An Essential Primer of Stochastic Processes
Su-Chan Park

Assignements

1. X_1 and X_2 are independent Gaussian r.v.'s. The mean of X_1 (X_2) is μ_1 (μ_2) and the variance is σ_1^2 (σ_2^2). Show $Y = X_1 + X_2$ is also a Gaussian r.v.. What are the mean and the variance of Y?

2. Let $X_1, \ldots, X_n$ be i.i.d. r.v.'s with the probability density function (pdf)
$$P(x) = \frac{1}{\sqrt{2\pi x^3}} \exp\left(-\frac{1}{2x}\right) \ (x > 0).$$
   (a) Find the generating function $G(\mu) = \langle e^{-\mu x}\rangle$ ($\mu > 0$).
   (b) Show that the pdf of an r.v. $Y_n = (X_1 + \cdots + X_n)/n^2$ is $P(x)$.

3. The Wiener process is defined by the conditional probability
$$P(x,t|y,t') = \frac{1}{\sqrt{4\pi D(t-t')}} \exp\left(-\frac{(x-y)^2}{4D(t-t')}\right),$$
and the Cauchy process by
$$P(x,t|y,t') = \frac{t-t'}{\pi\left[(x-y)^2 + (t-t')^2\right]},$$
where $t > t'$.
   (a) Show that both the Wiener process and the Cauchy process satisfy the CK equation.
   (b) By investigating the Lindberg condition, show that the sample paths of the Wiener (Cauchy) process is continuous (discontinuous).
   (c) Calculate $W(x|z,t)$, $A(z,t)$, and $B(z,t)$ for the above two processes.

4. In this problem, we will calculate the extinction probability of the Galton-Watson branching process (X_m) with the initial condition $X_0 = 1$.
   (a) For $p(k) = e^{-1-s}(1+s)^k/k!$, show that the extinction probability for $0 < s \ll 1$ is approximately $\xi = 1 - 2s$.
   (b) For $p(k) = q(1-q)^k$ ($0 < q < 1$), find the probability that no individual is left at generation m (that is, find ξ_m defined in the lecture). Find also the generating function $\mathcal{G}_m(z)$.

5. Consider a Markov chain with the transition probability
$$T_{mn} = \begin{cases} \dfrac{n}{N}\dfrac{(N-n)(1-s)}{n+(N-n)(1-s)} \equiv a_n, & m = n-1, \\ \dfrac{N-n}{N}\dfrac{n}{n+(N-n)(1-s)} \equiv c_n, & m = n+1, \\ 1 - a_n - c_n, & m = n, \\ 0 & \text{otherwise.} \end{cases}$$
Since $a_0 = c_0 = a_N = c_N = 0$, the states 0 and N are absorbing, so the state space $\mathfrak{X}$ should be finite [$\mathfrak{X} = \{0, 1, 2, \ldots, N\}$]. Thus, the system eventually should fall into one of these two absorbing states. By "winning" ("losing"), we mean that the system eventually falls into the $n = N$ ($n = 0$) absorbing state. Note that this process with $s = 0$ is martingale in that $\langle n\rangle$ does not change with time. Hence the probability of "winning" can be calculated as
$$\langle n\rangle_0 = N \times \mathbb{P}(\text{“winning”}) + 0 \times \mathbb{P}(\text{“losing”}),$$
$$\Rightarrow \mathbb{P}(\text{“winning”}) = \frac{\langle n\rangle_0}{N}.$$
   (a) In case the system starts from $n = 1$ at $t = 0$, what is the "winning" probability ?
   (b) Find the expected time to arrive at $n = N$ state for $s = 0$ provided "winning" will happen (see Prob. 8d).

