Cosmological Perturbations in Palatini Formalism

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11/20/20@String theory, gravity and cosmology (SGC2020) in APCTP

Mio Kubota, Kin-ya Oda, Keigo Shimada, MY: arXiv 2010.07867 [hep-th]

(I thank Keigo for sharing his many slides with me.)

$$c = \hbar = M_G^2 = 1/(8\pi G) = 1$$

Contents

- **Introduction**

  What's Palatini formalism ? Why do we consider ?

- **Palatini formalism and cosmological perturbations**

  Einstein gravity

  Non-minimal coupling case (L2, L4)

  Galileon (L3)

- **Discussion and conclusions**

**(There are many many references related to Palatini formalim.
I omit most of them in this talk because of no enough space, sorry.
Please see the references in our paper and so on.)**

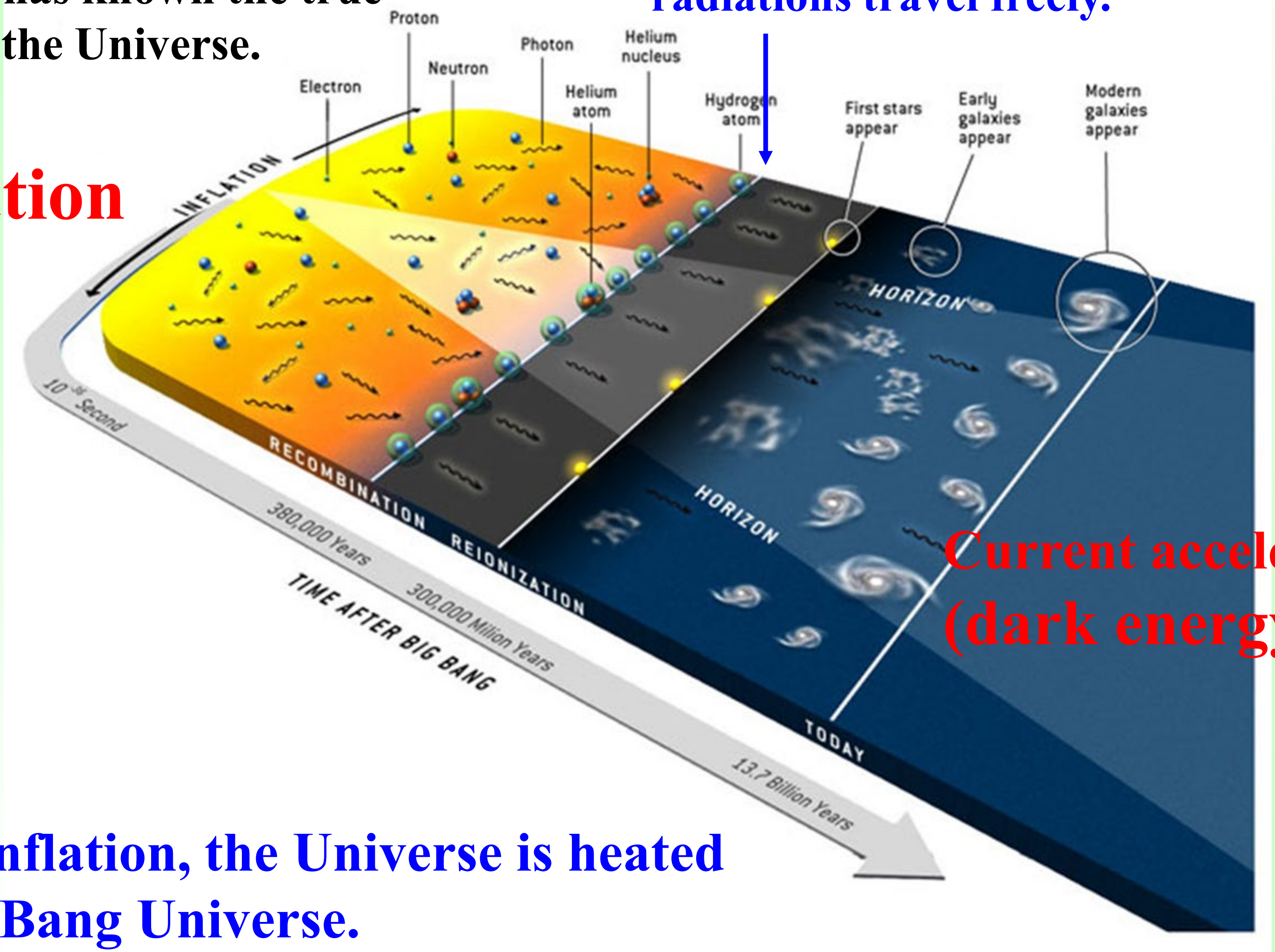
Introduction

Brief history of the Universe

Cosmic microwave background (CMB) radiations travel freely.

Nobody has known the true onset of the Universe.

Inflation



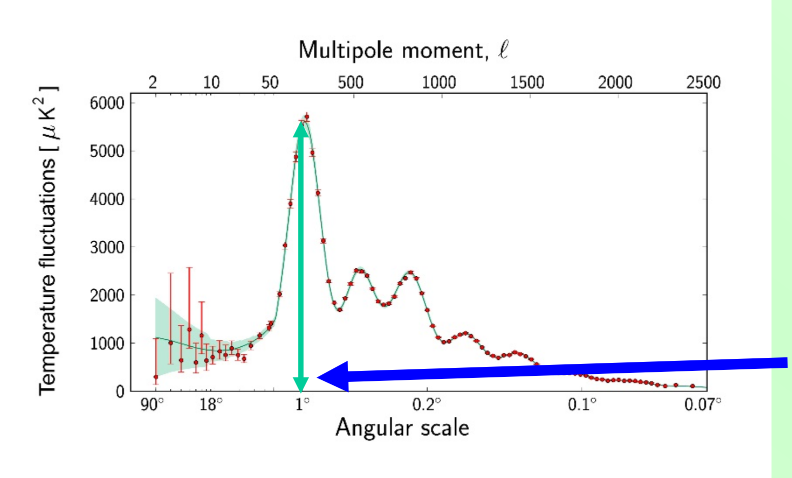
Current acceleration (dark energy)

