Spherical Branes, Supersymmetric Localization, and Holography

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Motivation

Three easy pieces

1. Develop precision holography for non-conformal QFTs.
2. Harness the power of supersymmetric localization to understand and test holography.
3. Understand D-branes with curved worldvolume in string theory and supergravity.

Goal: Describe recent progress which relates these 3 topics.

Synopsis

- SYM theory can be formulated on S^d while preserving 16 supercharges for $2 \leq d \leq 7$.
- The path integral can be reduced to a matrix model via supersymmetric localization.
- The free energy and the $\frac{1}{2}$-BPS Wilson loop vev simplify in the planar limit and can be computed for large 't Hooft coupling.
- There are explicit smooth spherical Dp-brane supergravity solutions $(p = d - 1)$ which are holographically dual to the SYM theory.
- The usual asymptotically locally AdS holographic renormalization procedure does not apply but can be modified to study new supergravity solutions.
- Explicit precision test of the gauge/gravity duality.

General lessons

S^d is a useful IR regulator, compatible with supersymmetry, for the path integral of a QFT.

S^d serves as a simple and effective way to resolve some naked singularities arising from D-branes in string theory.

Plan

- Introduction and motivation ✓
- Maximal SYM on S^d and localization
- Spherical branes
- Non-conformal precision holography

Maximal SYM on S^d

Lagrangian and symmetries

The SYM Lagrangian on S^d is [Blau], [Minahan-Zabzine]

$$\mathcal{L} = -\frac{1}{2g_{\mathrm{YM}}^2}\mathrm{Tr}\Bigg(\frac{1}{2}F_{MN}F^{MN} - \Psi \not{D}\Psi + \frac{(d-4)}{2\mathcal{R}}\Psi\Lambda\Psi + \frac{2(d-3)}{\mathcal{R}^2}\phi^A\phi_A$$
$$+ \frac{(d-2)}{\mathcal{R}^2}\phi^i\phi_i + \frac{2\,\mathrm{i}}{3\mathcal{R}}(d-4)[\phi^A,\phi^B]\phi^C\varepsilon_{ABC} - K_m K^m\Bigg)\,.$$

The indices are $M, N = 0, \dots 9$, $A, B = 0, 8, 9$ and $i, j = d+1, \dots 7$. The 7 auxiliary fields K_m allow for off-shell formulation of supersymmetry. Take $\mathrm{SU}(N)$ gauge group for this talk.

- The terms proportional to $\mathcal{R}^{-1}$ and $\mathcal{R}^{-2}$ break the R-symmetry from $\mathrm{SO}(1, 9-d)$ to $\mathrm{SU}(1,1) \times \mathrm{SO}(7-d)$ (except for $d = 4$).
- The dimensionless 't Hooft coupling is $\lambda = \mathcal{R}^{4-d} g_{\mathrm{YM}}^2 N$.
- The theory is asymptotically free for $d < 4$, conformal for $d = 4$, and non-renormalizable for $d > 4$.
- The expected UV completions are: the $(2,0)$ 6d SCFT for $d = 5$; $(1,1)$ little string theory for $d = 6$; type IIA string theory for $d = 7$.

SUSY Localization - I

Use supersymmetric localization to compute the path integral. [Pestun], [Kim-Kim], [Minahan-Zabzine],...

The path integral reduces to a matrix model in all integer dimensions $2 \leq d \leq 7$.

Ignore instantons in the large N limit. The path integral simplifies and can be treated with $d = p + 1$ as a parameter.

$$Z = \int_{\text{Cartan}} [\mathrm{d}\sigma] \ \exp\left(-\frac{4\pi^{\frac{p+2}{2}} N}{\lambda \Gamma(\frac{p-2}{2})} \mathrm{Tr}\,\sigma^2\right) Z_{1-\text{loop}}(\sigma)\,,$$

where $\sigma = \mathcal{R}\phi_0$.

