Stochastic Processes on Complex Networks - Problems

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ver. 1

Subcritical branching processes with diverging moments

- Let us consider the branching probability $q_k = \begin{cases} \kappa \frac{k^{-\gamma}}{\zeta(\gamma-1)} & (k \geq 1) \\ 1 - \kappa \frac{\zeta(\gamma)}{\zeta(\gamma-1)} & (k = 0) \end{cases}$
- The branching ratio κ can be controlled to change between subcritical critical and supercritical phase.
- Let us consider the subcritical phase with $\Delta = 1 - \kappa \ll 1$.
- **Problem1:**

Obtain the tree-size distribution $P(s)$ and the lifetime distribution $\ell(t)$ analytically (and numerically by simulations if time allows) and discuss how their behaviors depend on Δ.

First-return probability and the conditional mean first return time on complex networks

Problem 2:

Obtain the first-return probability $F_{ss}(n)$ at a starting node s and the conditional mean first return time $\tau_{ss}(n)$ in the random walks on scale-free networks with the degree distribution $P_d(k) \sim k^{-\gamma}$ and the spectral dimension d_s. Discuss how they depend on the network properties of the starting node and the whole network. If time allows, compare the analytic results with numerical simulations.

- The first-return probability's generating function is related to the return probability as $\mathcal{F}_{ss}(z) = 1 - \frac{1}{\mathcal{P}_{ss}(z)}$
- The singularity of $\mathcal{P}_{ss}(z)$ is related to that of $\mathcal{F}_{ss}(z)$ and in turn to the large-n behaviors of the first-return probability $F_{ss}(n)$.

Conditional mean first passage time

- The probability to reach s from i in n steps : $R_{is}(n) = \sum_{n'=1}^{n} F_{is}(n')$
- $\tau_{is}(n)$: The mean first passage time from s to i of the walkers which reach i in n steps.
- $\tau_{is}(n) = \frac{\sum_{n'=1}^{n} n' F_{is}(n')}{\sum_{n'=1}^{n} F_{is}(n')} = \frac{\sum_{n'=1}^{n} n' F_{is}(n')}{R_{is}(n)}$
- $Q_{is}(n) = R_{is}(n)\tau_{is}(n)$
- $\mathcal{Q}_{is}(z) = \sum_{n=1}^{\infty} \tau_{is}(n) R_{is}(n) z^n = \sum_{n=1}^{\infty} \sum_{n'=1}^{n} n' F_{is}(n') z^n = \sum_{n'=1}^{\infty} n' F_{is}(n') \sum_{n=n'}^{\infty} z^n = \frac{1}{1-z} \sum_{n=1}^{\infty} n F_{is}(n) z^n = \frac{z}{1-z} \mathcal{F}_{is}(z)'$
- The singularity of $\mathcal{Q}_{is}(z)$ at $z = 1$ informs of the large-n behavior of $R_{is}(n)\tau_{is}(n)$, which is proportional to that of $\tau_{is}(n)$ as $R_{is}(n)$ converges to a constant 1 or less than 1.
- Ex. 1d lattice: $F_{ss}(z) = 1 - \sqrt{1 - z^2}$ and thus $\mathcal{Q}_{ss}(z) \sim (1 - z)^{-3/2}$ implying $\tau_{is}(n) \sim n^{1/2}$.

Mean number of visited nodes on complex networks

- Let us consider the average of $\langle S_s(n)\rangle$ over the starting node s defined as $\langle S(n)\rangle = \sum_s \frac{k_s}{2L}\langle S_s(n)\rangle$

Problem 3:

First derive the relation between the mean number of visited nodes $\langle S(n)\rangle$, averaged over the starting nodes as above, and the return-to-origin probability in the random walks on a network. Then use the result to obtain $\langle S(n)\rangle$ in the random walks on scale-free networks with the degree distribution $P_d(k) \sim k^{-\gamma}$ and the spectral dimension d_s. If time allows, compare the analytic results with numerical simulations.

SI model on random scale-free networks

Problem 4:

Obtain the behavior of the fraction of infectious individuals as a function of time in the SI model on random scale-free networks and discuss their behaviors in the early and late-time regime with focus on their variations with the degree exponent of the networks