6. Let $X_1, \ldots, X_n$ be i.i.d. r.v.'s with the exponential distribution $\mathbb{P}(X_i > t) = \exp(-\lambda t)$ ($t \geq 0$). Let $S_n = X_1 + \cdots + X_n$ and $N(t)$ be the number of indices $k \geq 1$ such that $S_k < t$.
   (a) Show that the pdf for S_n is
$$g_n(t) = \lambda \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}.$$

(b) From $\mathbb{P}(N(t)=n)=\mathbb{P}(S_n<t \text{ and } S_{n+1}>t)$, prove that

$$\mathbb{P}(N(t)=n)=\frac{(\lambda t)^n}{n!}e^{-\lambda t}.$$

(c) Show that the solution of the following master equation with $P_n(0)=\delta_{n0}$ is $P_n(t)=\mathbb{P}(N(t)=n)$.

$$\dot{P}_n(t)=-\lambda P_n(t)+\lambda P_{n-1}(t), \dot{P}_0(t)=-\lambda P_0(t).$$

(d) Let $t>0$ be fixed, but arbitrary. Show that the element X_k satisfying the condition $S_{k-1}<t\le S_k$ has the pdf

$$v_t(x)=\begin{cases}\lambda^2 x e^{-\lambda x} & \text{for } 0<x\le t,\\ \lambda(1+\lambda x)e^{-\lambda x} & \text{for } x>t,\end{cases}$$

whose mean approaches $2/\lambda$ as $t\to\infty$.

(e) Show that the waiting time distribution (for definition, see the lecture note) is $\mathbb{P}(W_t<y)=1-\exp(-\lambda y)$.

7. A master equation on a finite state space $\mathfrak{X}=\{1,\dots,C\}$ satisfies detailed balance iff

$$W_{n_1n_2}P^e_{n_2}=W_{n_2n_1}P^e_{n_1}, \tag{1}$$

where $W_{n_1n_2}$ is the transition rate from n_2 to n_1 and P^e_n is the stationary distribution.

(a) It is often necessary to check for the presence or absence of detailed balance in situations where the stationary distribution P^e_n is not explicitly known. To this end, consider a closed loop

$$\mathcal{L}=(n_1\to n_2\to n_3\to \dots\to n_N\to n_1)$$

in the state space along with its 'time-reversed' partner $\bar{\mathcal{L}}=(n_1\to n_N\to n_{N-1}\to\dots\to n_2\to n_1)$. Take the product of the transition rates along the loop,

$$\pi(\mathcal{L})\equiv\prod_{k=1}^{N}W_{n_kn_{k+1}}$$

with $n_{N+1}\equiv n_1$, and show that Eq. (1) holds if and only if $\pi(\mathcal{L})=\pi(\bar{\mathcal{L}})$ for *all* possible loops. Note that this requires in particular to construct (at least in principle) the stationary distribution P^e_n from the rates.

(b) Make sure that, if the detailed balance is satisfied,

$$V_{n_1n_2}\equiv\left[P^e_{n_1}\right]^{-1/2}W_{n_1n_2}\left[P^e_{n_2}\right]^{1/2}$$

is symmetric and can therefore be diagonalized. Conclude from this that W can be diagonalized and express its eigenfunctions and eigenvalues in those of V.

8. In the lecture, the generating function for the continuous time branching process was calculated. In this problem, the r.v. $X(t)$ will be termed as the number of individuals at time t and the system as a population. As in the lecture, the initial number of individuals is m. For simplicity, we set $\lambda=1$.

(a) Find the generating function when $\mu=\lambda=1$.

(b) What is the expected number of individuals at time t?

(c) Calculate the extinction probability that the population eventually dies out.

(d) Write down the master equation for the survived ensembles, that is, a set of sample paths which never hit 0. You should think over the conditional probability

$$\mathbb{P}_x(X(t)=y|S)=\frac{\mathbb{P}_x((X(t)=y)\cap S)}{\mathbb{P}_x(S)},$$

where $\mathbb{P}_x$(an event) means the probability that this event occurs provided there were x individuals initially and S means survival by which is meant that extinction will not happen forever. You should recall that this is a Markov process.

(e) What is the expected number of individuals at time t provided extinction will not happen?