Solve this by a saddle point approximation for the eigenvalue density $\rho(\sigma)$

$$\frac{8\pi^{\frac{p+2}{2}}}{\lambda\Gamma\left(\frac{p-2}{2}\right)}\sigma = \fint G(\sigma - \sigma')\rho(\sigma')\mathrm{d}\sigma' \ .$$

The kernel is

$$\frac{\mathrm{i}G(\sigma)}{\Gamma(4-d)} = \frac{\Gamma(-\mathrm{i}\,\sigma)}{\Gamma(4-d-\mathrm{i}\,\sigma)} - \frac{\Gamma(\mathrm{i}\,\sigma)}{\Gamma(4-d+\mathrm{i}\,\sigma)} - \frac{\Gamma(d-3-\mathrm{i}\,\sigma)}{\Gamma(1-\mathrm{i}\,\sigma)} + \frac{\Gamma(d-3+\mathrm{i}\,\sigma)}{\Gamma(1+\mathrm{i}\,\sigma)}\,.$$

SUSY Localization - II

The result for the density is

$$\rho(\sigma) = \frac{2\pi^{\frac{p+2}{2}}(b^2-\sigma^2)^{\frac{4-p}{2}}}{\pi\lambda\Gamma(5-p)\Gamma(\frac{p}{2})} \ .$$

Here b is the edge of the eigenvalue distribution and is determined by normalization

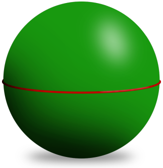
$$b^{5-p} = 32\lambda(4\pi)^{-\frac{p+2}{2}}\Gamma\left(\tfrac{7-p}{2}\right)\Gamma\left(\tfrac{5-p}{2}\right)\Gamma\left(\tfrac{p}{2}\right) \ .$$

Using this we find the free energy

$$F = -\frac{N^2 4\pi^{\frac{p+2}{2}}}{\lambda\Gamma(\frac{p-2}{2})}\frac{5-p}{(7-p)(p-3)}b^2 \ ,$$

and the expectation value of a $\frac{1}{2}$-BPS Wilson loop

$$\log\langle W\rangle = 2\pi b \ .$$



Special cases

The formulae above are valid for $3 < d < 6$. Results outside of this range can be obtained by careful regularization.

- For $d = 2$ one has

$$F = -\frac{4\sqrt{2\pi}}{3\sqrt{\lambda}} N^2 , \qquad \log\langle W\rangle = 2^{7/4}\pi^{3/4}\lambda^{1/4} .$$

- For $d = 3$ the calculation can be done for general λ

$$F = 0 , \quad \langle W\rangle = \frac{3}{\xi^3}(\xi\cosh\xi - \sinh\xi) , \qquad \xi^3 = 6\pi^2\lambda .$$

- For $d = 6$ one has to remove an exponential divergence (LST?) to find

$$F = -3N^2 \exp\left(-2 - \frac{16\pi^3}{3\lambda}\right) , \quad \log\langle W\rangle = 4\pi \exp\left(-1 - \frac{8\pi^3}{3\lambda}\right) .$$

- For $d = 7$ the divergence is more standard and one finds

$$F = \frac{16\pi^{10}}{3\lambda^3} N^2 , \qquad \log\langle W\rangle = -\frac{4\pi^4}{\lambda} .$$

Spherical branes

How to find the supergravity dual?

Naïve approach: Consider a suitable Ansatz in type II supergravity.

From symmetry considerations alone, we expect a background of type II supergravity of the form

$$
\begin{aligned}
\mathrm{d}s_{10}^2 \quad = \quad & \Delta\Big[\mathrm{d}r^2 + \mathcal{R}^2 \mathrm{e}^{2A}\mathrm{d}\Omega_{p+1}^2 + \mathrm{e}^{2B}\big(\mathrm{d}\theta^2 + P\cos^2\theta\,\mathrm{d}\widetilde{\Omega}_2^2 \\
& \qquad\qquad + Q\sin^2\theta\,\mathrm{d}\Omega_{5-p}^2\big)\Big]\,.
\end{aligned}
$$

where $\mathrm{d}\widetilde{\Omega}_2^2$ is the metric on dS$_2$. This realizes the $\mathrm{SO}(p+2)$ isometry and $\mathrm{SU}(1,1)\times\mathrm{SO}(6-p)$ R-symmetry.