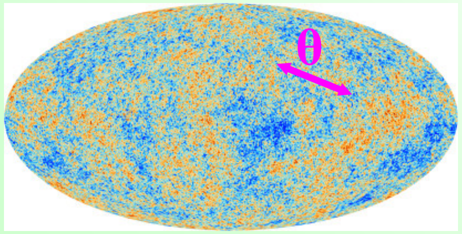
After inflation, the Universe is heated as Big-Bang Universe.

http://astrokatie.blogspot.jp/2012/08/the-long-dark-tea-time-of-cosmos.html

Inflation is strongly supported by CMB observations

Planck TT correlation :





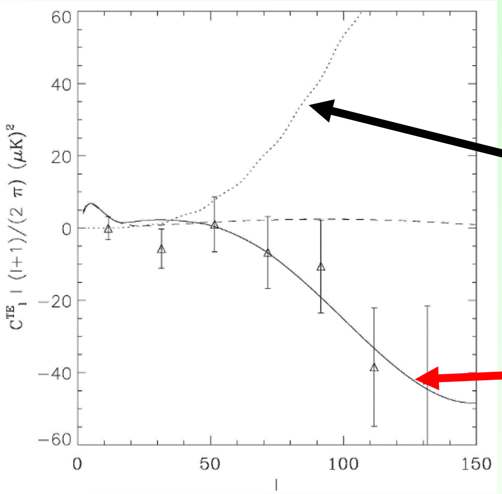
Green line : prediction by inflation

Red points : observation by PLANCK

Angle θ ~ 180° / 1

Total energy density ←→ Geometry of our Universe

WMAP TE correlation :



Our Universe is spatially flat as predicted by inflation !!

Causal seed models

Superhorizon models

(adiabatic perturbations)

We need a (scalar-like) dynamical degree of freedom responsible for inflation.

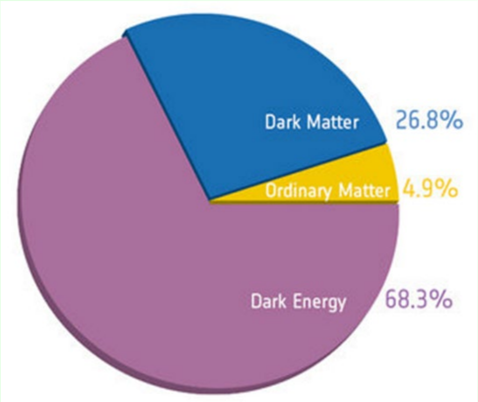
The presence of dark energy

The Universe is now accelerating !!

- Dark Energy is introduced

  or

- GR may be modified in the IR limit



PLANCK

In either case, we need a dynamical degree of freedom responsible for current acceleration.

We have almost confirmed the presence of inflation and dark energy, but, unfortunately, we know neither the identification of an inflaton nor that of dark energy.

Next task is to identify the inflaton and the origin of the dark energy.

→ **Introduce a new scalar d.o.f. in addition to spin 2 d.o.f.s (Modified gravity in a wider sense: Try to unify scalar and tensor d.o.f.s)**

Gravity

Two formalisms:
metric formalism & Palatini formalism

Metric formalism

A fundamental object (dynamical variable) is Riemann metric.

- **Riemann metric**

$g_{\mu\nu}$ **: a symmetric 2nd rank tensor determining the length**

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

⇨ **Connection (parallel transport) is given a priori in terms of a metric by requiring local Lorentz and the invariance of an angle between parallel transported vectors.**

- **symmetric ($\left\{ {\lambda \atop \mu\nu} \right\}_g = \left\{ {\lambda \atop \nu\mu} \right\}_g$)**
- **metric compatibility ($\nabla_\lambda g_{\mu\nu} = 0$)**

⇨ **Levi-Civita connection :** $\left\{ {\lambda \atop \mu\nu} \right\}_g := \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)$

Metric formalism II

A fundamental object (dynamical variable) is Riemann metric.

- Riemann metric

$g_{\mu\nu}$: a symmetric 2nd rank tensor determining the length

- Importantly, given an action,

➔ The variation of the action is taken only with respect to a metric in order to obtain the EOMs.

- But, a connection is fixed to be Levi-Civita one a priori.

Levi-Civita connection : $\left\{ {\lambda \atop \mu\nu} \right\}_g := \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)$

➔ The variation of the action is not taken with respect to a connection.

Palatini formalism

Fundamental objects (dynamical variables) are not only Riemann metric but also connection.

- **Riemann metric :**

  $g_{\mu\nu}$ **: a symmetric 2nd rank tensor determining the length**

- **Connection : (not confined to Levi-Civita one but arbitrary one)**

  $\Gamma^{\lambda}{}_{\nu\mu}$

  - ~~**symmetric (** $\left\{ {\lambda \atop \mu\nu} \right\}_g = \left\{ {\lambda \atop \nu\mu} \right\}_g$ **)**~~
  - ~~**metric compatibility (** $\nabla_\lambda g_{\mu\nu} = 0$ **)**~~

  ⇒

  - **Torsion :** $T^{\lambda}{}_{\mu\nu} := \Gamma^{\lambda}{}_{\nu\mu} - \Gamma^{\lambda}{}_{\mu\nu}$
  - **Non-metricity :** $Q_\sigma{}^{\mu\nu} := \overset{\Gamma}{\nabla}_\sigma g^{\mu\nu}$

(In general, torsion does not vanish, but, for simplicity, we consider only a torsion-less case later.)

Palatini formalism II

Fundamental objects (dynamical variables) are not only Riemann metric but also connection.

$$g_{\mu\nu} \qquad\qquad \Gamma^{\lambda}{}_{\nu\mu}$$

Importantly, given an action,

➔ The variations of the action with respect to not only a metric but also a connection are taken in order to obtain the EOMs.

c.f.) Electrodynamics :

$$\mathcal{L} = i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}^2$$

$$(D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi)$$

↑

connection = gauge field

We take the variation of the action with respect to not only an electron ψ but also a connection (gauge field) A in order to obtain the EOMs.

Lesson:

What happens to the Einstein gravity in Palatini formalism ?

