SGC2020， APCTP, 2020.11.17-19

No Inner Horizon Theorem

For Black Holes With Charged Scalar Hairs

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Ref: RGC, L.Li and R.Q. Yang, arXiv: 2009.05520

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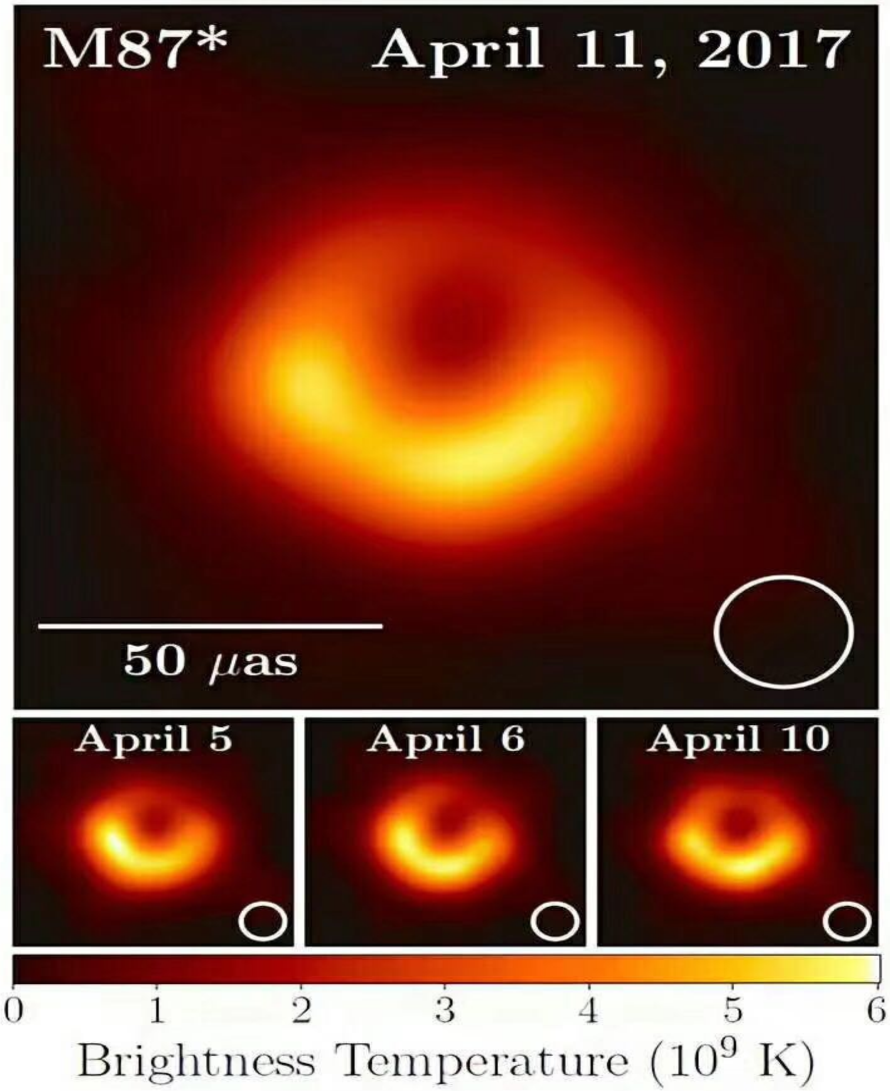
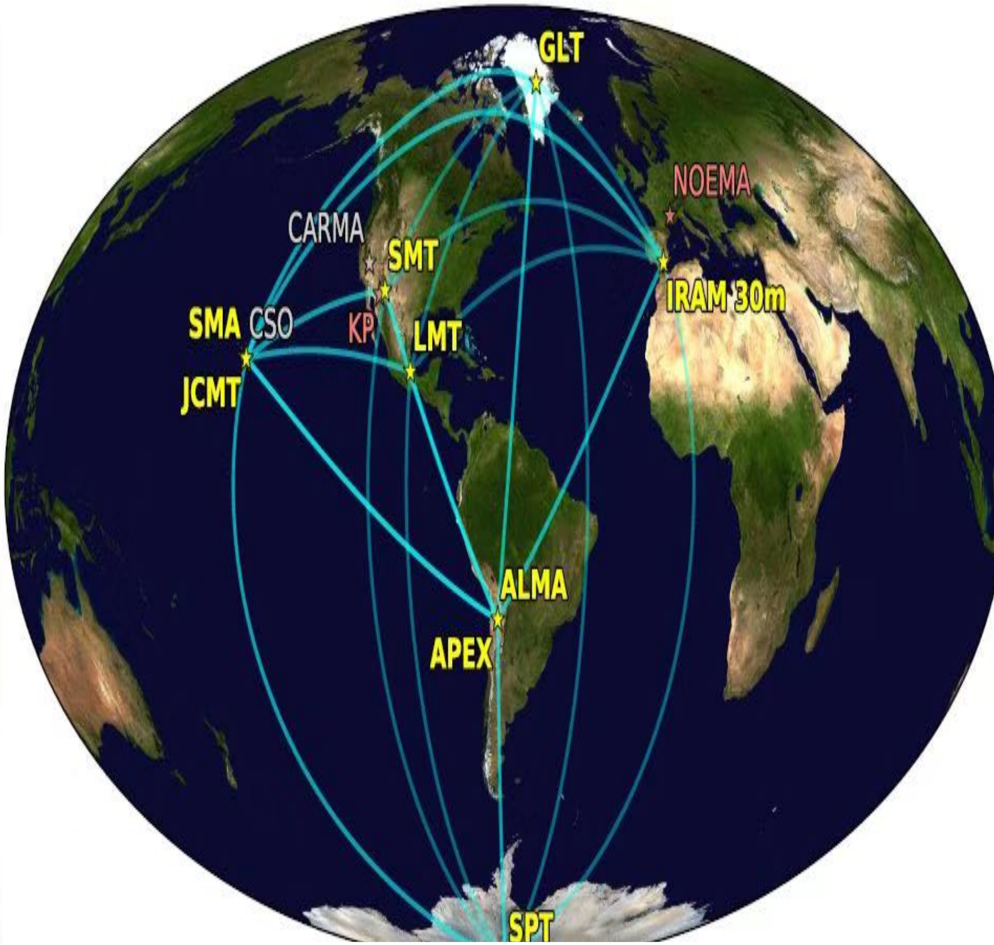
2020.10.6， Nobel prize in 2020

