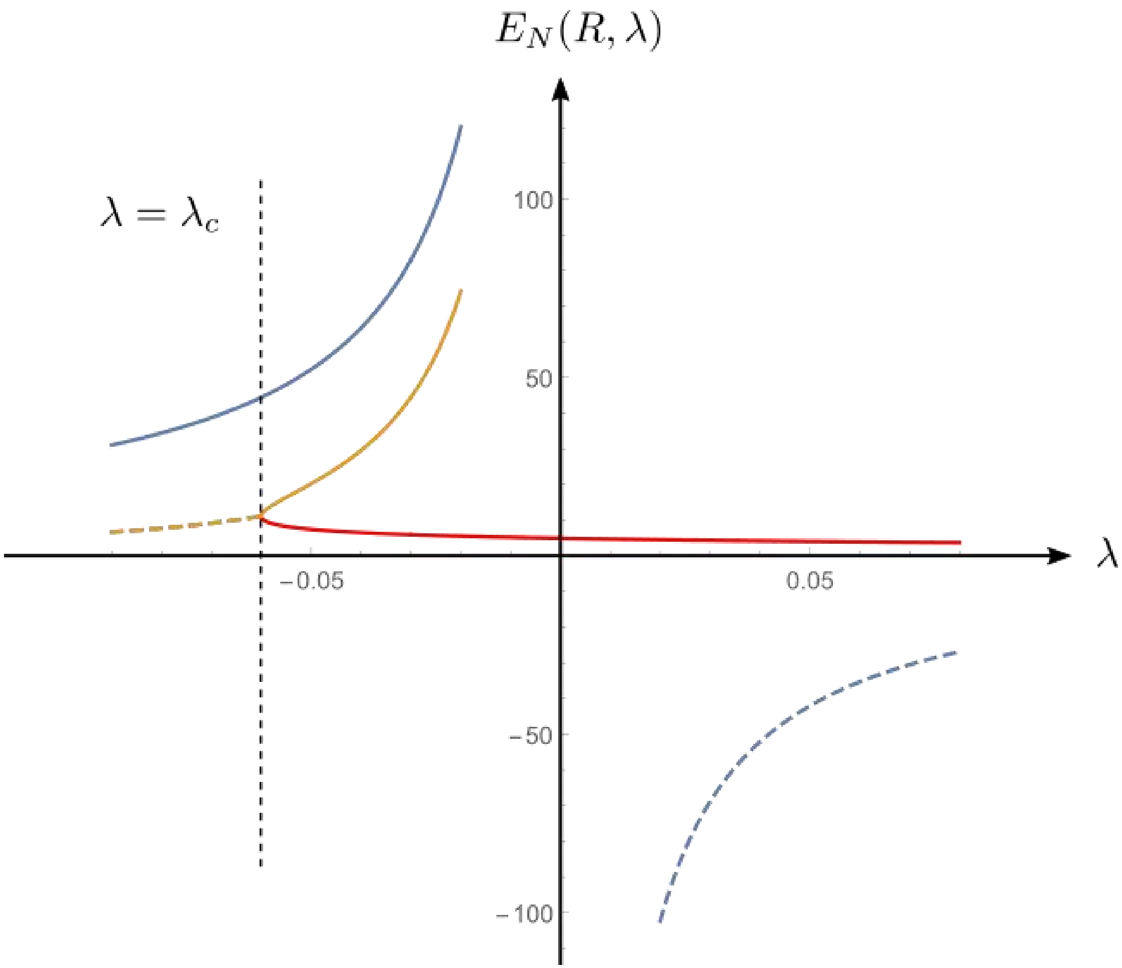
A cubic equation, solution with **several branches**