Turn on all supergravity fluxes compatible with this symmetry.

Impose that the supersymmetry variations of type II supergravity vanish. This leads to nonlinear PDEs in (r,θ) for the functions in the ansatz.

Comment: The solutions should asymptote to the standard flat near horizon Dp-brane solutions in the UV and be regular in the IR.

[Itzhaki-Maldacena-Sonnenschein-Yankielowicz]

This is hard!

Gauged supergravity - I

Better idea: Reduce the problem to a supergravity in $p+2$ dimensions and uplift the solution found there back to ten dimensions.

The supergravity theory should be a truncation of maximal Euclidean supergravity with gauge group $G \supset \mathrm{SO}(1,8-p)$. The truncation should retain only the metric $g_{\mu\nu}$ and three scalar fields dual to the operators

$$\mathrm{Tr}\,|F|^2 \ , \quad \mathrm{Tr}\,\phi_a\phi^a \ , \quad \mathrm{Tr}\,\bar{\Psi}\Lambda\Psi \ .$$

The scalars should implement the breaking of the gauge group as in the field theory

$$\mathrm{SO}(1,8-p) \to \mathrm{SU}(1,1) \times \mathrm{SO}(6-p) \ .$$

The relevant supergravity theories are

- $p=6$, eight-dimensional $\mathrm{SO}(3)$ gSUGRA [Salam-Sezgin]
- $p=5$, seven-dimensional $\mathrm{SO}(4)$ gSUGRA [Samtleben-Weidner]
- $p=4$, seven-dimensional $\mathrm{SO}(5)$ gSUGRA reduced on S^1 [Pernici-Pilch-van Nieuwenhuizen]
- $p=2$, four-dimensional $\mathrm{SO}(7)$ gSUGRA [Hull]

Note: We have to Wick rotate these theories to Euclidean signature.

Gauged supergravity - II

Unified description extracted from various gauged SUGRA papers

$$\mathcal{L} = \frac{1}{2\kappa^2}\left[R - \frac{3p}{2(6-p)}|\partial\eta|^2 - 2\mathcal{K}_{\tau\tilde{\tau}}\partial\tau\cdot\partial\tilde{\tau} - V\right],$$

$$V = \frac{1}{2}\mathrm{e}^{\mathcal{K}}\left(\frac{6-p}{3p}\,|\partial_\eta\mathcal{W}|^2 + \frac{1}{4}\mathcal{K}^{\tau\tilde{\tau}}D_\tau\mathcal{W}D_{\tilde{\tau}}\widetilde{\mathcal{W}} - \frac{p+1}{2p}\,|\mathcal{W}|^2\right),$$

$$\mathcal{W} = \begin{cases} -g\ \mathrm{e}^{\frac{1}{2}\eta}\left(3\tau + (6-p)\mathrm{i}\mathrm{e}^{-\frac{p}{6-p}\eta}\right) & \text{for } p<3\,, \\ -g\ \mathrm{e}^{\frac{3(2-p)}{2(6-p)}\eta}\left(3\mathrm{i}\mathrm{e}^{\frac{p}{6-p}\eta} + (6-p)\tau\right) & \text{for } p>3\,. \end{cases}$$

The limit $p \to 6$ has to be taken with care. The scalar field has to be removed.

Comment: The theory for $p = 1$ is obtained by "analytic continuation". Would be interesting to derive it.