April 10, 2019：first time to directly see a black hole in Nature

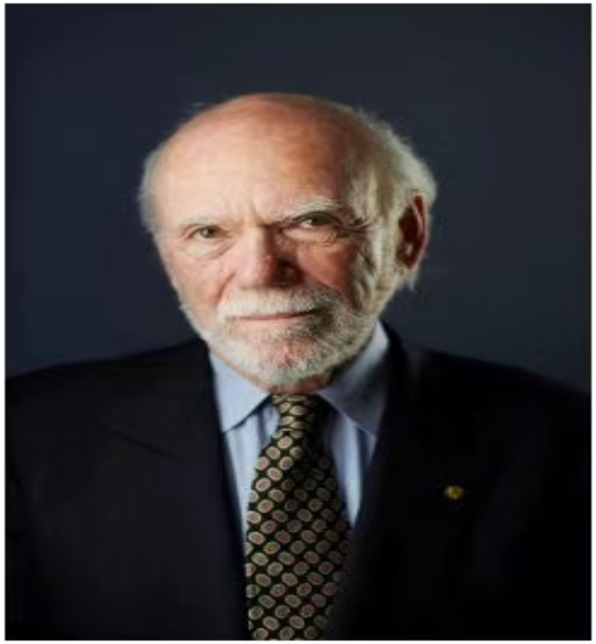




M87: 5300万光年~4.5x 10^23m，
65亿个太阳质量~1x10^13m

The Nobel Prize in Physics 2017

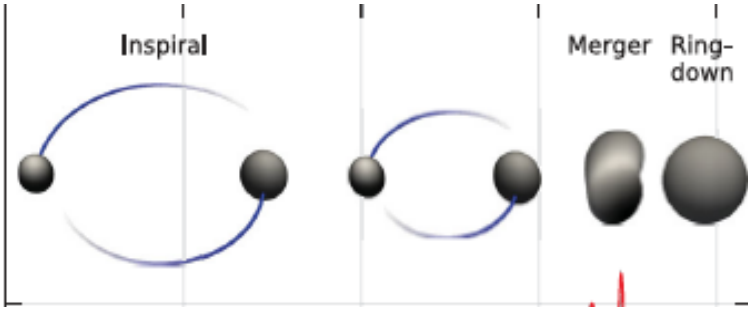
Rainer Weiss



Barry C. Barish

Kip S. Thorne

GW 150914:



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Questions:

What is the exterior geometry of a BH?

What is the inner structure of a BH?

“ A black hole has no hair” (R.Ruffini and J. Wheeler, 1971)

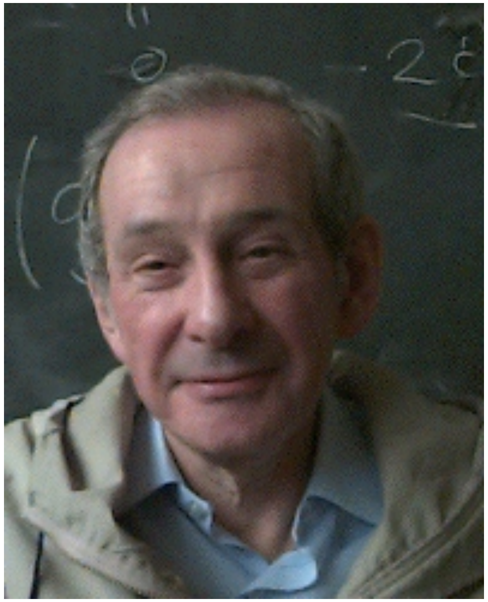
No-hair theorem of black holes (uniqueness theorem):

The most general, asymptotically flat stationary solution of Einstein-Maxwell equations is the Kerr-Newman solution!

Ref: M. Heusler

Black Hole Uniqueness Theorem

Cambridge University Press, 1996



(W. Israel)

Black hole in GR: Kerr-Newman solution

$$ds^2 = -(1 - \frac{2Mr - Q^2}{\rho^2})dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

$$+[(r^2 + a^2) + \frac{(2Mr - Q^2)a^2 \mathrm{Sin}^2\theta}{\rho^2}]\sin^2\theta d\varphi^2 - 2\frac{(2Mr - Q^2)a\mathrm{Sin}^2\theta}{\rho^2}dtd\varphi$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 + a^2 - 2Mr + Q^2$$

1) When a=0, Reissner-Nordstrom black hole solution
2) When Q=0. Kerr black hole solution
3) When a=Q=0, Schwarzschild black hole solution

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Three features:

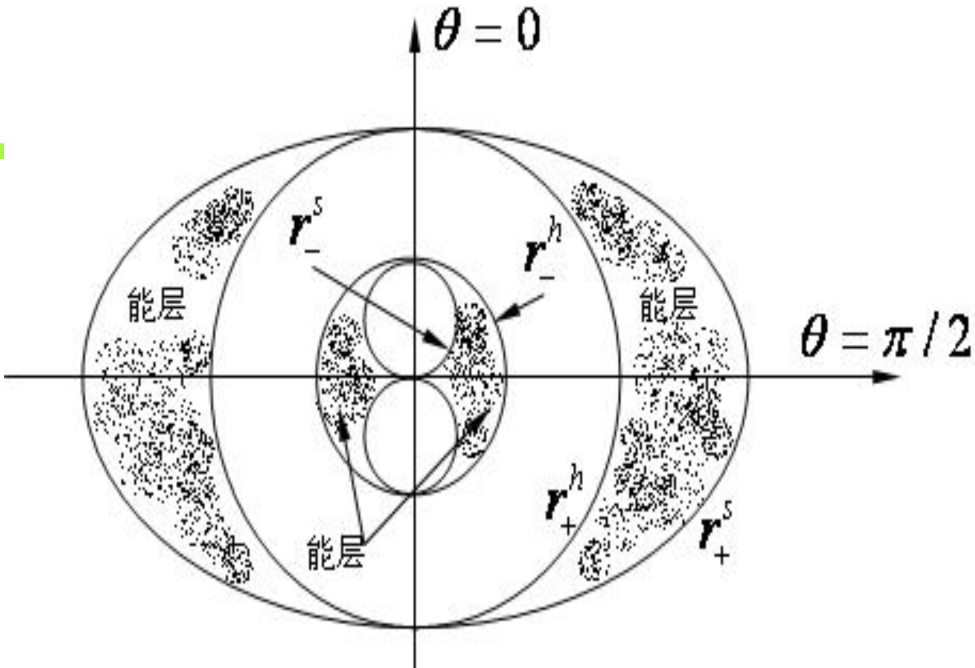
1) three parameters: M, J, Q

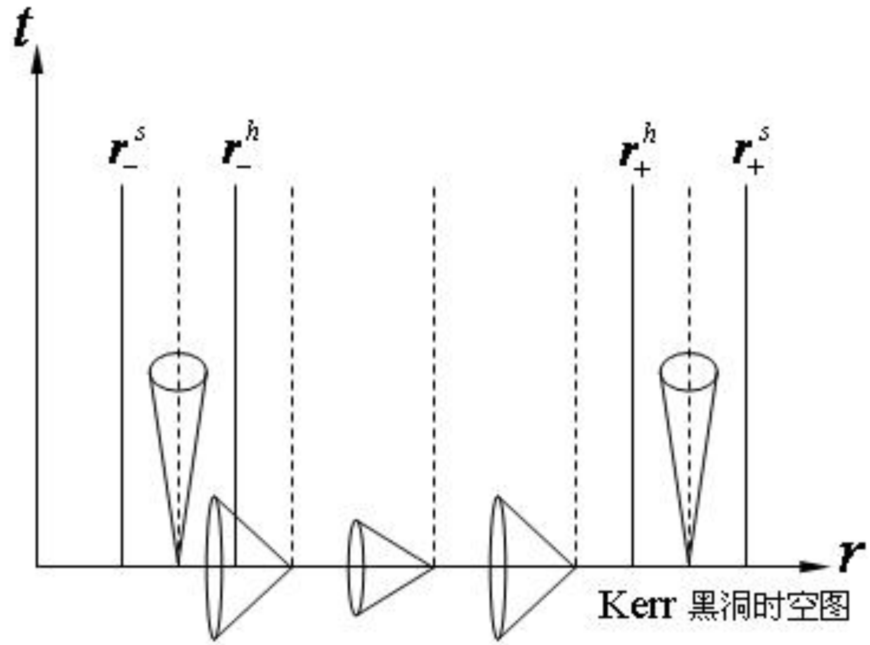
2) r= 0 is singular : singularity ring, x^2 +y^2 =a^2

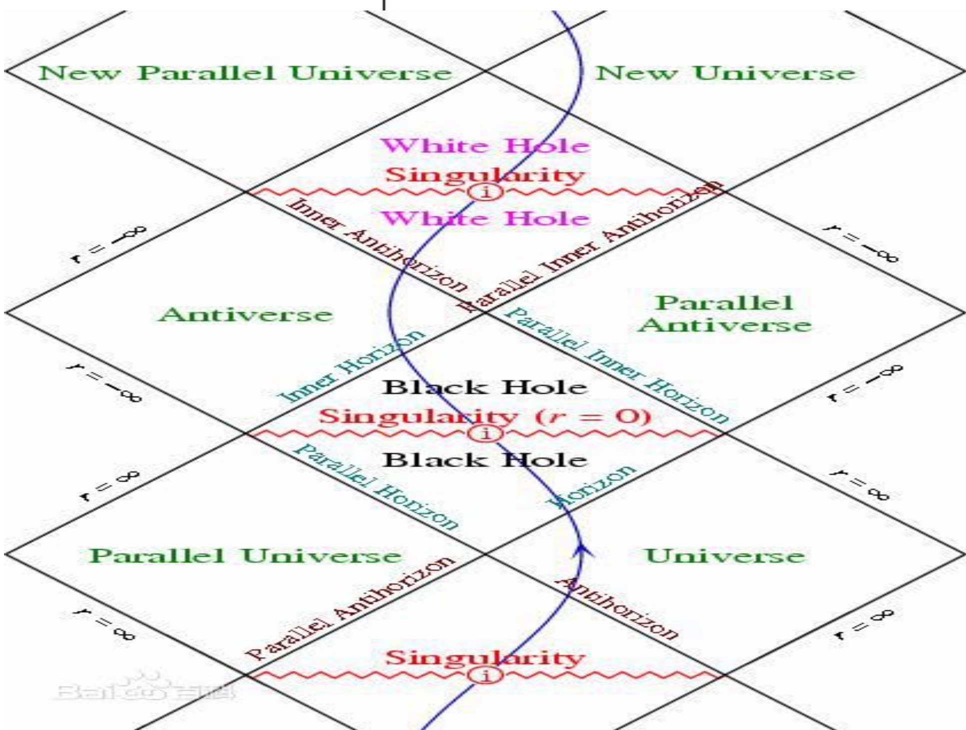
3) Two horizons:

$$\Delta = 0 \longrightarrow r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}$$

If the inner (Cauchy) horizon r_- is stable, then some interesting things happen!


$\theta = 0$
$\theta = \pi / 2$
r_-^s
r_-^h
r_+^h
r_+^s
能层
能层
能层


t
r
r_-^s
r_-^h
r_+^h
r_+^s
Kerr 黑洞时空图


New Parallel Universe
New Universe
White Hole
Singularity
White Hole
Inner Antihorizon
Parallel Inner Antihorizon
Antiverse
Parallel Antiverse
Inner Horizon
Parallel Inner Horizon
Black Hole
Singularity ($r = 0$)
Black Hole
Parallel Horizon
Horizon
Parallel Universe
Universe
Parallel Antihorizon
Antihorizon
Singularity
$r = -\infty$
$r = -\infty$
$r = -\infty$
$r = -\infty$
$r = \infty$
$r = \infty$
$r = \infty$
$r = \infty$

时空穿越和星际旅行

1) No hair theorem !?

If some matter presents

2) Cauchy horizon is unstable (mass inflation)

what will the CH turn to be?

2. No hair theorem (no scalar hair theorem)

In its original form, "black holes have no hairs' held that a black hole can be dressed only by fields like the EM one, which are associated with a Gauss-like law. Early ho hair theorem exclude scalar, massive vector and spinor fields from a stationary black hole's exterior.

But, in 1990's, various black hole with hairs were found, including black holes with Yang-Mills, Proca-type YMs, and Skyrme fields in various combination with Higgs fields.

In PRD 51 (1995) R 6608, J. Bekenstein showed:

Novel "no-scalar-hair" theorem for black holes

We formulate a new "no-hair" theorem for black holes in general relativity which rules out a multicomponent scalar field dressing of any asymptotically flat, static, spherically symmetric black hole. The field is assumed to be minimally coupled to gravity, and to bear a non-negative energy density as seen by any observer, but its field Lagrangian need not be quadratic in the field derivatives. The proof centers on energy-momentum conservation and the Einstein equations. One kind of field ruled out is the Higgs field with a double (or multiple) well potential. The theorem is also proved for scalar-tensor gravity.

PACS number(s): 97.60.Lf, 04.70.Bw, 11.15.Ex, 95.30.Tg

In PRD 54 (1996) 5059, Mayo and Bekenstein proved

No hair for spherical black holes: charged and nonminimaly coupled scalar field with self-interaction

They prove three theorems in general relativity which rule out classical scalar hair of static, spherically symmetric, possibly electrically charged black holes. They first generalize Bekenstein's no–hair theorem for a multiplet of minimally coupled real scalar fields with not necessarily quadratic action to the case of a charged black hole. They then use a conformal map of the geometry to convert the problem of a charged (or neutral) black hole with hair in the form of a neutral self–interacting scalar field nonminimally coupled to gravity to the preceding problem, thus establishing a no–hair theorem for the cases with nonminimal coupling parameter $\xi < 0$ or $\xi \geq 1$. The proof also makes use of a causality requirement on the field configuration. Finally, from the required behavior of the fields at the horizon and infinity they exclude hair of a charged black hole in the form of a charged self–interacting scalar field nonminimally coupled to gravity for any ξ.

Simply say:

There exists no non-extremal static and spherical charged black hole endowed with hair in the form of a charged scalar field, whether minimally or nonminimally coupled to gravity, and with a regular positive semi-definite self-interaction potential.

However, very recently, it has been found:

Spherically symmetric scalar hair for charged black hole

PRL125 (2020) 111104 by J.P. Hong, M. Suzuki and M. Yamada

A mass term is important at an asymptotic infinity, which was omitted in the Mayo and Bekenstein's proof. In this work, it shows there indeed exists static and spherically symmetric black holes with charged scalar hairs, dubbed as Q-hairs, by considering the backreaction of metric and gauge field.

On the other hand, in PRL 76 (1996) 571
by D. Nunez H. Quevedo and D. Sudarsky

Black holes have no short hair

It was showed that in all theories in which black hole hair has been discovered, the region with non-trivial structure of the non-linear matter fields must extended beyond 3/2 the horizon radius, independently of all other parameters present in the theory. It was argued that this is a universal lower bound that applies in every theory where the hair is present.