Different branches of
the solution

- For $\lambda > 0$, deformed spectrum well-defined, approaches to zero

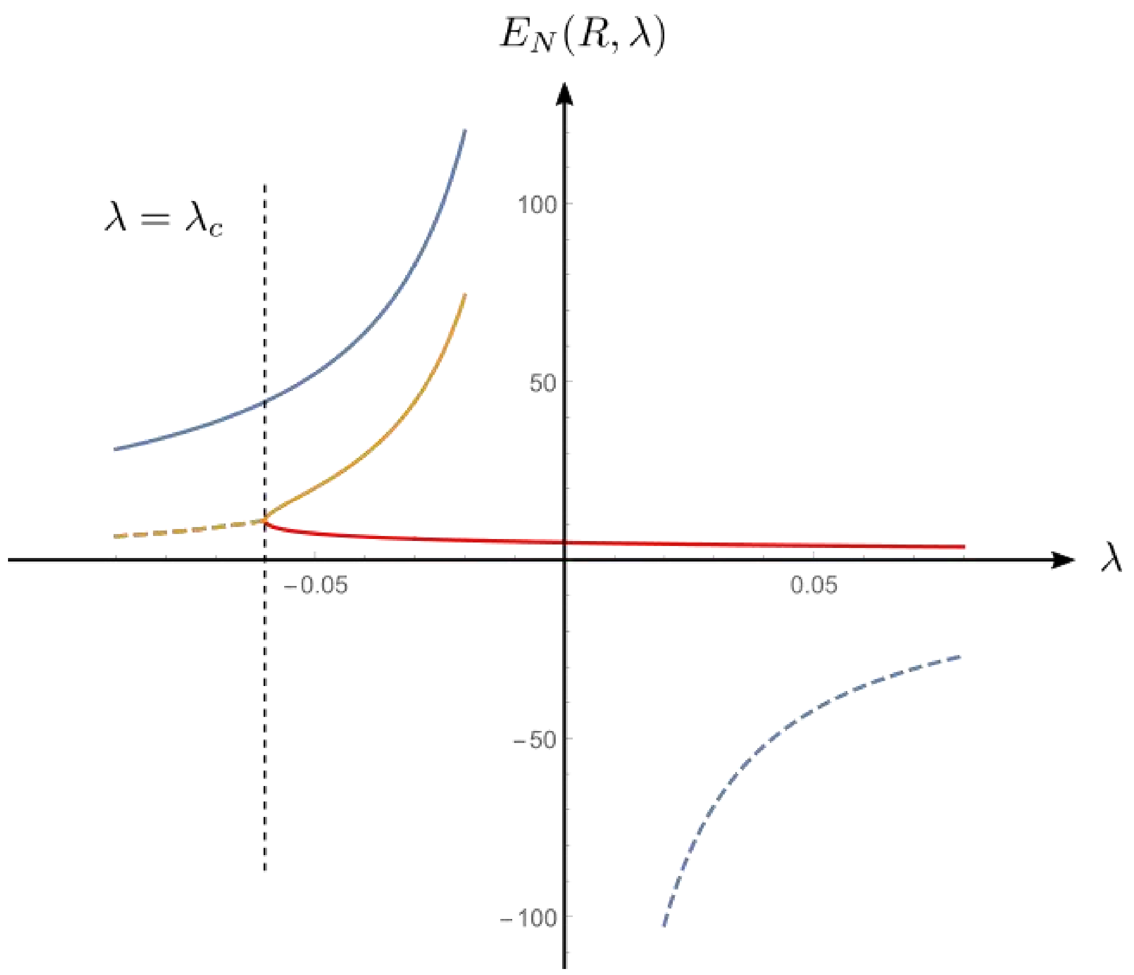


Different branches of the solution

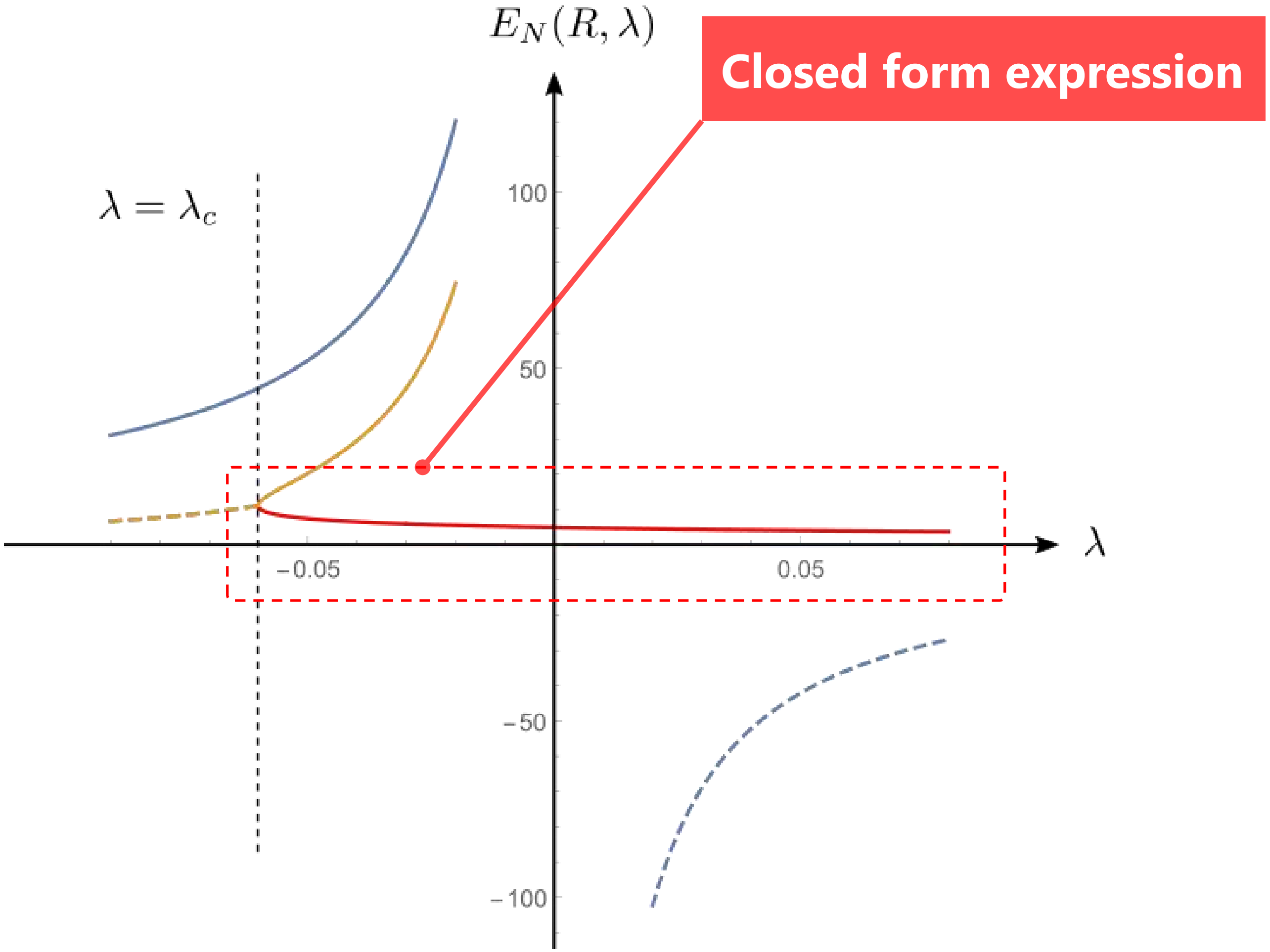
- For $\lambda > 0$, deformed spectrum well-defined, approaches to zero