(Assume torsion-less)

Einstein gravity in Palatini formalism

(Einstein 1925)

$$S = S_{\mathrm{EH}} + S_{\mathrm{matter}} = \int d^4x \sqrt{-g}\,\frac{1}{2}\overset{\Gamma}{R} + \int d^4x \sqrt{-g}\,\mathcal{L}_{\mathrm{m}}(g_{\mu\nu}, \Psi).$$

(Assume no dependence on Γ)

$$\begin{cases} \overset{\Gamma}{R} := g^{\mu\nu}\overset{\Gamma}{R}_{\mu\nu}, \\ \overset{\Gamma}{R}_{\mu\nu} := \overset{\Gamma}{R}{}^{\lambda}{}_{\mu\lambda\nu}, \\ \overset{\Gamma}{R}{}^{\lambda}{}_{\sigma\mu\nu} := \partial_\mu \Gamma^\lambda{}_{\sigma\nu} - \partial_\nu \Gamma^\lambda{}_{\sigma\mu} + \Gamma^\lambda{}_{\rho\mu}\Gamma^\rho{}_{\sigma\nu} - \Gamma^\lambda{}_{\rho\nu}\Gamma^\rho{}_{\sigma\mu}. \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \overset{\Gamma}{R}_{(\mu\nu)} - \frac{1}{2}\overset{\Gamma}{R} g_{\mu\nu} - T_{\mu\nu} = 0, \\ \frac{\delta S}{\delta \Gamma^\lambda{}_{\nu\mu}} = \frac{1}{2}\overset{\Gamma}{\nabla}_\sigma\left[\sqrt{-g}\left(g^{\nu\sigma}\delta^\mu_\lambda - g^{\mu\nu}\delta^\sigma_\lambda\right)\right] = 0, \end{cases}$$

$$\left(T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{matter}}}{\delta g^{\mu\nu}}\right)$$

$$\Rightarrow \overset{\Gamma}{\nabla}_\lambda g_{\mu\nu} = 0 \Rightarrow \Gamma^\lambda{}_{\nu\mu} = \left\{ {\lambda \atop \mu\nu} \right\}_g = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)$$

Different from metric formalism, a connection is dynamically fixed to be the Levi-Civita connection as the result of the EOM.

Now, let’s try to extend gravity to a scalar-tensor theory in Palatini formalism

But, before going to Palatini formalism, let’s briefly remember a scalar-tensor theory in metric formalism.

Generalized Galileon = Horndeski

Deffayet et al. 2009, 2011, Charmousis et al. 2012

Horndeski 1974

equivalence

Kobayashi, MY, Yokoyama 2011

$$\begin{cases} \mathcal{L}_2 = K(\phi, X) \\ \mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \\ \mathcal{L}_4 = G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\ \mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\ \quad\quad -\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]. \end{cases}$$

$$X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \qquad G_{iX} \equiv \partial G_i/\partial X.$$

This is the most general scalar tensor theory whose Euler-Lagrange EOMs are up to second order though the action includes second derivatives.
Many of inflation and dark energy models can be understood in a unified manner.

NB :
- **$G_4 = M_G^2 / 2$ yields the Einstein-Hilbert action**
- **$G_4 = f(\varphi)$ yields a non-minimal coupling of the form $f(\varphi)R$**
- **The new Higgs inflation with $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ comes from $G_5 \propto \varphi$ after integration by parts.**

Cosmological perturbations of Horndeski theory in metric formalism

(Kobayashi, MY, Yokoyama 2011)

- **Tensor perturbations:**

$$S_T^{(2)} = \frac{1}{8}\int \mathrm{d}t\mathrm{d}^3x\, a^3 \left[\mathcal{G}_T \dot{h}_{ij}^2 - \frac{\mathcal{F}_T}{a^2}(\nabla h_{ij})^2\right].$$

$$\begin{cases} \mathcal{F}_T := 2\left[G_4 - X\left(\ddot{\phi}G_{5X} + G_{5\phi}\right)\right], \\ \mathcal{G}_T := 2\left[G_4 - 2XG_{4X} - X\left(H\dot{\phi}G_{5X} - G_{5\phi}\right)\right] \end{cases}$$

$$c_T^2 := \frac{\mathcal{F}_T}{\mathcal{G}_T}$$

If this Horndeski field is responsible for dark energy, the sound speed of tensor perturbations (GWs) must be very close to unity.

$$c_T^2 = c_{\mathrm{GW}}^2 \simeq 1.$$

(e.g. Creminelli & Vernizzi 2017)
(Kimura & Yamamoto 2012)

(GW170817 & GRB170817A)

(gravitational Cherenkov radiation)

$$\begin{cases} \mathcal{L}_2 = K(\phi, X), \\ \mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \\ \mathcal{L}_4 = G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\ \mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\ \quad -\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]. \end{cases}$$

$$G_{4X} \simeq 0, \quad G_5 \simeq 0$$

Let's try to extend this Horndeski (Generalized Galileon) action to Palatini case.

$$
\begin{cases}
\mathcal{L}_2 = K(\phi, X), \\
\mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \\
\mathcal{L}_4 = G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\
\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\
\qquad -\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right].
\end{cases}
$$

$$
\left(X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,\ G_{iX} \equiv \partial G_i/\partial X\right)
$$

As a dark energy, only magenta boxes are allowed in metric formalism.

Horndeski correspondence in Palatini formalism

A non-minimal coupling of a scalar field to the Ricci scalar

In metric formalism,

$$\begin{cases} \mathcal{L}_2 = K(\phi, X), \\ \mathcal{L}_4 = G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]. \end{cases}$$

(Later, we will discuss L3)

$\left(c_T^2 \neq 1 \text{ for } G_{4X} \neq 0\right)$ $\left(X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,\ G_{iX} \equiv \partial G_i/\partial X\right)$

In Palatini formalism, $\mathcal{L}_4$ $\mathcal{L}_2$

$$S_4^{\text{Jordan}} := \int d^4x\sqrt{-g}\left[G_4(\phi, X)\overset{\Gamma}{R} + K(\phi, X)\right],$$

(The counter terms are unnecessary to keep the second order EOMs for the metric & φ.)