For the case with a cosmological constant:
PRD 58 (1998) 024002 by R.G. Cai and J.Y. Ji

Hairs on the cosmological horizon

We investigate the possibility of having hairs on the cosmological horizon. The cosmological horizon shares similar properties of black hole horizons in the aspect of having hairs on the horizons. For those theories admitting haired black hole solutions, the nontrivial matter fields may reach and extend beyond the cosmological horizon. For Q-stars and boson stars, the matter fields cannot reach the cosmological horizon. The no short hair conjecture keeps valid, despite the asymptotic behavior (de Sitter or anti-de Sitter) of black hole solutions. We prove the no scalar hair theorem for anti-de Sitter black holes. Using the Bekenstein's identity method, we also prove the no scalar hair theorem for the de Sitter space and de Sitter black holes if the scalar potential is convex.

Theorem 1: In the spherically symmetric, asymptotically (anti-)de Sitter black hole spacetime with matter fields satisfying the weak energy condition, the energy density going to zero faster than r−4, and the trace of stress-energy tensor being non-positive, if the nontrivial matter configuration reaches the black hole horizon, it must extend beyond a universal critical point satisfying r(crit) = 3m(rcrit), where m(r) is the mass function in the metric. Or the no short hair conjecture keeps valid for asymptotically (anti-)de Sitter black holes.

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Theorem 2: In the Einstein-minimally coupled scalar field system with a positive semi-definite scalar potential and a negative cosmological constant, the static, spherically symmetric black hole solution with a regular horizon and possessing asymptotically anti-de Sitter behavior is the Schwarzschild-anti-de Sitter spacetime and the scalar field is a constant corresponding to a local extremum of this potential.

Theorem 3: In the Einstein-minimally coupled scalar field system with a positive semi-definite, convex scalar potential and a positive cosmological constant, the spherically symmetric, asymptotically de Sitter solution (with a cosmological horizon and no black hole horizon) and the spherically symmetric, asymptotically de Sitter black hole solution (with a regular black hole horizon and a cosmological horizon) are the de Sitter space and the Schwarzschild–de Sitter black hole, respectively. And the scalar field is a constant corresponding to the point of the minimum of this potential.

PRL 99 (2007) 201101 by S. Bhattacharya and A. Lahiri

No hair theorems for a positive Λ

Abstract

We extend all known black hole no-hair theorems to space-times endowed with a positive cosmological constant Λ. Specifically, we prove that static spherical black holes with $\Lambda > 0$ cannot support scalar fields in convex potentials and Proca-massive vector fields in the region between black hole and cosmic horizons. We also demonstrate the existence of at least one type of quantum hair, and of exactly one charged solution for the Abelian Higgs model. Our method of proof can be applied to investigate other types of quantum or topological hair on black holes in the presence of a positive Λ.

3. No inner horizon theorem

The model: we consider a (d+2)-dimensional gravity theory coupled with a Maxwell field and a charged scalar field:

$$S = \frac{1}{2\kappa_N^2}\int d^{d+2}x\sqrt{-g}\,[\mathcal{R} + \mathcal{L}_M]\,,$$

$$\mathcal{L}_M = -\frac{Z(|\Psi|^2)}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\Psi)^* D^\mu\Psi - V(|\Psi|^2)$$

The black hole solution ansatz:

$$ds^2 = \frac{1}{z^2}\left[-f(z)e^{-\chi(z)}dt^2 + \frac{dz^2}{f(z)} + d\Sigma_{d,k}^2\right],$$

$$\Psi = \psi(z) \quad A = A_t(z)dt\,,$$

$$d\Sigma_{d,k}^2 = \begin{cases} d\theta^2 + \sin^2\theta d\Omega_{d-1}^2, & k = 1\,, \\ \sum_{i=1}^{d} dx_i^2, & k = 0\,, \\ d\theta^2 + \sinh^2\theta d\Omega_{d-1}^2, & k = -1\,, \end{cases}$$

RGCai and Y.Z. Zhang, PRD 54 (1996) 4891

The equations of motion:

$$V_{\text{eff}}(x) = V(x) - \tfrac{1}{2} Z(x) z^4 e^{\chi} A_t'^2$$

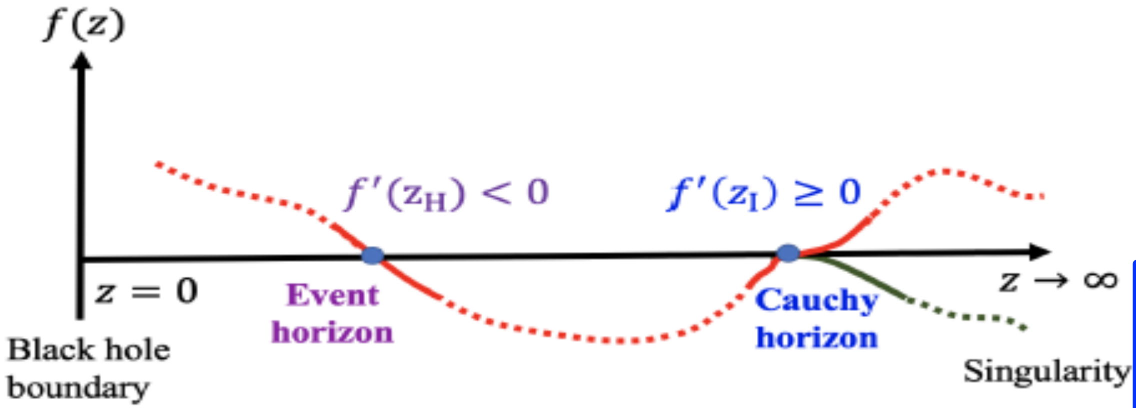
$$z^{d+2} e^{\chi/2} (e^{-\chi/2} z^{-d} f \psi')' = \left[\dot{V}_{\text{eff}}(\psi^2) - \frac{q^2 z^2 e^{\chi} A_t^2}{f} \right] \psi$$