9. Prove the following:

$$\begin{aligned}&\operatorname*{ms-lim}_{n\to\infty}\sum_{i=1}^{n}W\left(\alpha t_i+(1-\alpha)t_{i-1}\right)\left(W(t_i)-W(t_{i-1})\right)\\&=\frac{1}{2}\left[W(t)^2-W(t_0)^2\right]+\left(\alpha-\frac{1}{2}\right)(t-t_0),\end{aligned}$$

where $W(t)$ is the Wiener process and $0\le\alpha\le1$. One has to prove

$$\left\langle\left(\sum_i\Delta W_i^2-(t-t_0)\right)^2\right\rangle=2\sum_i(t_i-t_{i-1})^2\to0$$

or a similar relation.

10. Show that if a many-variable Itô equation is

$$dx_i = A_i(\boldsymbol{x}, t)dt + B_{ij}(\boldsymbol{x}, t)dW_j,$$

where W_j's are the independent Wiener processes, then the equivalent FPE is

$$\frac{\partial P}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}\left[A_i(\boldsymbol{x}, t)P\right] + \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\left[\sum_k B_{ik}B_{jk}P\right],$$

where $P = P(\boldsymbol{x}, t|\boldsymbol{x}_0, t_0)$. Also prove that if $C_{ij} = B_{ik}S_{kj}$, where S is orthogonal, that is $SS^{\mathrm{T}} = 1$, the Itô equation

$$dx_i = A_i(\boldsymbol{x}, t)dt + C_{ij}(\boldsymbol{x}, t)dW_j,$$

has the same FPE as above.

11. Using $\langle \exp(z)\rangle = \exp\left(\frac{1}{2}\langle z^2\rangle\right)$ for any Gaussian variable z with zero mean, calculate the mean and the autocorrelation function for the geometric Brownian motion (see the lecture note). If the equation for the geometric Brownian motion is interpreted as a Stratonovich one, show that

$$\begin{aligned}\langle x(t)\rangle &= \langle x(0)\rangle \exp\left[\frac{1}{2}c^2(t-t_0)\right], \\ \langle x(t)x(s)\rangle &= \langle x(0)^2\rangle \times \\ &\times \exp\left\{\frac{1}{2}c^2\left[t+s-2t_0+2\mathrm{min}(t-t_0, s-t_0)\right]\right\}.\end{aligned} \tag{2}$$

12. Consider the following equations (Itô interpretation)

$$\begin{aligned} d\xi_\gamma(t) &= -\gamma^2\xi_\gamma dt + \gamma^2 dW, \\ \frac{dx}{dt} &= cx\xi_\gamma(t). \end{aligned}$$

   (a) Show that $\lim_{\gamma\to\infty}\langle \xi_\gamma(t)\xi_\gamma(t')\rangle_s = \delta(t-t')$.

   (b) Show that in the limit of $\gamma \to \infty$, $x(t)$ is a Gaussian process with mean and autocorrelation function given by Eq. (2).

13. Solve the following SDE (Stratonovich interpretation):

$$(\mathrm{S})dx = \sqrt{x}dW, \quad x(t=0) = 0,$$

and find the generating function $G(k, t) = \langle e^{ikx(t)}\rangle$. Can you conclude that $P(x, t) = \delta(x)$? If not, how do you interprete the result?

14. **Project:** Considering the following reaction dynamics

$$X \underset{k_3}{\overset{k_1}{\rightleftharpoons}} 0\ , \quad X \xrightarrow{k_2} 2X\ ,$$

whose generating function can be obtained exactly, find the moment generating function $G(\mu, t)$ of the following SDE

$$dx = (a + bx)dt + \sqrt{2cx}dW.$$

with $a > 0$, $c > 0$, and $c > b$.

$$G(\mu, t) \equiv \int dx e^{-\mu x}\mathbb{P}(x, t)dx.$$

The initial condition is $P(x, t = 0) = \delta(x - 1)$.