Analysis in three frames :

- **Einstein frame** : Minimal coupling, Einstein gravity (Calculation is well-known) (commonly used in the literatures, especially, in the context of Higgs inflation)
- **Jordan frame** : Non-minimal coupling (Calculation is tedious but straightforward)
- **Riemann frame** : Geometry is Riemannian (Calculation is done in metric formalism)

The central question : is c_T (GW speed) unity or not ?

Einstein frame

Analysis in Einstein frame

$$S_4{}^{\text{Jordan}} := \int d^4x\sqrt{-g}\left[G_4(\phi, X)\overset{\Gamma}{R} + K(\phi, X)\right],$$

Conformal transformation : $\tilde{g}_{\mu\nu} = G_4(\phi, X)g_{\mu\nu}$

$$\begin{pmatrix} \sqrt{-\tilde{g}} = G_4^2\sqrt{-g}, & \overset{\Gamma}{R} = g^{\mu\nu}\overset{\Gamma}{R}_{\mu\nu} = G_4\tilde{g}^{\mu\nu}\overset{\Gamma}{R}_{\mu\nu}, \\ X = G_4\tilde{X}, & \tilde{X} = -\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \end{pmatrix} \qquad \left(X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)$$

$$\begin{aligned} S_4{}^{\text{Jordan}} &= \int d^4x\sqrt{-g}\left[G_4(\phi, X)g^{\mu\nu}\overset{\Gamma}{R}_{\mu\nu} + K(\phi, X)\right], \\ &= \int d^4x\sqrt{-\tilde{g}}\left[\tilde{g}^{\mu\nu}\overset{\Gamma}{R}_{\mu\nu} + \frac{K(\phi, G_4\tilde{X})}{G_4^2(\phi, G_4\tilde{X})}\right] := S_4{}^{\text{Einstein}} \end{aligned}$$

This action is nothing but the k-essence action and the Einstein-Hilbert action with respect to $\tilde{g}_{\mu\nu} = G_4(\phi, X)g_{\mu\nu}$ **.**

$$\Gamma^\lambda{}_{\nu\mu} = \left\{{\lambda \atop \mu\nu}\right\}_{\tilde{g}} = \frac{1}{2}\tilde{g}^{\lambda\sigma}\left(\partial_\mu\tilde{g}_{\nu\sigma} + \partial_\nu\tilde{g}_{\mu\sigma} - \partial_\sigma\tilde{g}_{\mu\nu}\right)$$

The connection is given by the Levi-Civita one with respect to $\tilde{g}_{\mu\nu}$ **.**

Cosmological perturbations in Einstein frame

$$S_4^{\text{Einstein}} = \int d^4x \sqrt{-\tilde{g}} \left[\tilde{g}^{\mu\nu} \overset{\Gamma}{R}_{\mu\nu} + \frac{K(\phi, G_4 \tilde{X})}{G_4^2(\phi, G_4 \tilde{X})} \right]$$

Metric perturbations : $d\tilde{s}^2 = -\tilde{N}^2 d\tilde{t}^2 + \tilde{\gamma}_{ij} \left(d\tilde{x}^i + \tilde{N}^i d\tilde{t} \right) \left(d\tilde{x}^j + \tilde{N}^j d\tilde{t} \right)$

$$\begin{cases} \tilde{N} = 1 + \tilde{\alpha}, \\ \tilde{N}_i = \partial_i \tilde{\beta}, \\ \tilde{\gamma}_{ij} = \tilde{a}(\tilde{t})^2 e^{-2\tilde{\zeta}} \left(e^{\tilde{h}} \right)_{ij}. \end{cases}$$

(unitary gauge ←→ δφ = 0)

3 scalar perturbations : $\tilde{\alpha}, \ \tilde{\beta}, \ \tilde{\zeta}$

1(x2) tensor perturbations : $\tilde{h}_{ij}$

Conformal transformation for the background : $d\tilde{s}^2 = G_4(t) ds^2$

$$\begin{cases} d\tilde{t} = \sqrt{G_4(t)}\, dt, \\ d\tilde{x} = dx, \\ \tilde{a}(\tilde{t}) = \sqrt{G_4(t)}\, a(t). \end{cases}$$

Cosmological perturbations in Einstein frame II

$$S_4{}^{\text{Einstein}} = \int d^4x \sqrt{-\tilde{g}} \left[\tilde{g}^{\mu\nu} \overset{\Gamma}{R}_{\mu\nu} + \frac{K(\phi, G_4 \tilde{X})}{G_4^2(\phi, G_4 \tilde{X})} \right]$$

We have only to perturb the metric with the Levi-Civita connection.

- **Expand the action up to quadratic order of perturbations.**
- **Solve the constraints for lapse $\tilde{\alpha}$, shift $\tilde{\beta}$.**
- $$\delta^{(2)} S_4{}^{\text{Einstein,tensor}} = \frac{1}{4} \int d\tilde{t}\, d^3\tilde{x}\, \tilde{a}^3 \left[\tilde{h}'^2_{ij} - \frac{1}{\tilde{a}^2} (\tilde{\partial}_k \tilde{h}_{ij})^2 \right]$$

  → **$c_T = 1$ (GW speed = light speed)**

  $$\delta^{(2)} S_4{}^{\text{Einstein,scalar}} = \int d\tilde{t}\, d^3\tilde{x}\, \tilde{a}^3 \left[\tilde{\mathcal{G}}_S \zeta'^2 - \frac{\tilde{\mathcal{F}}_S}{\tilde{a}^2} (\tilde{\partial}_i \tilde{\zeta})^2 \right],$$