$$z^d [Z(\psi^2) e^{\chi/2} z^{2-d} A_t']' = \frac{2 q^2 \psi^2 e^{\chi/2}}{f} A_t \,,$$

$$\frac{d}{2} \chi' = z \psi'^2 + \frac{z e^{\chi} q^2 \psi^2 A_t^2}{f^2} \,,$$

$$\frac{d}{2} \frac{f'}{f} - \frac{z}{2} \psi'^2 - \frac{d(d+1)}{2z} = \frac{V_{\text{eff}}(\psi^2)}{2 z f} - \frac{k d (d-1) z}{2f} + \frac{z e^{\chi} q^2 A_t^2}{2 f^2} \psi^2 + \frac{Z(\psi^2) z^3 e^{\chi} A_t'^2}{2f} \,,$$

$$A_t(z_H) = A_t(z_I) = 0$$



$$f'(z_H) < 0, \quad f'(z_I) \geq 0 \,.$$

The key observation: the existence of the conserved quantity

$$\mathcal{Q}(z) = z^{2-d} e^{\chi/2}\left[z^{-2}(f e^{-\chi})' - Z A_t A_t'\right] + 2k(d-1)\int^z y^{-d} e^{-\chi(y)/2} dy\,.$$

Namely, $\boxed{\mathcal{Q}'(z) = 0.}$

Let us evaluate this quantity both at the horizons

$$\mathcal{Q}(z_j) = \frac{f'(z_j)}{z_j^d} e^{-\chi(z_j)/2} + 2k(d-1)\int^{z_j} y^{-d} e^{-\chi(y)/2} dy$$

$$\Rightarrow \boxed{\frac{f'(z_H)}{z_H^d} e^{-\chi(z_H)/2} - \frac{f'(z_I)}{z_I^d} e^{-\chi(z_I)/2} = 2k(d-1)\int_{z_H}^{z_I} y^{-d} e^{-\chi(y)/2} dy}$$

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$$\frac{f'(z_H)}{z_H^d}e^{-\chi(z_H)/2}-\frac{f'(z_I)}{z_I^d}e^{-\chi(z_I)/2}$$
$$=2k(d-1)\int_{z_H}^{z_I} y^{-d}e^{-\chi(y)/2}dy$$

(1) when k=0 or k=1: The lhs <0, while the rhs >=0;

(2) when k=-1, the both sides have the same sign:
counter partner exists, see later. But there only exists at most one inner horizon with non zero surface gravity.

The proof is as follows:

(i) The horizon z_I must be a single root, otherwise we must have f''(z_I) <=0

$$Q'(z_I) = \frac{f''(z_I)}{z_I^d} e^{-\chi(z_I)/2} - Z z_I^{2-d} e^{\chi(z_I)/2} A_t'(z_I)^2 - 2(d-1) z_I^{-d} e^{-\chi(z_I)/2} < 0 .$$

This is not consistent with Q'(z)=0! Thus z_i should be a single root with f'(z_I) >0.

(ii) Suppose there exists a second inner horizon z_II

$$\frac{f'(z_I)}{z_I^d} e^{-\chi(z_I)/2} - \frac{f'(z_{II})}{z_{II}^d} e^{-\chi(z_{II})/2} = -2(d-1) \int_{z_I}^{z_{II}} y^{-d} e^{-\chi(y)/2} dy$$

$$\frac{f'(z_I)}{z_I^d}e^{-\chi(z_I)/2} - \frac{f'(z_{II})}{z_{II}^d}e^{-\chi(z_{II})/2} = -2(d-1)\int_{z_I}^{z_{II}} y^{-d}e^{-\chi(y)/2}dy$$

The lhs is positive, while the rhs is negative, therefore the second inner horizon is impossible.

4. Singularity

For simplicity, consider Z=1 and the kinetic term of the scalar field dominates. In that case, the potential can be neglected.

$$\psi = \sqrt{d}\alpha \ln z + \cdots\,, \ A_t' = E_s z^{d-2-\alpha^2} + \cdots\,,$$
$$e^{\chi} = \chi_s z^{2\alpha^2} + \cdots\,, \ f = -f_s z^{1+d+\alpha^2} + \cdots\,,$$

In that case,

$$Q(z) = \frac{1}{2\kappa_N^2}\int_\Sigma Z\,{}^\star F = -\frac{\omega_{(d)}}{2\kappa_N^2} Z z^{2-d} e^{\chi/2} A_t'\,,$$

The charge approaches to a constant at the singularity

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Introduce the proper time $\tau \sim z^{-(1+d+\alpha^2)/2}$

$$\boxed{\begin{aligned} ds^2 &= -d\tau^2 + c_t \tau^{2p_t} dt^2 + c_s \tau^{2p_s} d\Sigma_{d,k}^2 \,, \\ \psi(z) &= -p_\psi \ln \tau \,, \end{aligned} \qquad (18) \\ \text{where} \\ p_t = \frac{1-d+\alpha^2}{1+d+\alpha^2},\ p_s = \frac{2}{1+d+\alpha^2},\ p_\psi = \frac{2\sqrt{d}\alpha}{1+d+\alpha^2}\,. \qquad (19)}$$

We have

$$p_t + dp_s = 1, \quad p_t^2 + dp_s^2 + p_\psi^2 = 1\,,$$

The metric has a (generalized) Kasner form!