- For $\lambda < 0$, there is a critical value at

$$\boxed{\lambda_c = -\frac{4R^3}{27\alpha_N}}$$



Deformed spectrum



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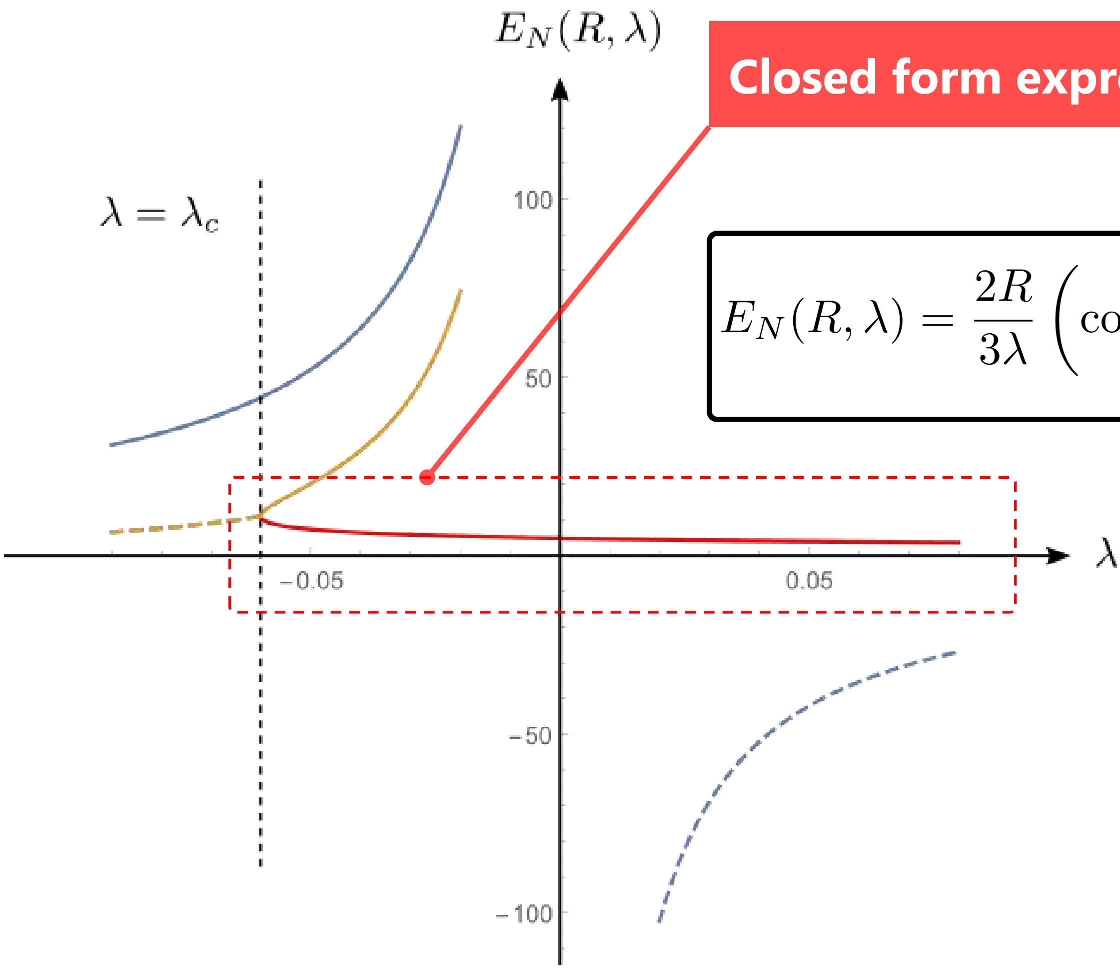
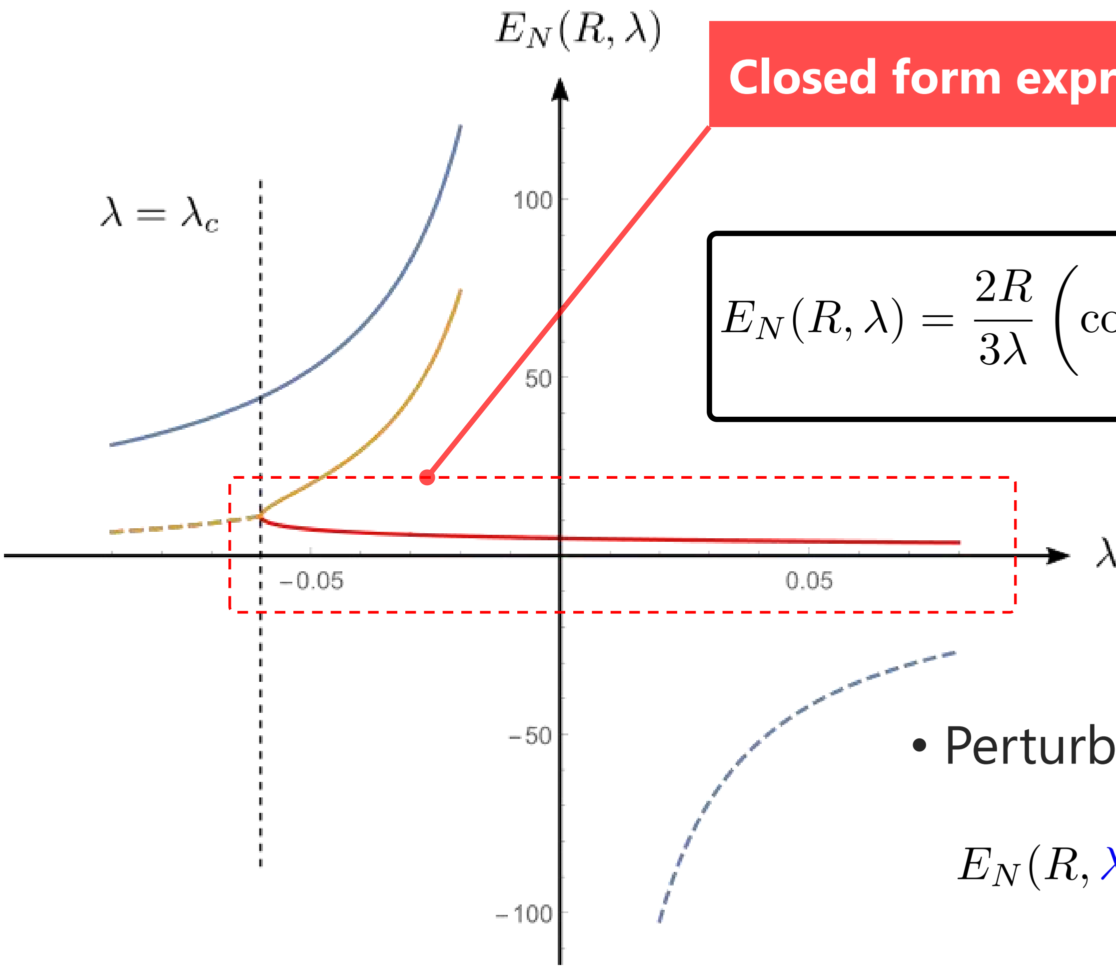


Closed form expression

$$E_N(R,\lambda) = \frac{2R}{3\lambda}\left(\cosh\left[\frac{2}{3}\text{arcsinh}\left(\frac{3\sqrt{3\lambda}\alpha_N}{2R^{3/2}}\right)\right] - 1\right)$$

Deformed spectrum



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- Perturbatively expansion

$$E_N(R,\lambda) = \frac{\alpha_N}{R^2} - \frac{2\alpha_N^2}{R^5}\lambda + \frac{7\alpha_N^3}{R^8}\lambda^2 + \cdots$$