(background quantities)

$$\begin{aligned}
\tilde{\mathcal{F}}_S &= \frac{6\tilde{X} K_{\tilde{X}}}{-\tilde{K} + 2\tilde{X}\tilde{K}_{\tilde{X}}} = 2\tilde{c} \\
&= \frac{6X(2KG_{4X} - K_X G_4)}{(K - 2XK_X)G_4 + 3KG_{4X}X}, \\
\tilde{\mathcal{G}}_S &= \frac{6(\tilde{X}\tilde{K}_{\tilde{X}} + 2\tilde{X}^2 \tilde{K}_{\tilde{X}\tilde{X}})}{-\tilde{K} + 2\tilde{X}\tilde{K}_{\tilde{X}}} \\
&= \frac{6X}{(G_4 - G_{4X}X)^2 \{-K(G_4 + 3G_{4X}) + 2XK_X G_4\}} \\
&\quad \times \left[-6X^2 K G_{4X}^3 + X(8K + 5K_X X) G_4 G_{4X}^2 + (K_X + 2K_{XX}X) G_4^3 \right. \\
&\quad \left. -2\{K(G_{4X} + 2G_{4XX}X) + XK_X(3G_{4X} - XG_{4XX}) + X^2 K_{XX} G_{4X}\} G_4^2 \right],
\end{aligned}$$

$$\left(\tilde{c}_S^2 = \frac{\tilde{\mathcal{F}}_S}{\tilde{\mathcal{G}}_S} \right)$$

Jordan frame

Connection in Jordan frame

$$S_4{}^{\text{Jordan}} := \int d^4x\sqrt{-g}\left[G_4(\phi, X)\overset{\Gamma}{R} + K(\phi, X)\right], \qquad \left(X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)$$

$$\Rightarrow \begin{cases} -2\dfrac{1}{\sqrt{-g}}\dfrac{\delta S}{\delta g^{\mu\nu}} = (K_X + G_{4X}\overset{\Gamma}{R})\partial_\mu\phi\partial_\nu\phi + (K + G_4\overset{\Gamma}{R})g_{\mu\nu} - 2G_4\overset{\Gamma}{R}_{(\mu\nu)} = 0. \\ \dfrac{\delta S}{\delta \Gamma^\lambda{}_{\nu\mu}} = \overset{\Gamma}{\nabla}_\sigma\left[\sqrt{-g}\,G_4\,(g^{\nu\sigma}\delta^\mu\lambda - g^{\mu\nu}\delta^\sigma_\lambda)\right] = 0. \end{cases}$$

$$\Rightarrow \overset{\Gamma}{\nabla}_\lambda g_{\mu\nu} = g^{\mu\nu}\partial_\lambda(\ln G_4) \neq 0.$$

$$\Rightarrow \Gamma^\lambda{}_{\mu\nu} = \left\{{\lambda \atop \mu\nu}\right\}_g + \frac{1}{2}g^{\lambda\sigma}\left(2g_{\sigma(\mu}\partial_{\nu)}\ln G_4 - g_{\mu\nu}\partial_\sigma \ln G_4\right).$$

The connection does not coincide with the Levi-Civita one in general.

Cosmological perturbations in Jordan frame

$$S_4{}^{\mathrm{Jordan}} := \int d^4x\sqrt{-g}\left[G_4(\phi, X)\overset{\Gamma}{R} + K(\phi, X)\right],$$

Metric perturbations : $ds^2 = -N^2dt^2 + \gamma_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$

$$\begin{cases} N &= 1+\alpha, \\ N_i &= \partial_i\beta, \\ \gamma_{ij} &= a(t)^2 e^{-2\zeta}\left(e^h\right)_{ij}. \end{cases}$$

(unitary gauge ←→ δφ = 0)

- **3 scalar perturbations : α, β, ζ**
- **1(x2) tensor perturbations : hij**

Connection perturbations :

$$\begin{cases} \delta\Gamma^0{}_{00} &= c_1, \\ \delta\Gamma^0{}_{i0} &= \partial_i c_2, \\ \delta\Gamma^0{}_{ij} &= D_{1,ij} + \delta_{ij}c_3 + \partial_i\partial_j c_4, \\ \delta\Gamma^i{}_{00} &= \partial^i c_5 \\ \delta\Gamma^i{}_{j0} &= D^i_{2,j} + \delta^i_j c_6 + \partial^i\partial_j c_7, \\ \delta\Gamma^i{}_{jk} &= \partial^i D_{3,jk} + \partial_{(j}D^i_{4,k)} + \delta_{jk}\partial^i c_8 + \delta^i_{(j}\partial_{k)}c_9 + \partial^i\partial_j\partial_k c_{10}. \end{cases}$$

- **10 scalar perturbations : cn**
- **4(x2) tensor perturbations : Dm,ij**

Cosmological perturbations in Jordan frame II

$$S_4{}^{\text{Jordan}} := \int d^4x\sqrt{-g}\left[G_4(\phi, X)\overset{\Gamma}{R} + K(\phi, X)\right],$$

Metric perturbations :

- **3 scalar perturbations : α, β, ζ**
- **1(x2) tensor perturbations : hij**

Connection perturbations :

- **10 scalar perturbations : cn**
- **4(x2) tensor perturbations : Dm,ij**

⇒ Expand the action up to quadratic order of perturbations

⇒ Solve the constraints for lapse α, shift β, and connections

⇒

$$\delta^{(2)}S_4{}^{\text{Jordan,tensor}} = \frac{1}{4}\int dtd^3xG_4a^3\left[\dot{h}_{ij}^2 - \frac{1}{a^2}(\partial_k h_{ij})^2\right] \quad \left(= \delta^{(2)}S_4{}^{\text{Einstein,tensor}}\right)$$

⇒ $c_T = 1$ (GW speed = light speed)

$$\delta^{(2)}S_4{}^{\text{Jordan,scalar}} = \int dt\, d^3x\, a^3\left[\mathcal{G}_S\dot{\zeta}^2 - \frac{\mathcal{F}_S}{a^2}(\partial_i\zeta)^2\right] \quad \left(= \delta^{(2)}S_4{}^{\text{Einstein,scalar}}\right)$$

$$\left(\mathcal{G}_S = G_4(t)\tilde{\mathcal{G}}_S, \quad \mathcal{F}_S = G_4(t)\tilde{\mathcal{F}}_S\right) \qquad \left(h_{ij} = \tilde{h}_{ij}, \quad \zeta = \tilde{\zeta}\right)$$