In the above, the following constraint must be obeyed:

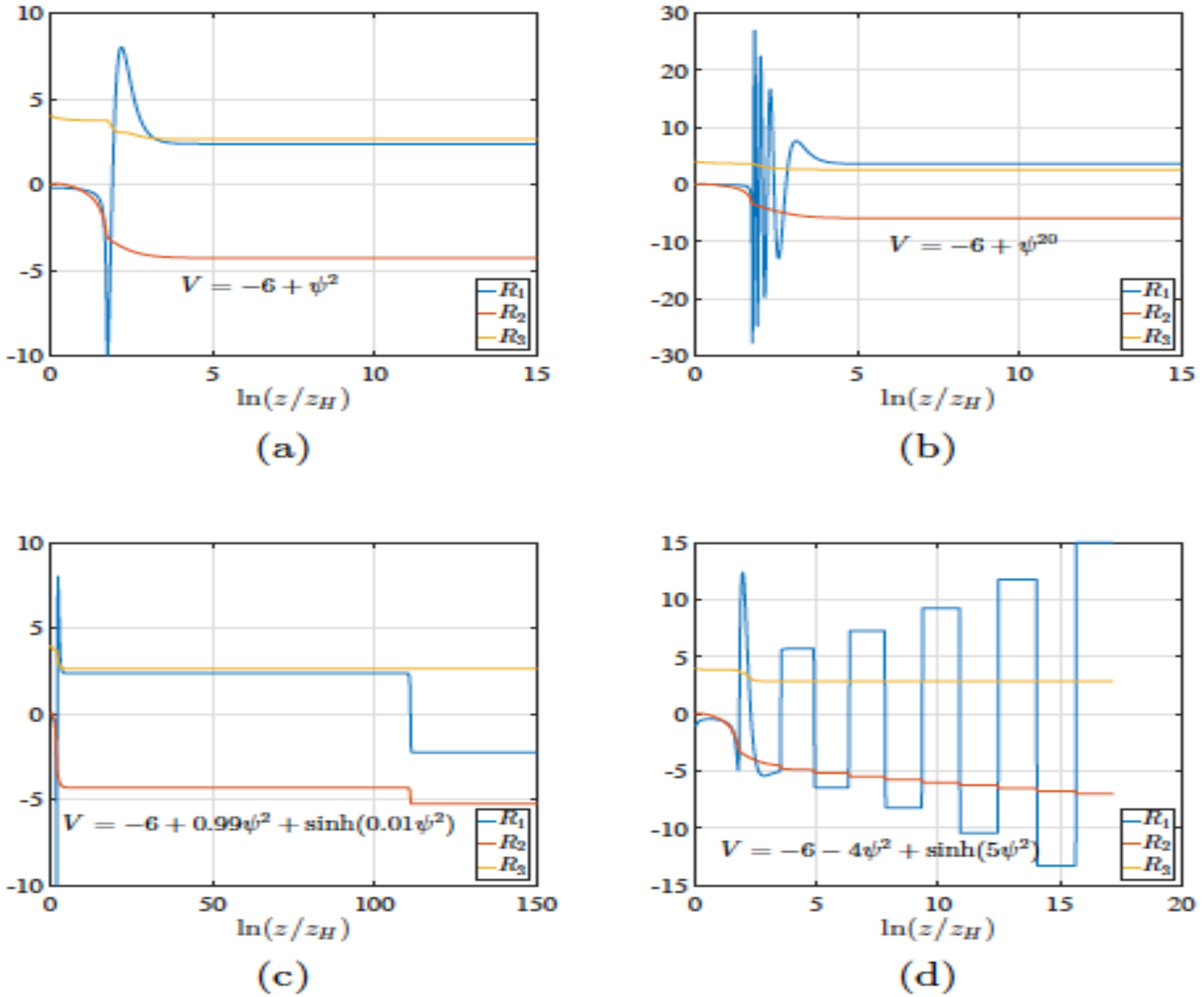
$$\lim_{z\to\infty} \frac{|V(\psi^2)|}{z^{d+1+\alpha^2}} \ll 1\,,$$

which allows the potential V to be arbitrary algebraic function, including polynomial function, for example, V= m^2 \phi^2.

Some examples: we define

$$R_1 = z\psi'\,,\ R_2 = \ln\left(\frac{z_H^2}{z^2} - h\right),\ R_3 = 4z^{2-d}e^{\chi/2}A_t'\,, \quad \text{with } h = e^{-\chi/2}f/z^{1+d}.$$

Note that for Kasner geometry, they are constants when $z\to\infty$


$V = -6 + \psi^2$
$V = -6 + \psi^{20}$
$V = -6 + 0.99\psi^2 + \sinh(0.01\psi^2)$
$V = -6 - 4\psi^2 + \sinh(5\psi^2)$
R_1
R_2
R_3
$\ln(z/z_H)$
(a)
(b)
(c)
(d)

5. Hyperbolic black hole with inner horizon

Consider a model in four dimensions:

$$V(\psi^2) = -6 + m^2\psi^2 \,, \quad Z = 1 \,,$$

The equations of motion:

$$\chi' = \frac{2}{d}\left[\frac{\psi^2 A_t^2 q^2}{h^2 z^{2d+1}} + z\psi'^2\right],$$

$$h' = -\frac{(d-1)k}{z^d} e^{-\chi/2} + \frac{e^{-\chi/2}}{d}\left(\frac{\tilde{Q}^2 z^{d-2}}{2} + \frac{V(\psi^2)}{z^{d+2}}\right),$$

$$\tilde{Q}' = \frac{2\psi^2 A_t q^2}{z^{2d+1} h}, \qquad A_t' = \tilde{Q} e^{-\chi/2} z^{d-2} \,,$$

$$\psi'' = -\left(\frac{h'}{h} + \frac{1}{z}\right)\psi' + \left(\frac{\dot{V}(\psi^2) e^{-\chi/2}}{z^{d+3} h} - \frac{A_t^2 q^2}{h^2 z^{2d+2}}\right)\psi \,,$$