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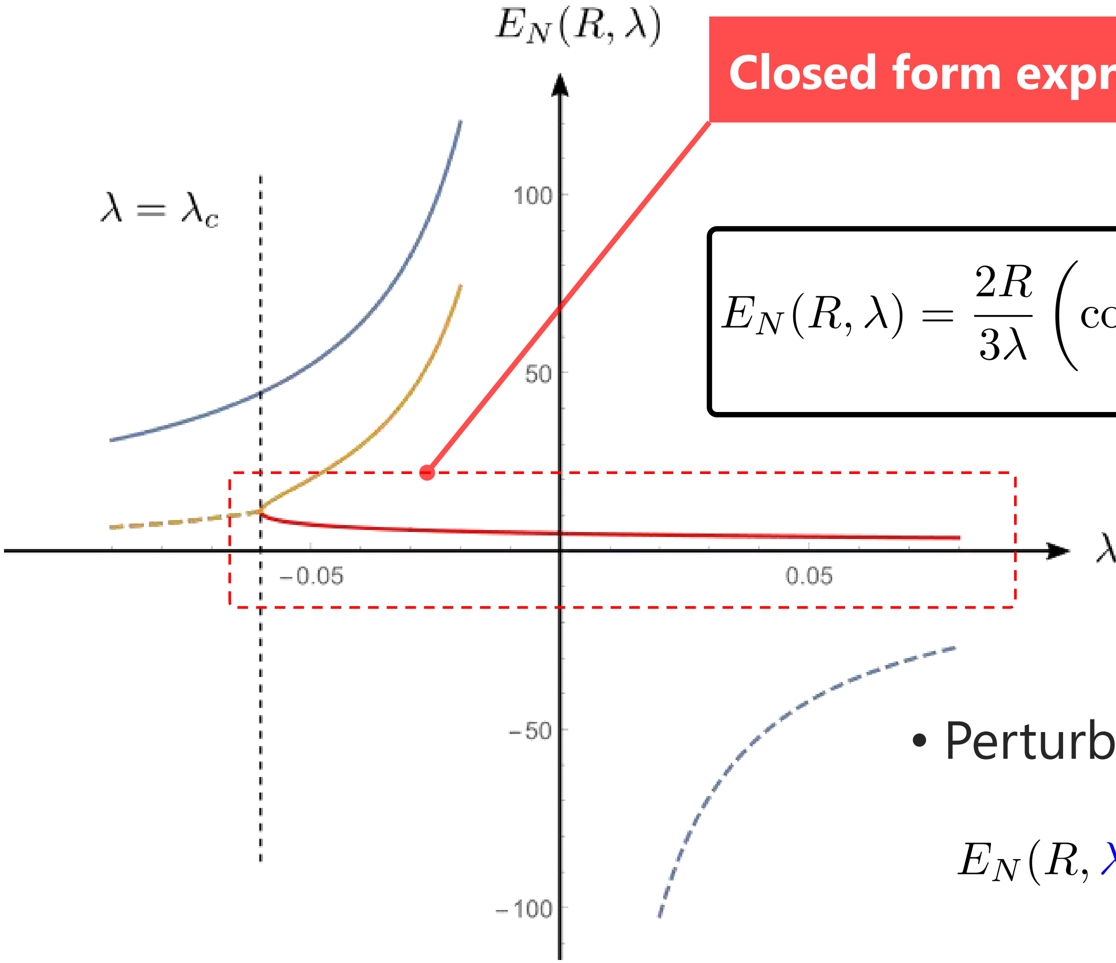

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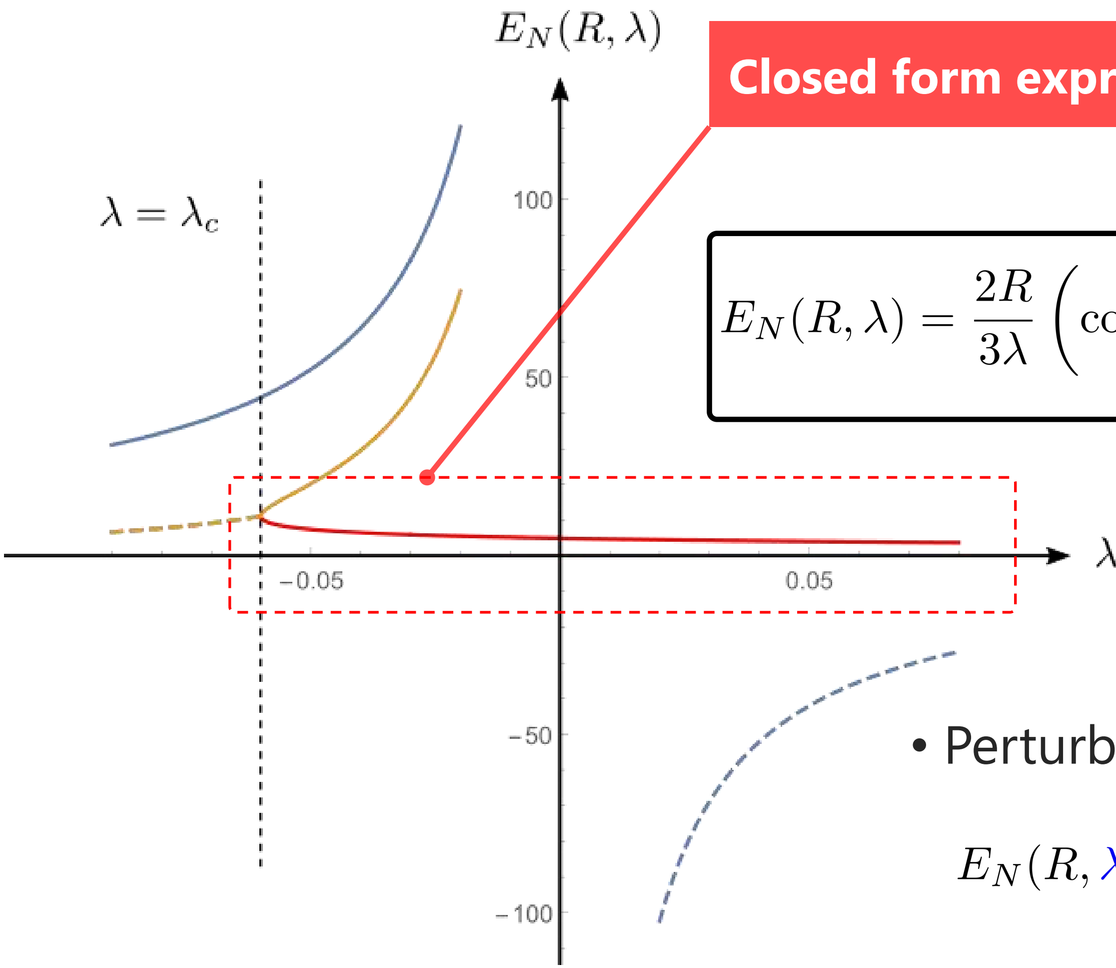
- Decrease monotonically, maximal

$$E_N(R,\lambda_c) = \frac{9}{4}E_N(R,0)$$

Deformed spectrum



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- **A branch point at** $\lambda = \lambda_c$ for fixed R

An alternative interpretation

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For fixed $\lambda < 0$, we must have $R \geq R_c$

$$R_c(\lambda) = 3\left(\frac{|\lambda|\alpha_N}{4}\right)^{1/3}$$

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A break down of UV physics ！