Riemann frame

Analysis in Riemann frame

$$S_4{}^{\text{Jordan}} := \int d^4x\sqrt{-g}\left[G_4(\phi, X)\overset{\Gamma}{R} + K(\phi, X)\right], \qquad \left(X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)$$

$$\Rightarrow \quad \Gamma^\lambda{}_{\mu\nu} = \left\{{\lambda \atop \mu\nu}\right\}_g + \frac{1}{2}g^{\lambda\sigma}\left(2g_{\sigma(\mu}\partial_{\nu)}\ln G_4 - g_{\mu\nu}\partial_\sigma \ln G_4\right).$$

$$\left(\Rightarrow \quad \overset{\Gamma}{R} = \overset{g}{R} - \overset{g}{\nabla}_\sigma\left(\partial_\sigma \log G_4\right) - \frac{3}{2}g^{\mu\nu}\left(\partial_\mu \log G_4\right)\left(\partial_\nu \log G_4\right), \right)$$

$$\Rightarrow \quad S_4{}^{\text{Jordan}} = \sqrt{-g}\left[G_4\overset{g}{R} + \frac{3}{2}\frac{(\overset{g}{\nabla}G_4)^2}{G_4} + K\right]$$

$$= \sqrt{-g}\left[G_4\overset{g}{R} - \frac{3}{2G_4}\left(2G_{4\phi}^2X + 2G_{4\phi}G_{4X}\phi^\alpha\phi_{\alpha\beta}\phi^\beta - G_{4X}^2\phi^\alpha\phi_{\alpha\beta}\phi^{\beta\gamma}\phi_\gamma\right) + K\right] := S_4{}^{\text{Riemann}}$$

In this frame, the connection is a priori fixed to the Levi-Civita one.

⇒ **But, this is nothing but simple rewriting and hence both g and φ obey the same EOMs as those in Jordan frame.**

(Langlois & Noui 2016, Crisostomi et al. 2016, Ben Achour et al. 2016 …)

In fact, this action reduces to the so-called DHOST action and the quadratic actions for perturbations are shown to coincide with those in Jordan frame.

Cosmological perturbations in three frames

- **The quadratic actions for tensor and scalar perturbations in three different frames (Einstein, Jordan, Riemann) are obtained and also shown to be the same.**

- **Even if G4 has X-dependence, the speed of GWs is unity, in sharp contrast with the case of metric formalism.**

$$S_4^{\text{Jordan}} := \int d^4x\sqrt{-g}\left[G_4(\phi, X)\overset{\Gamma}{R} + K(\phi, X)\right],$$

Let's finally discuss L3 (Galileon) action in Palatini formalism.

$$
\left\{
\begin{aligned}
\mathcal{L}_2 &= K(\phi, X), \\
\mathcal{L}_3 &= -G_3(\phi, X)\Box\phi, \\
\mathcal{L}_4 &= G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\
\mathcal{L}_5 &= G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\
&\quad -\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right].
\end{aligned}
\right.
$$

$$\left(X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,\ G_{iX} \equiv \partial G_i/\partial X\right)$$

As a dark energy, only magenta boxes are allowed in metric formalism.

L3 term (KGB or G-inflation) in metric formalism

(Kobayashi, MY, Yokoyama 2010, 2011
Cedric, Pujolas, Sawicki, Vikman 2010)

- The L3 term is **uniquely** determined in **metric** formalism

$$\overset{g}{\Box}\phi = g^{\mu\nu}\overset{g}{\nabla}_\mu\overset{g}{\nabla}_\nu\phi = \overset{g}{\nabla}_\mu(g^{\mu\nu}\overset{g}{\nabla}_\nu\phi) = \overset{g}{\nabla}_\mu\left(\overset{g}{\nabla}_\nu(g^{\mu\nu}\phi)\right) = \frac{1}{4}g^{\mu\nu}g^{\alpha\beta}\overset{g}{\nabla}_\mu\left(g_{\alpha\beta}\overset{g}{\nabla}_\nu\phi\right) \cdots$$

$$\begin{cases} \mathcal{L}_2 = K(\phi, X), \\ \mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \\ \mathcal{L}_4 = G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\ \mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\ \qquad -\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]. \end{cases}$$

All of these expressions are the same thanks to **the metricity.**

$$\left(\overset{g}{\nabla}_\lambda g_{\mu\nu} = 0\right)$$

- The L3 term does **not** affect the speed of GWs at all in metric formalism.

Tensor perturbations: $S_T^{(2)} = \frac{1}{8}\int \mathrm{d}t\mathrm{d}^3x\, a^3\left[\mathcal{G}_T\dot{h}_{ij}^2 - \frac{\mathcal{F}_T}{a^2}(\nabla h_{ij})^2\right].$

$$\begin{cases} \mathcal{F}_T := 2\left[G_4 - X\left(\ddot{\phi}G_{5X} + G_{5\phi}\right)\right], \\ \mathcal{G}_T := 2\left[G_4 - 2XG_{4X} - X\left(H\dot{\phi}G_{5X} - G_{5\phi}\right)\right] \end{cases}$$

$$c_T^2 := \frac{\mathcal{F}_T}{\mathcal{G}_T}$$

L3 term (KGB or G-inflation) in Palatini formalism

- **The L3 term is not uniquely determined in Palatini formalism**

$$\overset{\Gamma}{\Box}\phi \overset{?}{=} g^{\mu\nu}\overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu\phi \overset{?}{=} \overset{\Gamma}{\nabla}_\mu(g^{\mu\nu}\overset{\Gamma}{\nabla}_\nu\phi) \overset{?}{=} \overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu(g^{\mu\nu}\phi) \cdots$$

⟷ **g** → $\nabla \ \nabla \ \phi$

Non-metricity : $Q_\lambda{}^{\mu\nu} := \overset{\Gamma}{\nabla}_\lambda g^{\mu\nu} \neq 0$

Fortunately, there are only finite (10) number of types.

$$\overset{\Gamma}{\Box}\phi = \begin{cases} g \times \nabla\nabla\phi & \textbf{1 term} \\ \nabla g \times \nabla\phi & \textbf{2 terms} \\ \nabla g \times \nabla g \times \phi & \textbf{5 terms} \\ \nabla\nabla g \times \phi & \textbf{2 terms} \end{cases}$$