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$$h := f e^{-\chi/2} z^{-1-d}, \quad \tilde{Q} := -\frac{2\kappa_N^2}{\omega_{(d)}} Q = z^{2-d} e^{\chi/2} A_t' \,.$$

Take the parameter and initial conditions:

$m^2 = -0.18388$ and $q = 1.5$. At the event horizon $z_H = 1.193936$,

$$\psi(z_H) \approx 1.10683410, \quad \psi'(z_H) \approx 0.115816263, \quad \tilde{Q}(z_H) \approx 0.650999915, \quad \chi(z_H) = h(z_H) = A_t(z_H) = 0.$$

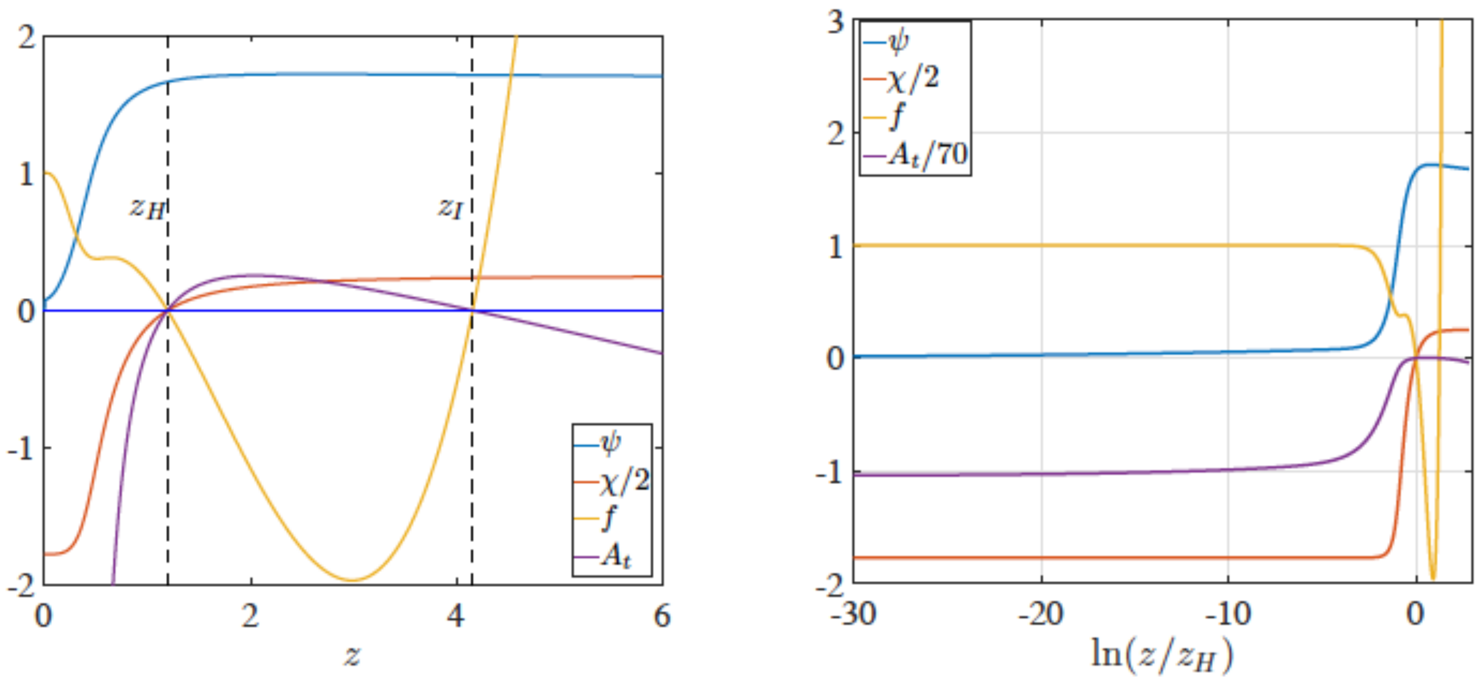

FIG. 3. Numerical solution for hyperbolic case ($k = -1$) with the boundary conditions (30). The hairy black hole has the event horizon at $z_H = 1.193936$. There is a Cauchy horizon at $z_I \approx 4.15699837$ and all functions are smooth at two horizons. From the right panel, we see that $f(0) = 1, \psi(0) = 0$, $\chi(0)$ and $A_t(0)$ are both finite, which implies that this solution is indeed an asymptotically AdS black hole. We have considered the four dimensional model with $V(\psi^2) = -6 - 0.18388\psi^2$, $Z = 1$ and $q = 1.5$.

Conclusions:

We establish a no inner-horizon theorem for black holes with charged scalar hair. Considering a general gravitational theory with a charged scalar eld, we prove that there exists no inner Cauchy horizon for both spherical and planar black holes with non-trivial scalar hair. The hairy black holes approach a spacelike singularity at late interior time. This result is independent of the form of scalar potentials as well as the UV completion of spacetime. For a large class of solutions, we show that the geometry near the singularity resembles the Kasner type solution. When the potentials become important, the behaviors that are quite distinct from the Kasner form are observed. For the hyperbolic horizon case, we show that it only has at most one inner horizon. All these features are also valid for the Einstein gravity coupled with a neutral scalar.

谢谢！

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