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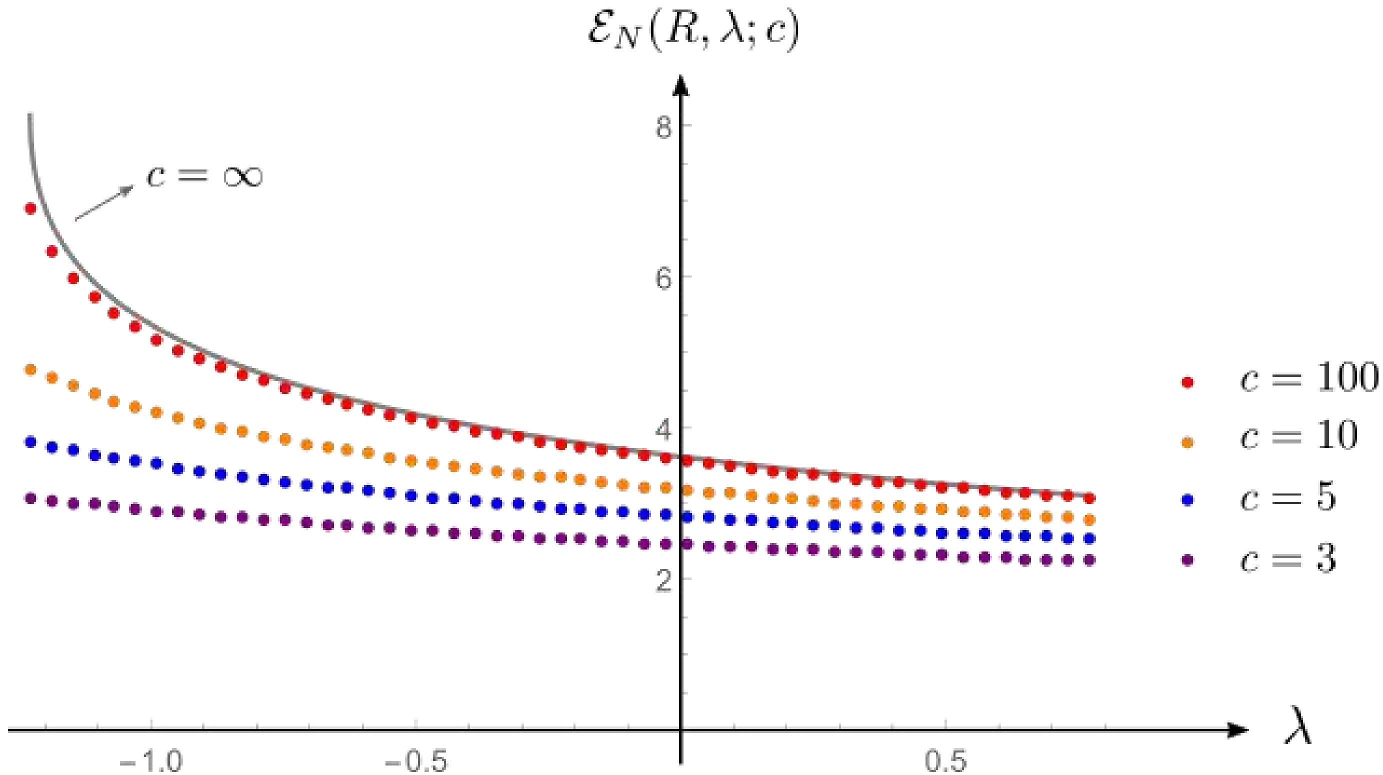
Comments

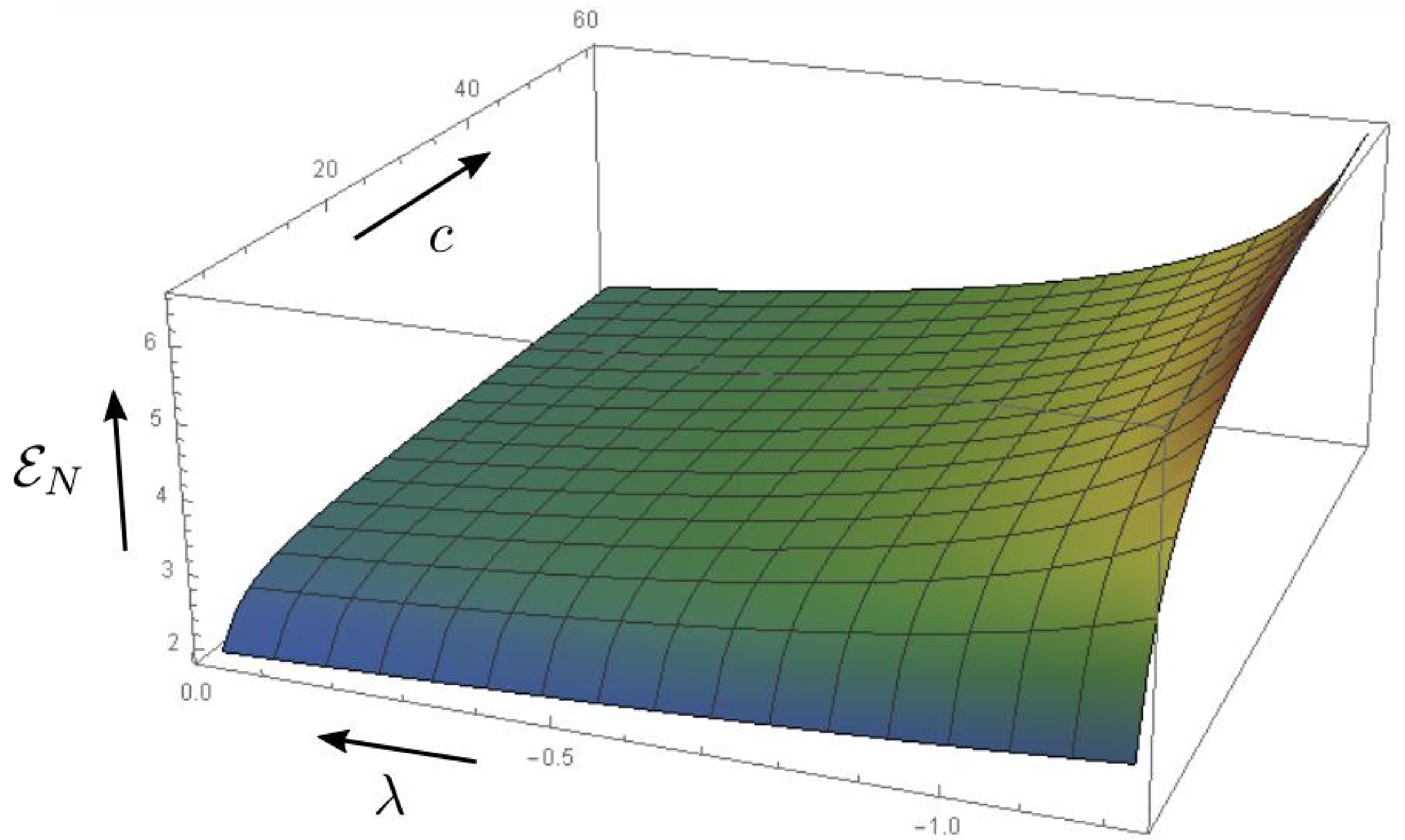
- The same behavior as deformed QFT
- Consistent with shock formation of Burgers' equation
- Can be explained with the generalized hard rod picture

Away from free fermion point



Find spectrum numerically. Qualitatively the same.