⟵ **(Aoki & Shimada 2018, 2019 Helpin & Volkov 2019, 2020)**

$$\begin{aligned}\mathcal{L}_3{}^{\text{Palatini}} &:= G_{3,0}\overset{g}{\Box}\phi + G_{3,1}Q^\mu\partial_\mu\phi + G_{3,2}\bar{Q}^\mu\partial_\mu\phi \\ &\quad + G_{3,3}\phi Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} + G_{3,4}\phi Q_{\alpha\beta\gamma}Q^{\beta\gamma\alpha} + G_{3,5}\phi Q^\mu Q_\mu + G_{3,6}\phi Q_\mu\bar{Q}^\mu \\ &\quad + G_{3,7}\phi\bar{Q}_\mu\bar{Q}^\mu + G_{3,8}\phi\overset{g}{\nabla}_\mu Q^\mu + G_{3,9}\phi\overset{g}{\nabla}_\mu\bar{Q}^\mu \\ &=: \sum_{i=0}^{9} G_{3,i}\,\overset{\Gamma}{\Box}_{(i)}\phi\end{aligned}$$

$$\left(Q_\mu := \frac{1}{4}Q_{\mu\nu}{}^\nu,\ \bar{Q}^\mu := Q_\nu{}^{\nu\mu}\right)$$

L3 term (KGB or G-inflation) in Palatini formalism II

$$\mathcal{L}_{2+3+4}^{\text{Palatini}} := K + \sum_{i=0}^{9} G_{3,i} \overset{\Gamma}{\Box}_{(i)}\phi + G_4 \overset{\Gamma}{R}$$

This action has, in general,

- **no Einstein frame**
- **an (Ostrogradsky) ghost mode**
- **non-unity sound speed of GWs.**

→ **If we remove the ghost by suitable choice of G3,i , this model reduces to the DHOST model with the sound speed of GWs being unity (like the case in metric formalism).**

Connection of L3 term (KGB or G-inflation) in Palatini formalism

$$\mathcal{L}_{2+3+4}^{\text{Palatini}} := K + \sum_{i=0}^{9} G_{3,i} \overset{\Gamma}{\Box}_{(i)}\phi + G_4 \overset{\Gamma}{R}$$

$$\Rightarrow \quad \Gamma^{\lambda}{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\}_g + \frac{1}{D}\left[\left\{A^X \partial^\lambda X + A^\phi \partial^\lambda \phi\right\} g_{\mu\nu} + 2\left\{B^X \partial_{(\mu} X + B^\phi \partial_{(\mu}\phi\right\} \delta^\lambda_{\nu)}\right],$$

$$\left\{ \begin{aligned}
A^\phi &= -6G_4(2G_{4\phi} + G_{3,1} + 2G_{3,2} - G_{3,8} - 2G_{3,9}) \\
&\quad - \Big\{-6(G_{3,8\phi} + 2G_{3,9\phi})G_4 + 2[32(G_{3,3} + G_{3,4}) + 5G_{3,5} + 17G_{3,6} + 56G_{3,7}]G_{4,\phi} \\
&\qquad + (8G_{3,3} + 12G_{3,4} + 5G_{3,6} + 28G_{3,7})G_{3,1} - 2(16G_{3,3} + 8G_{3,4} + 5G_{3,5} + 7G_{3,6})G_{3,2} \\
&\qquad - 8G_{3,3}G_{3,8} - 12G_{3,4}G_{3,8} - 5G_{3,6}G_{3,8} - 28G_{3,7}G_{3,8} + 32G_{3,3}G_{3,9} + 16G_{3,4}G_{3,9} \\
&\qquad + 10G_{3,5}G_{3,9} + 14G_{3,6}G_{3,9}\Big\}\phi \\
&\quad + \{(8G_{3,3} + 12G_{3,4} + 5G_{3,6} + 28G_{3,7})G_{3,8\phi} - 2(16G_{3,3} + 8G_{3,4} + 5G_{3,5} + 7G_{3,6})G_{3,9\phi}\}\phi^2, \\
A^X &= 6G_4\{-2G_{4X} + (G_{3,8X} + 2G_{3,9X})\phi\} \\
&\quad - 2\{32(G_{3,3} + G_{3,4}) + 5G_{3,5} + 17G_{3,6} + 56G_{3,7}\}G_{4X}\phi \\
&\quad + \{(8G_{3,3} + 12G_{3,4} + 5G_{3,6} + 28G_{3,7})G_{3,8X} - 2(16G_{3,3} + 8G_{3,4} + 5G_{3,5} + 7G_{3,6})G_{3,9X}\}\phi^2, \\
B^\phi &= 4G_4(6G_{4\phi} + G_{3,1} - 2G_{3,2} - G_{3,8} + 2G_{3,9}) \\
&\quad + 2\Big\{-2(G_{3,8\phi} - 2G_{3,9\phi})G_4 + 2[16(G_{3,3} + G_{3,4}) + G_{3,5} + 7G_{3,6} + 40G_{3,7}]G_{4,\phi} \\
&\quad + (8G_{3,3} + 4G_{3,4} + G_{3,6} + 20G_{3,7})G_{3,1} - 2(8G_{3,4} + G_{3,5} + 5G_{3,6})G_{3,2} - 8G_{3,3}G_{3,8} \\
&\quad - 4G_{3,4}G_{3,8} - G_{3,6}G_{3,8} - 20G_{3,7}G_{3,8} + 16G_{3,4}G_{3,9} + 2G_{3,5}G_{3,9} + 10G_{3,6}G_{3,9}\Big\}\phi \\
&\quad + 2\{(8G_{3,3} + 4G_{3,4} + G_{3,6} + 20G_{3,7})G_{3,8\phi} + 2(8G_{3,4} + G_{3,5} + 5G_{3,6})G_{3,9\phi}\}\phi^2, \\
B^X &= 4G_4\{6G_{4X} - (G_{3,8X} - 2G_{3,9X})\phi\} + 4\{16(G_{3,3} + G_{3,4}) + G_{3,5} + 7G_{3,6} + 40G_{3,7}\}G_{4X}\phi \\
&\quad + \{2(8G_{3,3} + 4G_{3,4} + G_{3,6} + 20G_{3,7})G_{3,8X} - 4(8G_{3,4} + G_{3,5} + 5G_{3,6})G_{3,9X}\}\phi^2, \\
D &= 24G_4^2 + 4(8G_{3,3} + 20G_{3,4} - G_{3,5} + 8G_{3,6} + 44G_{3,7})G_4\phi \\
&\quad - 2\Big\{64G_{3,3}^2 - 32G_{3,4}^2 + 16G_{3,3}(2G_{3,4} + G_{3,5} + G_{3,6} + 10G_{3,7}) - 9(G_{3,6}^2 - 4G_{3,5}G_{3,7}) \\
&\qquad + 4G_{3,4}[G_{3,5} - 8(G_{3,6} + G_{3,7})]\Big\}\phi^2.
\end{aligned} \right.$$