Constructing the solutions

We look for spherical domain walls of this theory:

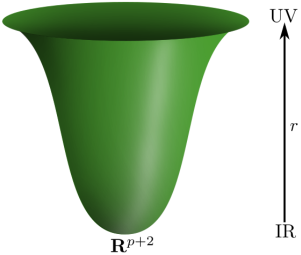
$$\mathrm{d}s^2_{p+2} = \mathrm{d}r^2 + \mathrm{e}^{2A(r)}\mathrm{d}\Omega^2_{p+1}\,.$$

Supersymmetry reduces the problem to three nonlinear ODEs and an algebraic relation for e^{2A}.

$$\begin{aligned}
(\eta')^2 &= \mathrm{e}^{\mathcal{K}}\left(\frac{6-p}{3p}\right)^2(\partial_\eta\mathcal{W})(\partial_\eta\widetilde{\mathcal{W}})\,,\\
(\eta')(\tau') &= \mathrm{e}^{\mathcal{K}}\left(\frac{6-p}{12p}\right)(\partial_\eta\mathcal{W})\,\mathcal{K}^{\tau\tilde{\tau}}D_{\tilde{\tau}}\widetilde{\mathcal{W}}\,,\\
(\eta')(\tilde{\tau}') &= \mathrm{e}^{\mathcal{K}}\left(\frac{6-p}{12p}\right)(\partial_\eta\widetilde{\mathcal{W}})\mathcal{K}^{\tilde{\tau}\tau}D_{\tau}\mathcal{W}\,,\\
\mathrm{e}^{2A} &= \mathrm{e}^{\mathcal{K}}\frac{4}{9g^4}\frac{(\tilde{\tau}-\tau)^2}{(\tilde{\tau}+\tau)^2}\mathrm{e}^{\frac{4(p-3)}{6-p}\eta}(\partial_\eta\mathcal{W})(\partial_\eta\widetilde{\mathcal{W}})\,.
\end{aligned}$$

Results

A 1-parameter family of explicit solutions parametrized by the value of the scalar η_{IR}.



UV: The solutions approach the near horizon limits of the flat Dp-branes supergravity solutions.

IR: Smooth R^{p+2} cap-off.

Note: Analytic solutions for $p = 1, 2, 3, 4, 6$. Numerical solution for $p = 5$.

Uplifting to 10d

The 10d string frame metric is

$$\mathrm{d}s_{10}^2 = Q^{-\frac{1}{2}}\mathrm{e}^{\eta}\left[\mathrm{d}s_{p+2}^2 + g^{-2}\mathrm{e}^{\frac{2(p-3)}{6-p}\eta}\left(\mathrm{d}\theta^2 + P\cos^2\theta\mathrm{d}\tilde{\Omega}_2^2 + Q\sin^2\theta\mathrm{d}\Omega_{5-p}^2\right)\right]$$

where

$$P = \begin{cases} X\Big(X\sin^2\theta + (X^2-Y^2)\cos^2\theta\Big)^{-1} & \text{for } p<3\,, \\ X\Big(\cos^2\theta + X\sin^2\theta\Big)^{-1} & \text{for } p>3\,, \end{cases}$$

$$Q = \begin{cases} X\Big(\sin^2\theta + X\cos^2\theta\Big)^{-1} & \text{for } p<3\,, \\ X\Big(X\cos^2\theta + (X^2-Y^2)\sin^2\theta\Big)^{-1} & \text{for } p>3\,. \end{cases}$$

and

$$\tau = \mathrm{i}\mathrm{e}^{-\frac{p}{6-p}\eta}(X+Y)\,, \quad \tilde{\tau} = -\mathrm{i}\mathrm{e}^{-\frac{p}{6-p}\eta}(X-Y)\,, \quad \text{for } p<3\,,$$
$$\tau = \mathrm{i}\mathrm{e}^{\frac{p}{6-p}\eta}(X+Y)\,, \quad \tilde{\tau} = -\mathrm{i}\mathrm{e}^{\frac{p}{6-p}\eta}(X-Y)\,, \quad \text{for } p>3\,.$$