A 3D plot for the deformed spectrum



III. Thermodynamics

TBA in one slide

TBA in one slide

1

Pseudo-energy

TBA = **Thermodynamic** **+** **Bethe ansatz**

Central quantity : $\varepsilon(u)$

TBA in one slide

1

Pseudo-energy

TBA = **Thermodynamic** + **Bethe ansatz**

Central quantity : $\varepsilon(u)$

2

TBA equation

$$\varepsilon(u) = u^2 - \mu - \frac{1}{2\pi\beta} \int_{-\infty}^{\infty} \varphi(u,v) \ln\left(1 + e^{-\beta\varepsilon(v)}\right) \mathrm{d}v$$

TBA in one slide

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Thermal quantities

Free energy $\quad F = N\mu - \dfrac{R}{2\pi\beta}\displaystyle\int_{-\infty}^{\infty} \ln\left(1 + e^{-\beta\varepsilon(u)}\right) \mathrm{d}u$

TBA in one slide

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Thermal quantities

Pressure $\quad \mathrm{P} = \frac{1}{2\pi\beta}\int_{-\infty}^{\infty}\ln\left(1+e^{-\beta\varepsilon(u)}\right) du$

Deformed TBA

Deformed TBA

TTbar deformation changes TBA kernel

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Degenerate kernel, can be **solved analytically**

Analytical solution for pseudo-energy

$$\boxed{\varepsilon(u) = u^2 - \mu + \lambda(2u\,G_1 - G_2)}$$

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The quantities G_k satisfy self-consistency relations

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We can show that $G_1 = 0$

$$G_2 = \frac{1}{2\pi\beta} \int_{-\infty}^{\infty} u^2 \ln\left(1 + e^{-\beta(u^2 - \mu - \lambda G_2)}\right) \mathrm{d}u$$

Conclusion TTbar deformation **shifts chemical potential**.

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The shift is determined by a **self-consistency relation**

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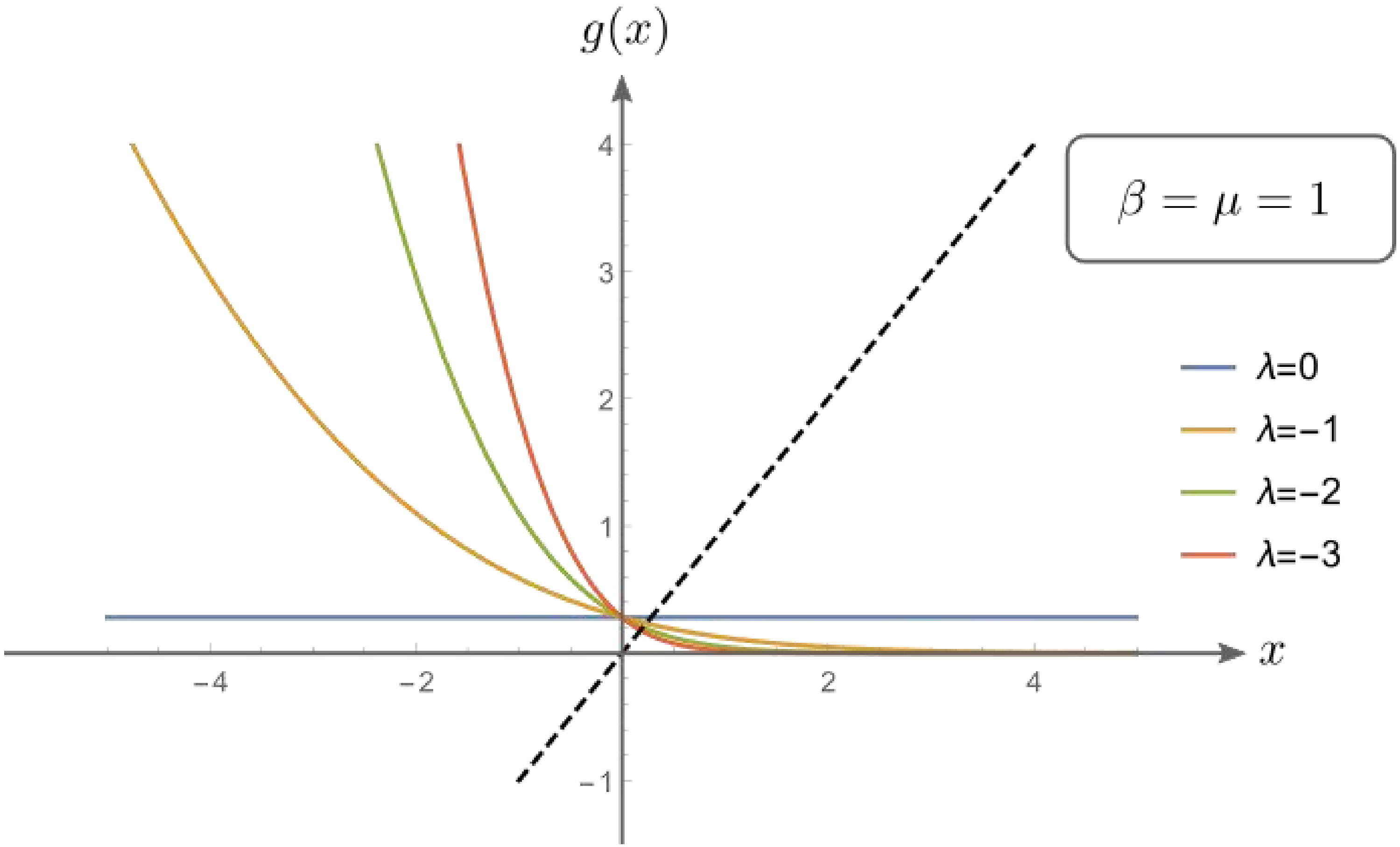
A **transcendental equation**

$$G_2 = \frac{1}{4\sqrt{\pi}\beta^{5/2}}\mathcal{F}_{3/2}\left[\beta(\mu+\lambda G_2)\right]$$

$$\mathcal{F}_s(\eta) = -\mathrm{Li}_{s+1}(-e^\eta)$$

$$g(x) = \frac{1}{4\sqrt{\pi}\beta^{5/2}}\mathcal{F}_{3/2}[\beta(\mu+\lambda x)]$$