Riemann frame of L3 term (KGB or G-inflation) in Palatini formalism

$$\mathcal{L}_{2+3+4}^{\text{Palatini}} := K + \sum_{i=0}^{9} G_{3,i} \overset{\Gamma}{\Box}_{(i)}\phi + G_4 \overset{\Gamma}{R}$$

$$\Rightarrow \quad \Gamma^{\lambda}{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\}_g + \frac{1}{D}\left[\left\{A^X \partial^\lambda X + A^\phi \partial^\lambda \phi\right\} g_{\mu\nu} + 2\left\{B^X \partial_{(\mu} X + B^\phi \partial_{(\mu}\phi\right\}\delta^\lambda_{\nu)}\right],$$

$$\Rightarrow \quad \mathcal{L}_{2+3+4}^{\text{Riemann}} = G_4 \overset{g}{R} + K + G_{3,0} \overset{g}{\Box}\phi + E_{\phi\phi} + E_{\phi X}\phi^\alpha \phi_{\alpha\beta}\phi^\beta + E_{XX}\phi^\alpha\phi_{\alpha\beta}\phi^{\gamma\delta}\phi_\delta$$

$$\begin{aligned}
E_{\phi\phi} = & -\frac{X}{8D^2}\Big[4A^{\phi 2}\left\{12G_4 + (40G_{3,3} + 28G_{3,4} + G_{3,5} + 10G_{3,6} + 100G_{3,7})\phi\right\} \\
& + B^{\phi 2}\left\{12G_4 + (136G_{3,3} + 124G_{3,4} + 25G_{3,5} + 70G_{3,6} + 196G_{3,7})\phi\right\} \\
& + 4A^\phi B^\phi\left\{24G_4 + (56G_{3,3} + 68G_{3,4} + 5G_{3,5} + 32G_{3,6} + 140G_{3,7})\phi\right\} \\
& - 8DA^X\{6G_{4\phi} - G_{3,1} - 10G_{3,2} + G_{3,8} + 10G_{3,9} + (G_{3,8\phi} + 10G_{3,9\phi})\phi\} \\
& + 4DB^X\{6G_{4\phi} + 5G_{3,1} + 14G_{3,2} - 5G_{3,8} - 14G_{3,9} - (5G_{3,8\phi} + 14G_{3,9\phi})\phi\}\Big], \\
E_{\phi X} = & -\frac{1}{8D^2}\Big[A^\phi A^X\left\{48G_4 + 4(40G_{3,3} + 28G_{3,4} + G_{3,5} + 10G_{3,6} + 100G_{3,7})\phi\right\} \\
& + (B^\phi A^X + A^\phi B^X)\left\{48G_4 + 2(56G_{3,3} + 68G_{3,4} + 5G_{3,5} + 32G_{3,6} + 140G_{3,7})\phi\right\} \\
& + B^\phi B^X\left\{12G_4 + (136G_{3,3} + 124G_{3,4} + 25G_{3,5} + 70G_{3,6} + 196G_{3,7})\phi\right\} \\
& - 4DA^\phi(6G_{4X} + G_{3,8X}\phi + 10G_{3,9X}\phi) \\
& + 2DB^\phi(6G_{4X} - 5G_{3,8X}\phi - 14G_{3,9X}\phi) \\
& - 4DA^X\left\{6G_{4\phi} - G_{3,1} - 10G_{3,2} + G_{3,8} + 10G_{3,9} + (G_{3,8\phi} + 10G_{3,9\phi})\phi\right\} \\
& + 2DB^X\left\{6G_{4\phi} + 5G_{3,1} + 14G_{3,2} - 5G_{3,8} - 14G_{3,9} - (5G_{3,8\phi} + 14G_{3,9\phi})\phi\right\}\Big], \\
E_{XX} = & \frac{1}{16D^2}\Big[4A^{X2}\left\{12G_4 + (40G_{3,3} + 28G_{3,4} + G_{3,5} + 10G_{3,6} + 100G_{3,7})\phi\right\} \\
& + B^{X2}\left\{12G_4 + (136G_{3,3} + 124G_{3,4} + 25G_{3,5} + 70G_{3,6} + 196G_{3,7})\phi\right\} \\
& + 4A^X B^X\left\{24G_4 + (56G_{3,3} + 68G_{3,4} + 5G_{3,5} + 32G_{3,6} + 140G_{3,7})\phi\right\} \\
& - 8DA^X\{6G_{4X} + (G_{3,8X} + 10G_{3,9X})\phi\} \\
& + 4DB^X\{6G_{4X} - (5G_{3,8X} + 14G_{3,9X})\phi\}\Big].
\end{aligned}$$

Ghost-free :

$$E_{XX} = \frac{3G_{4X}^2}{2G_4}$$

(at most, 2 tensor & 1 scalar d.o.f.)

⇒ **qDHOST class of ²N-I/Ia**

⇒ **$c_T = 1$**

Summary

- **We considered Palatini formalism, where the variation of an action is taken with respect to not only metric but also connection.**
- **We considered the case of a non-minimal coupling of a scalar field to the Ricci scalar (L4) plus k-essence action (L2) and discussed cosmological perturbations, yielding their quadratic actions in three different frames.**
- **The sound speed of GWs is always unity in Palatini formalism even if G4 includes X-dependence, in sharp contrast with that in metric formalism.**
- **We classified the Galileon action in Palatini formalism and found that there are essentially 10 different terms.**
- **An action consisting of these terms as well as L2+L4, it generally leads to a ghost d.o.f. and the deviation from unity of the sound speed of GWs. However, once we eliminate such a ghost, the sound speed of GWs becomes unity, which coincides with that in metric formalism.**