The non-vanishing type II supergravity fields are given by

$$B_2 = \mathrm{e}^{\frac{p}{6-p}\eta}\frac{YP}{g^2X}\cos^3\theta\ \mathrm{vol}_2\,, \qquad \mathrm{e}^{2\Phi} = g_s^2\mathrm{e}^{\frac{p(7-p)}{6-p}\eta}\,P\,Q^{\frac{1-p}{2}}\,,$$

$$C_{5-p} = \mathrm{i}\mathrm{e}^{-\frac{p}{6-p}\eta}\frac{YQ}{g_s g^{5-p}X}\sin^{4-p}\theta\ \mathrm{vol}_{5-p}\,,$$

$$C_{7-p} = \frac{\mathrm{i}}{g_s g^{7-p}}\Big(\omega(\theta) + P\cos\theta\ \sin^{6-p}\theta\Big)\mathrm{vol}_2\wedge\mathrm{vol}_{5-p}\,.$$

Special cases

- The spherical type IIA D4-brane solution has a large dilaton in the UV. Uplift to M-theory to find the well-known $\mathrm{AdS}_7 \times S^4$ background. Note, the boundary is $S^5 \times S^1$.

- The D5-brane solution has a linear dilaton in the UV. It should be appropriate for studying the $\mathcal{N} = (1,1)$ little string theory on S^6.

- The spherical type IIA D6-brane solution also has a region with a large dilaton. Uplift to M-theory to find $\mathrm{S}_7 \times \mathbb{H}_{2,2}/\mathbb{Z}_N$.

$$
\begin{aligned}
\mathrm{d}s^2_{11} &= \frac{L^2}{4}\left(4\mathrm{d}\Omega_7^2 + \mathrm{d}s_4^2\right) , \quad L = \mathcal{R}/g_s^{1/3} \\
\mathrm{d}s_4^2 &= \mathrm{d}\rho^2 - \frac{\sinh^2\rho}{4}\left(\mathrm{d}t^2 - \cosh^2 t\,\mathrm{d}\psi^2 + (N^{-1}\mathrm{d}\omega - \sinh t\,\mathrm{d}\psi)^2\right) .
\end{aligned}
$$

The eleven-dimensional 4-form is given by

$$
G_4 = \frac{6\,\mathrm{i}}{L}\,\mathrm{vol}_{\mathbf{H}^{2,2}} ,
$$

Note: One can Wick rotate to $(1,10)$ signature with real flux.

Non-conformal holography

Holographic free energy - I

Set up the stage by defining a few key quantities.

The effective 't Hooft coupling is

$$\lambda_{\mathsf{hol}} = g^2_{\mathsf{YM,hol}} N E^{p-3}_{\mathsf{hol}} \ ,$$

where E is some energy scale. Following the QFT we set $E = \mathcal{R}^{-1}_{\mathsf{hol}}$.

Key point: Both g_{YM} and $\mathcal{R}_{\mathsf{hol}}$ run with energy.

We take

$$g^2_{\mathsf{YM,hol}} = \frac{2\pi}{(2\pi\ell_s)^{3-p}} \mathrm{e}^{\Phi} \ , \quad \mathcal{R}_{\mathsf{hol}} = Q^{-1/4} \mathrm{e}^{A+\eta/2} \ .$$

Putting all this together we find

$$\lambda_{\mathsf{hol}} = \lim_{r\to\infty} \frac{2\pi g_s N}{(2\pi\ell_s)^{3-p}} \mathrm{e}^{(3-p)A} \mathrm{e}^{\frac{9-p}{6-p}\eta} \ .$$

Note: This is finite for all p.

Holographic free energy - II

The holographic free energy is computed by evaluating the $p+2$-dimensional supergravity action on-shell.

$$F_{\text{hol}} = S^{\text{Ren.}}_{\text{on-shell}} \ .$$

Problem: The backgrounds we have are no asymptotically locally AdS.
Solution: Employ the dual frame of [Kanitscheider-Skenderis-Taylor]

$$g_{\mu\nu} = \mathrm{e}^{2a\eta} \tilde{g}_{\mu\nu} \,, \qquad \text{where} \quad a = \frac{p-3}{6-p} \,.$$