$\lambda < 0$ For negative sign, there's always a real solution

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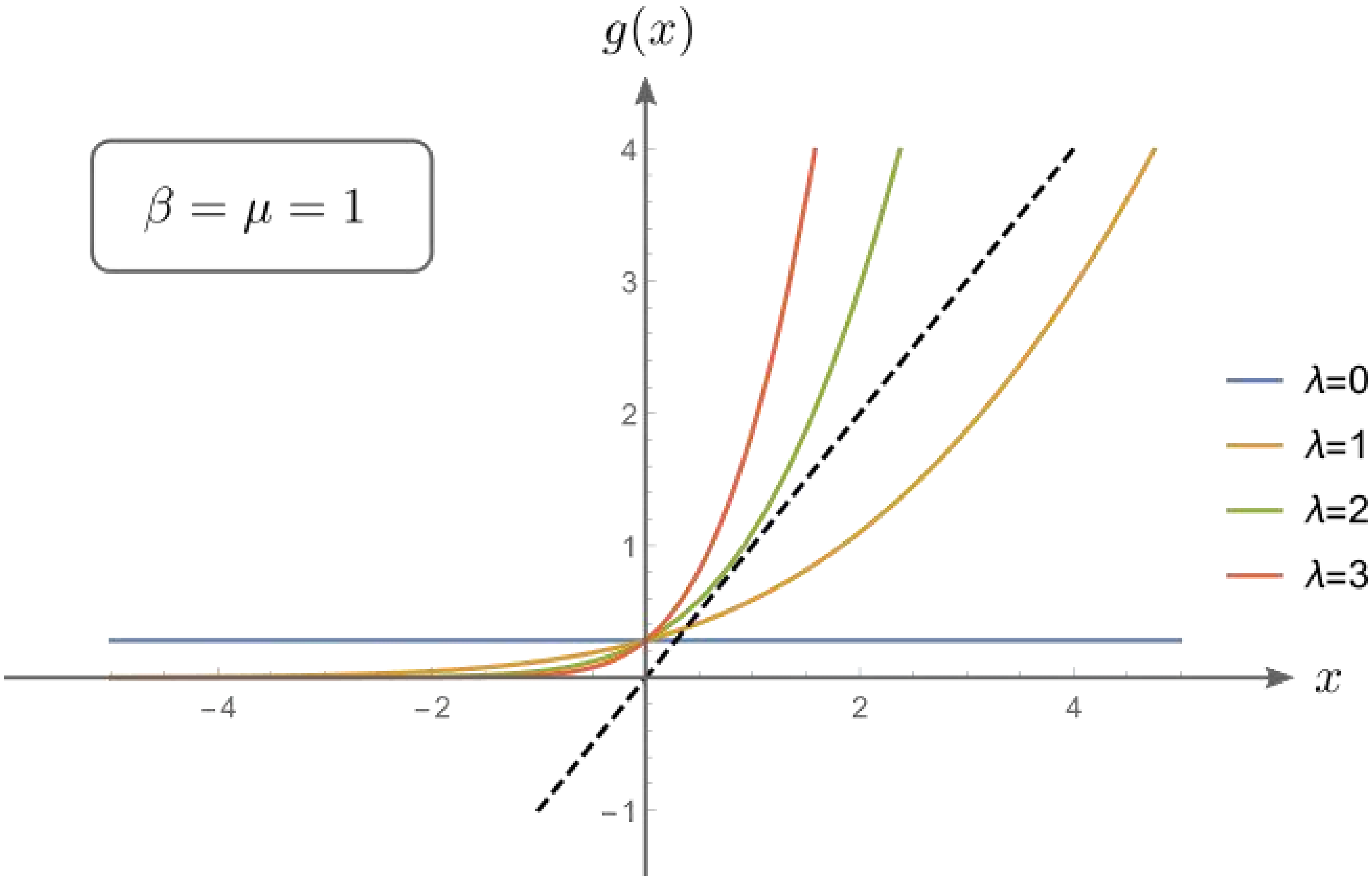

$\lambda > 0$ For positive sign, there's a critical value

More analytic study

Self-consistency relation can be written

$$G_2 = \frac{2}{3\pi} \int_{-\infty}^{\infty} \frac{u^4}{1 + e^{\beta(u^2 - \mu - \lambda G_2)}} \mathrm{d}u$$

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$$G_2 = \frac{2}{3\pi}\int_{-\infty}^{\infty} \frac{u^4}{1 + e^{\beta(u^2-\mu-\lambda G_2)}}\mathrm{d}u$$

Classical limit [low density, high temperature]

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Define $W = -\beta\lambda G_2$

$$We^W = z$$

$$z = -\frac{e^{\beta\mu}\lambda}{2\sqrt{\pi}\beta^{3/2}}$$

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Can be solved by **Lambert's *W*-function** $W(z)$

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Real solution only exist for

$$\boxed{z \geq -e^{-1}}$$

$$We^{W} = z$$

$$z = -\frac{e^{\beta\mu}\lambda}{2\sqrt{\pi}\beta^{3/2}}$$

Can be solved by **Lambert's *W*-function** $W(z)$

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$$\boxed{z \geq -e^{-1}}$$

For fixed β and μ, this implies

$$\lambda \leq \lambda_c(\beta, \mu) = 2\sqrt{\pi}\beta^{3/2}e^{-\beta\mu-1}$$

$$We^W = z \qquad z = -\frac{e^{\beta\mu}\lambda}{2\sqrt{\pi}\beta^{3/2}}$$

Can be solved by **Lambert's *W*-function** $W(z)$

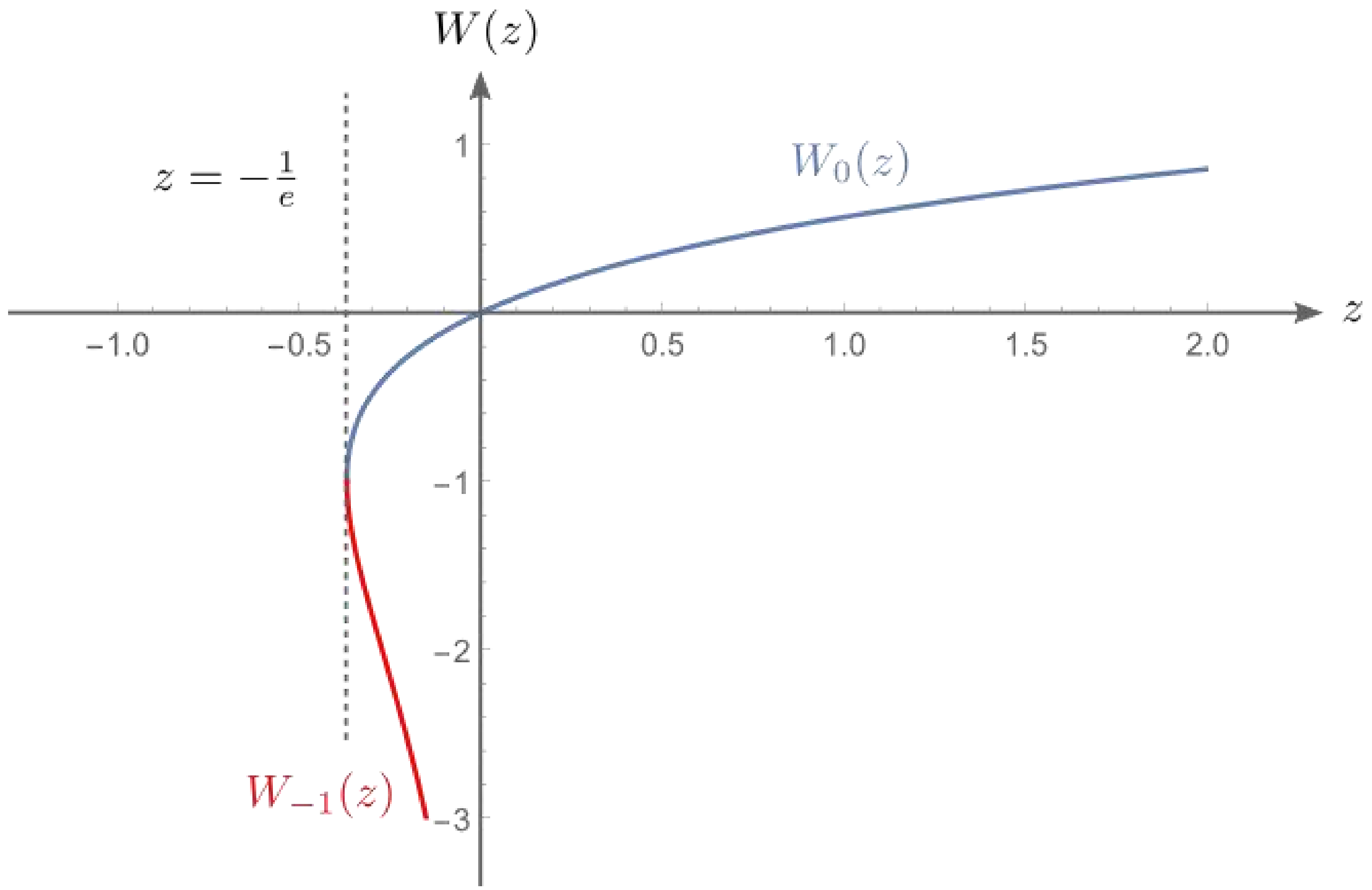
Real solution only exist for

$$\boxed{z \geq -e^{-1}}$$

For fixed β and μ, this implies

$$\lambda \leq \lambda_c(\beta, \mu) = 2\sqrt{\pi}\beta^{3/2}e^{-\beta\mu-1}$$

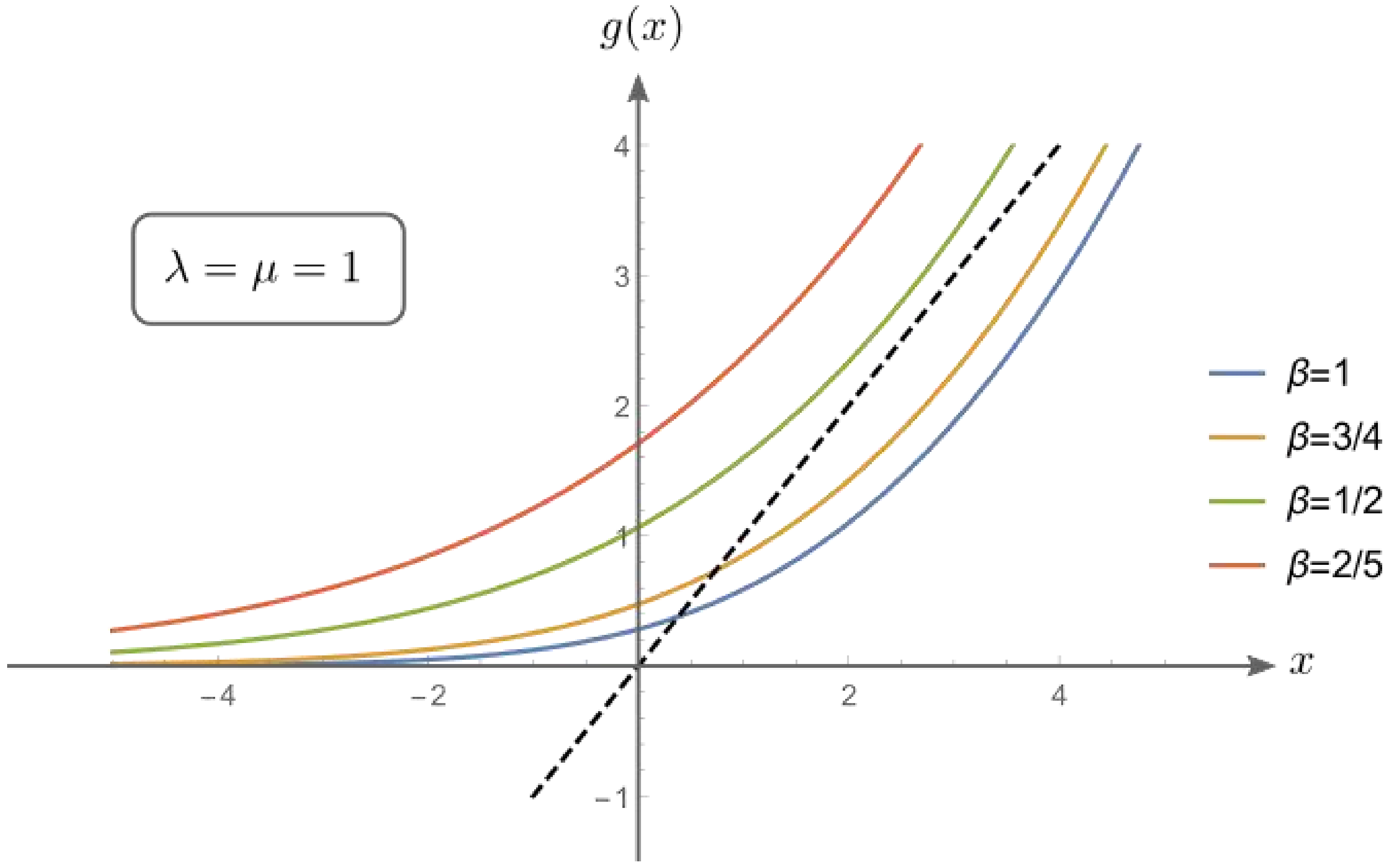
There exisit an **upper bound** for deformation parameter !



Plot for Lambert's *W*-function

Alternative explanation

For fixed $\lambda, \mu > 0$



An **upper bound for temperature**, the Hagedorn behavior

Conclusions

We can define **TTbar deformation for the Bose gas** as a special case of integrable bilinear deformation.

The deformation changes the **size of the particle**, or length of the system.

For finite volume spectrum, there is a **critical value for the negative sign** of the deformation parameter.

For thermodynamics, the TTbar deformation shifts the chemical potential. There's an **upper bound in temperature.**

Outlook

- Other quantities

  Compute correlation functions and
  other possible quantities

- Other interpretations

  Can we have an interpretation
  from non-relativistic gravity

- Relation to other models

  Bethe / gauge duality,
  attractive regime and matrix model