The case $p = 6$ needs to be treated separately. The gSUGRA action takes the form

$$S = \frac{1}{2\kappa^2_{p+2}} \int \mathrm{d}^{p+2}x \sqrt{\tilde{g}}\, \mathrm{e}^{pa\eta} \Big\{ \tilde{R} + \left(\tfrac{3p}{2(p-6)} + a^2 p(p+1) \right) |\mathrm{d}\eta^2| \\ -2\mathcal{K}_{\tau\tilde{\tau}} |\mathrm{d}\tau|^2 - \mathrm{e}^{2a\eta} V \Big\} \,.$$

Now apply the standard holographic renormalization to the fields in the new action. This works for all cases $p \neq 5$.
Upshot: For all values of p we are able to show

$$F_{\text{hol}} = F_{\text{QFT}} \ .$$

Example

For $p = 1, 4$ there are finite counterterms that can be added to the action. Tune their coefficients to get a match with the localization results.

Take D4-branes, i.e. 5d SYM. The finite counterterms are

$$S_{\text{ct,fin}} = \frac{1}{\kappa_6^2} \int \mathrm{d}^5 x \sqrt{\tilde{h}} \mathrm{e}^{2\eta} \left(c_1 \left(\frac{1}{g} \tilde{R} Y^2 - 20 g Y^4 \right) + c_2 g Y^6 + \frac{c_3}{g} \tilde{R} Y^4 + \frac{c_4}{g^3} \tilde{R}^2 Y^2 \right) .$$

Fix

$$c_1 = c_2 = c_4 = 0 \,, \qquad c_3 = -\frac{1}{10} \,,$$

to get a match with holography.

Resolves a puzzle in the literature where the naive AdS$_7$ action with $S^5 \times S^1$ boundary was evaluated. [Kallen-Minahan-Nedelin-Zabzine]

Similar problem with the on-shell action of AdS$_5$ with $S^3 \times S^1$ boundary. [Benetti Genolini-Cassani-Martelli-Sparks]

Should be made more rigorous using a proper supersymmetric regularization scheme. [Papadimitriou]

Holographic Wilson loop

The vev of a (fundamental) Wilson loop is given by the renormalized on-shell action of a probe string placed in the supergravity background [Maldacena], [Rey-Yee]

$$\log\langle W\rangle = -S^{\text{Ren.}}_{\text{string}} \ .$$

We must first determine a stable configuration for the probe string

$$S_{\text{string}} = \frac{1}{2\pi\ell_s^2}\int(\sqrt{|g|}\mathrm{d}^2\sigma - B_2)\,.$$

The string action is minimized for a string wrapping the great circle of the S^{p+1}. On the internal space it sits at $\theta = 0$ and a point on $\mathrm{d}\widetilde{\Omega}_2^2$.

For this configuration the action simplifies to

$$S_{\text{string}} = \frac{1}{\ell_s^2}\int \mathrm{e}^{A+\eta}\ \mathrm{d}r\ .$$

This diverges in the UV and should be regularized by using the counterterm

$$\mathcal{L}_{\text{ct}} = \frac{1}{\ell_s^2}\mathrm{e}^{A+\frac{3}{6-p}\eta}|_{r\to\infty}\ .$$

A precise match with the localization results for all p!

Summary

- Studied maximal SYM on S^d for $2 \leq d \leq 7$.
- Explicit results for the free energy and the $\frac{1}{2}$-BPS Wilson loop vev in the large N, large λ limit.
- For $d = 3$ exact result for the $\frac{1}{2}$-BPS Wilson loop vev in the large N limit as a function of λ.
- Explicit supergravity spherical brane solutions.
- Non-trivial application of holographic renormalization which leads to precision tests of non-conformal holography.

Outlook

- Generalize to SYM theories with less supersymmetry.
- Other compact manifolds, i.e. other curved D-branes?
- For $d = 3$ we have a trivial free energy but an exact result for the Wilson loop as a function of λ. Understand this better. Integrability?
- Lessons from our results in $d = 6$ for little string theory?
  [Aharony-Evtikhiev-Feldman]
- Understand $d = 7$ better.
- Is it possible to compute $1/N$ and $1/\lambda$ corrections both in QFT and in supergravity? [Binder-Chester-Pufu-Wang]

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Thank You!